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Topology and combinatorics of arrangement covers and of nested set complexes

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Allegro giusto, nel modo russico; senza allegrezza, ma poco sostenuto.

M. P. Musorgskij, "Tableaux d'une exposition"

Foreword

DEFINITION 0.0.0. **Dissertatio**, onis, f.*: a spoken discourse, disquisition (not a written treatise), a debating speech. From* **disserto**, are, v.*, (to hold a speech over something, to argue, to debate a thing), frequentative of* **dissero**, ere, v.*: to set forth in order, to arrange distinctly, to discuss with words, to develop something (see* [**74**]*,* [**85**]*).*

Debate, colloquiality and (re)arrangement of ideas are therefore essential constitutives of a dissertation.

Even before the writing began, it was pretty clear that the reader of this thesis would not have missed arrangements (even though 'of hyperplanes' rather than 'of ideas'). Special care was nevertheless taken in order to fulfill all the above criteria. For us, colloquiality translates into 'readability': a quality that is especially challenging to reach in a discipline involving quite a lot of formalism. A true idea of debating discussion is the most tricky to conciliate with the soberty and the rigour that are required from a mathematical text.

This work tries its best to face all the above requirements. The abstract in its very beginning lists the main points of interest and gives precise references to the principal statements, hoping to help the reader's overall orientation. After a concise picture of the background on which the discussion will take place is sketched, the introduction gives a more detailed but still informal account of the various topics, trying to arrange them into a coherent picture - a goal that is of primary importance, especially for the first part. In the beginning of every chapter an even stronger magnifying glass is applied, in order to present a concise overview of the essential steps and, if needed, to introduce some minor characters.

Since self-containedness is a decisive propery of global pictures, the major methods and objects on which the work relies are introduced in specific chapters or sections.

Expressions of gratitude

There are words that easily sound duty-bound, and this impression is only reinforced, rather than weakened, by excessive emphasis.

Let me therefore invoke the delicate deepness of soberty in expressing my gratitude to all the people that accompained, sustained, trusted and helped me during the last three years.

First my family, especially my parents and my sister, with their support and their generous love.

Then my supervisor, Eva Maria Feichtner. She helped me to manage the scientific and human troubles of a doctoral study, led my mathematical research and introduced me to the research community of arrangements of hyperplanes.

Many interesting ideas originated during the conferences and workshops that I could attend. In particular, I profited very much from discussions with Graham Denham, Michael Falk, Dmitry Kozlov, Mario Salvetti, Sergey Yuzvinsky and Rade Živaljević. Not only mathematically.

I thank prof. Yuzvinsky also for accepting to be on my PhD committee together with prof. Giovanni Felder, whom I thank for his interest in my work and his support.

Part of the mathematics of this thesis (chapter 10) is due to joint work with Sonja Čukić, who is in a sense my ‘scientific sister’.

A grateful thought goes to the mathematics department of the ETH Zurich, where I spent the last 8 years - a common destiny that I share with Fausta Leonardi, Gabrio Caimi and Ivo Dell’Ambrogio, who became more than colleagues. For what concerns the doctoral study, let me thank particularily prof. Guido Mislin and prof. Paul Balmer for their patience in answering the questions of a novice about homological algebra, and the assistants of the *Gruppe 4* (particularly the ‘underground section’) for the friendly atmosphere.

These last years were probably the most enriching time I had. For this I cannot omit to thank all people with whom I shared so many different musical experiences: an invaluable luck. In particular, Anna Jelmorini could show us the profound humanity and intense vitality that lie at the core of the music we performed under her direction.

There is, I think, some fuzziness in the concept of a ‘friend’ - but absolutely no doubt about the uniqueness of people that vivify your life. A considerable amount of common interests, including music, and experiences I share with Benedetto, Paolo, Ulisse, Francesca, Fabrizio, Katharina, Thomas. The patient help of Markus cannot be overestimated.

But making lists of names is maybe a mistake. To anyone I have not mentioned I acknowledge a particular colour of the sky I see: beyond the mountains, between the clouds, over the rainbow.

Emanuele Delucchi
Zurich, June 2006.

Contents

Abstract

Part 1: Let $\mathcal{A}$ be a locally finite, possibly affine arrangement of n hyperplanes in $\mathbb{R}^d$, and let $\mathcal{M}(\mathcal{A})$ denote the complement of its complexification in $\mathbb{C}^d$.

Building on the classical theory of coverings of groupoids, we model any topological cover $\rho : X \to \mathcal{M}(\mathcal{A})$ by the homotopy colimit of a diagram of spaces $\mathscr{D}_\rho$ over the face poset $\mathcal{F}$ of $\mathcal{A}$ (Theorem 4.5.1). We can then write the given cover as the order complex of a poset $\mathcal{S}_\rho$ that can be explicitly constructed from the combinatorial data of the arrangement (Scholium 4.5.2) and, for $\rho = id$, reduces to the poset of cells of the well-known Salvetti complex of $\mathcal{A}$. The order complex of any $\mathcal{S}_\rho$ is the barycentric subdivision of a polyhedral complex tiled by copies of the zonotope of the arrangement. This generalizes and extends the results of Paris [**96, 97**].

Motivated by the above constructions, we give a derivation of homology and cohomology spectral sequences for diagrams of spaces over any poset (Theorems 5.2.6 and 5.4.1) that does not rely on the work of Segal [**104**] (which is commonly cited in this context) and allows to write the spectral sequences in a particularly compact form if the index category of the diagram is the poset of cells of a regular CW-complex (Theorem 5.2.4). We then express these spectral sequences in the language of sheaves over posets, showing how the theory of homotopy colimits for diagrams over posets of [**118**] naturally parallels the theory of sheaves over posets [**3, 33**] (Theorem 5.3.1, Proposition 5.3.3). When applied to the diagrams $\mathscr{D}$, these spectral sequences reduce to the cellular (co)chain complexes of the zonotopal complex mentioned above (Section 5.5).

We introduce a new family $(\mathscr{G}_\rho)_\rho$ of diagrams of spaces over the poset $\mathcal{O}$ of convex acyclic sets of the oriented matroid associated to $\mathcal{A}$. This family is again indexed by the topological covers of $\mathcal{M}(\mathcal{A})$. We prove homotopy equivalence $\mathrm{hocolim}\mathscr{D}_\rho \simeq \mathrm{hocolim}\mathscr{G}_\rho$ for any covering ρ (Proposition 6.2.7). The homotopy type of the diagrams $\mathscr{G}_\rho$, and thus of the corresponding cover of $\mathcal{M}(\mathcal{A})$, is given by an abstract simplicial flag complex that we call $U_\rho(\mathcal{A})$ (Proposition 6.2.12). It is tiled by copies of all order complexes $\Delta(\mathcal{P}_C(\mathcal{A}))$ (where $\mathcal{P}_C(\mathcal{A})$ denotes the poset of regions of $\mathcal{A}$ with base chamber C). If $\mathcal{A}$ is the reflection arrangement of a finite irreducible Coxeter system W and $\hat{\rho}$ denotes the universal cover, then $U_{\hat{\rho}}(\mathcal{A})$ is the contractible complex that is customarily used to construct finite $K(\pi, 1)$s for Garside groups, of which the finite type Artin group associated to W is an example. An application of the spectral sequence derived in chapter 5 to the diagram $\mathscr{G}_{id}$ allows to write the chain complex for the cohomology of $\mathcal{M}(\mathcal{A})$ in terms of the structure of the Varchenko-Gel'fand ring of the arrangement (Proposition 6.6.4. See also 6.6.2).

It is known that the poset $\mathcal{P}_C(\mathcal{A})$ is a lattice for all chambers C if and only if $\mathcal{A}$ is simplicial ([**12**, Theorem 3.5]). We revisit Deligne's classical proof of the decisive 'property D' for positive paths (see section 6.4.2) assuming the strong lattice property instead of simpliciality. Then, we prove that, in fact, property D is equivalent to simpliciality of the arrangement (Theorem 6.5.3), giving a negative answer to a question posed by Paris in [**97**, Section 4].

With a counterexample we show that, in order for $\mathcal{M}(\mathcal{A})$ to be aspherical, it is not sufficient that $\mathcal{P}_C(\mathcal{A})$ is a lattice for only one chamber C (section 7.2). Still, with this weak assumption we derive Proposition 7.5.5, which is the key step to show contractibility of certain subcomplexes $\mathbf{\Omega}(\gamma) \subset U_{\hat{p}}(\mathcal{A})$ indexed by elements γ of the fundamental group $\pi_1(\mathcal{M}(\mathcal{A}))$ (Corollary 7.5.8). In view of an application of the nerve lemma to this stratification, we determine the incidence relation of the $\mathbf{\Omega}$s (Proposition 7.6.3), and study the topology of their intersections (Section 7.7).

Part 2: In the context of the theory of building sets and nested set complexes developed by Feichtner and Kozlov [**60**], we extend the result of [**59**] relating stellar subdivisions and combinatorial blowups to the full generality of meet - semilattices (Theorem 8.4.7, joint with Sonja Čukić). Then we apply this theory to three different contexts.

First, given a finite group G and a natural number n, we study the structure of the complex of nested sets of the associated Dowling lattice $\mathcal{Q}_n(G)$ (see [**44**]) and of its subposet of the G-symmetric partitions $\mathcal{Q}_n^G$. Hultman [**81**] recently proved that this subposet is Cohen-Macaulay showing its homotopy equivalence with the complex of G-symmetric phylogenetic trees $\mathcal{T}_n^G$, and determined the rank of their top homology.
An application of the theory of building sets and nested set complexes shows that, in fact, $\mathcal{T}_n^G$ is subdivided by the order complex of $\mathcal{Q}_n^G$ (Theorem 9.3.3). We introduce the complex of Dowling trees $\mathcal{T}_n(G)$ (Definition 9.4.3). We prove that it is subdivided by the order complex of $\mathcal{Q}_n(G)$ (9.4.5) and contains $\mathcal{T}_n^G$ as a subcomplex (see 9.4.3). We show that $\mathcal{T}_n(G)$ is obtained from $\mathcal{T}_n^G$ by successive coning over certain subcomplexes (Corollary 9.4.9). We explicitly and independently calculate how many homology spheres are added in passing from $\mathcal{T}_n^G$ to $\mathcal{T}_n(G)$ (Corollary 9.4.11).

Second, we report on joint work with Sonja Čukić where the classical construction of the Bier sphere $\mathrm{Bier}(K)$ for a simplicial complex K (due to Bier, see [**7**] and [**89**]) is put into the general framework of the theory of nested set complexes (Proposition 10.1.2, Corollary 10.1.5). Björner, Paffenholz, Sjöstrand and Ziegler [**14**] generalize this construction to obtain a Bier poset $\mathrm{Bier}(P, I)$ from any bounded poset P and any proper ideal $I \subseteq P$. They show shellability of $\mathrm{Bier}(P, I)$ for the case $P = B_n$, the boolean lattice, and obtain thereby 'many shellable spheres' in the sense of Kalai [**68**].
We obtain 'more shellable spheres' by proving the general statement that combinatorial blowups, hence stellar subdivisions, preserve shellability (Theorem 10.2.5).

In view of the last application, let $\Pi^{(k)}_{(n-1)k+1}$ be the poset of partitions of $[(n-1)k+1]$ with block sizes congruent to 1 modulo k. We prove that the order complex $\Delta(\Pi^{(k)}_{(n-1)k+1})$ is a subdivision of the complex of k-trees $\mathcal{T}_n^k$ (Theorem 11.2.12), thereby answering a question posed by Feichtner [**58**, 5.2].
The result is obtained by an *ad-hoc* generalization of concepts from the theory of nested set complexes beyond lattices.

Riassunto

Prima parte: Sia $\mathcal{A}$ un arrangiamento localmente finito di n iperpiani (affini o lineari) in $\mathbb{R}^d$, e sia $\mathcal{M}(\mathcal{A})$ il complemento in $\mathbb{C}^d$ della complessificazione di $\mathcal{A}$.

Partendo talla teoria classica dei rivestimenti di gruppoidi, descriviamo ogni rivestimento topologico $\rho : X \to \mathcal{M}(\mathcal{A})$ con il colimite omotopico di un diagramma di spazi topologici $\mathscr{D}_\rho$ sull'ordine parziale $\mathcal{F}$ delle facce di $\mathcal{A}$ (Teorema 4.5.1). Possiamo quindi scrivere il rivestimento dato come il complesso d'ordine di un insieme parzialmente ordinato $\mathcal{S}_\rho$ che può essere costruito esplicitamente partendo dai dati combinatorici dell'arrangiamento (Scolio 4.5.2) e che, nel caso $\rho = id$, si riduce all'ordine parziale delle celle del noto complesso di Salvetti. Il complesso d'ordine di ogni $\mathcal{S}_\rho$ è la suddivisione baricentrica di un complesso poliedrale che, nel caso lineare, è tassellato da copie del poliedro che ha l'arrangiamento stesso come ventaglio normale. Ciò generalizza ed estende i risultati ottenuti da Paris [**96, 97**].

Motivati dalla situazione descritta sopra, deriviamo sequenze spettrali per l'omologia e la coomologia di diagrammi di spazi topologici su insiemi parzialmente ordinati (Teoremi 5.2.6 e 5.4.1) senza basarci sul lavoro di Segal [**104**] che è comunemente citato in questi casi. Il nostro metodo permette di scrivere la sequenza spettrale in una forma particolarmente compatta se l'insieme parzialmente ordinato in questione è l'ordine parziale delle celle di un CW-complesso regolare (Teorema 5.2.4). Esprimiamo poi queste sequenze spettrali nel linguaggio dei 'fasci su ordini parziali', mostrando come la teoria dei colimiti omotopici di diagrammi su ordini parziali di [**118**] sia la controparte naturale della teoria dei fasci su insiemi parzialmente ordinati di [**3, 33**] (Teorema 5.3.1, Proposizione 5.3.3). Se applicate ai diagrammi $\mathscr{D}$, queste sequenze spettrali danno il complesso cellulare della tassellazione poliedrale del rivestimento descritta sopra (Sezione 5.5).

Introduciamo poi una nuova famiglia $(\mathscr{G}_\rho)_\rho$ di diagrammi sull'ordine parziale $\mathcal{O}$ degli insiemi aciclici e convessi del matroide orientato associato ad $\mathcal{A}$. Questa famiglia è indicizzata dai rivestimenti topologici di $\mathcal{M}(\mathcal{A})$. Proviamo quindi che $\mathrm{hocolim}\mathscr{D}_\rho \simeq \mathrm{hocolim}\mathscr{G}_\rho$ per ogni rivestimento ρ (Proposizione 6.2.7). Il tipo di omotopia dei diagrammi $\mathscr{G}_\rho$, e quindi anche dei corrispondenti rivestimenti di $\mathcal{M}(\mathcal{A})$, è dato da un complesso simpliciale di bandiera che chiamiamo $U_\rho(\mathcal{A})$ (Proposizione 6.2.12). Questo complesso è tassellato da copie dei complessi d'ordine $\Delta(\mathcal{P}_C(\mathcal{A}))$ (dove $\mathcal{P}_C(\mathcal{A})$ è l'ordine parziale delle regioni di $\mathcal{A}$, basato alla regione C). Quando $\mathcal{A}$ è l'arrangiamento degli iperpiani fissati dalle riflessioni di un sistema di Coxeter finito e irriducibile W, e se $\hat{\rho}$ rappresenta il rivestimento universale, allora $U_{\hat{\rho}}(\mathcal{A})$ è il complesso contrattile comunemente usato per costruire spazi $K(\pi, 1)$ per i gruppi di Garside, dei quali fanno parte i gruppi di Artin di tipo finito. Applicando al diagramma $\mathscr{G}_{id}$ la sequenza spettrale ottenuta al capitolo 5 possiamo esprimere il complesso per la coomologia di $\mathcal{M}(\mathcal{A})$ attraverso l'anello di Varchenko e Gel'fand associato ad $\mathcal{A}$ (Proposizione 6.6.4. Cfr. anche 6.6.2).

È noto che l'insieme parzialmente ordinato $\mathcal{P}_C(\mathcal{A})$ è un reticolo per ogni camera di base C se, e solo se, l'arrangiamento $\mathcal{A}$ è simpliciale ([**12**, Teorema 3.5]). Ripercorriamo la famosa dimostrazione che Deligne dà della centrale 'proprietà D' soddisfatta dai cammini

positivi (vedi sezione 6.4.2) supponendo direttamente la proprietà 'del reticolo' in luogo della simplicialità dell'arrangiamento. Mostriamo poi che in effetti anche il converso vale: la proprietà D equivale alla simplicialità dell'arrangiamento (Teorema 6.5.3). Ciò risponde negativamente ad un quesito posto da Paris in [**97**, Sezione 4].

Mostrando un controesempio proviamo che il fatto di possedere una camera C tale che $\mathcal{P}_C(\mathcal{A})$ sia un reticolo non assicura l'asfericità del complemento $\mathcal{M}(\mathcal{A})$ (section 7.2). Utilizzando questa debole condizione proviamo però la proposizione 7.5.5, che è essenziale alla dimostrazione della contrattibilità di certi sottocomplessi $\mathbf{\Omega}(\gamma) \subset U_{\hat{\rho}}(\mathcal{A})$, definiti per ogni elemento γ del gruppo fondamentale $\pi_1(\mathcal{M}(\mathcal{A}))$ (Corollario 7.5.8). Per caratterizzare le proprietà combinatorie di questa nuova stratificazione determiniamo poi le relazioni di incidenza tra gli $\mathbf{\Omega}$ (Proposizione 7.6.3), e studiamo la topologia delle loro intersezioni (Sezione 7.7).

Seconda parte: All'interno della teoria combinatoria dei *building sets* e dei *nested sets*[1] sviluppata da Feichtner e Kozlov [**60**], estendiamo alla completa generalità il risultato di [**59**] che collega suddivisioni stellari e *blowups* combinatorï (Teorema 8.4.7, con Sonja Čukić). Applichiamo poi questa teoria a tre situazioni particolari.

Dapprima studiamo la struttura del complesso dei *nested sets* nel reticolo di Dowling $\mathcal{Q}_n(G)$ associato ad ogni gruppo finito G e ogni numero naturale n (cfr. [**44**]) e nel suo sottoinsieme dato dalle partizioni G-simmetriche. Hultman [**81**] ha provato recentemente che quest'ultimo ordine parziale è *Cohen-Macaulay* mostrandone l'equivalenza omotopica con il complesso $\mathcal{T}_n^G$ degli alberi filogenetici G-simmetrici, e ha calcolato il rango della loro omologia di grado massimo.

Applicando la teoria di Feichtner e Kozlov mostriamo che in effetti $\mathcal{T}_n^G$ suddivide il complesso d'ordine di $\mathcal{Q}_n^G$ (Teorema 9.3.3). Introduciamo il domplesso $\mathcal{T}_n(G)$ degli alberi di Dowling (Definizione 9.4.3) provando che esso è suddiviso dal complesso d'ordine di $\mathcal{Q}_n(G)$ (9.4.5) e che contiene $\mathcal{T}_n^G$ come sottocomplesso (vedi 9.4.3). Mostriamo poi come $\mathcal{T}_n(G)$ può essere ottenuto da $\mathcal{T}_n^G$ costruendo ricorsivamente dei coni su dei particolari sottocomplessi (Corollario 9.4.9). Calcoliamo poi esplicitamente quante sfere (in senso topologico) vengono create passando da $\mathcal{T}_n^G$ a $\mathcal{T}_n(G)$ (Corollario 9.4.11).

In secondo luogo presentiamo un lavoro fatto con Sonja Čukić che pone la costruzione classica della sfera $\mathrm{Bier}(K)$ associata da T. Bier ad ogni complesso simpliciale K ([**7, 89**]) nella cornice della teoria dei complessi di *nested sets* (Proposizione 10.1.2, Corollario 10.1.5). Björner, Paffenholz, Sjöstrand e Ziegler [**14**] generalizzano l'idea di Bier e costruiscono un 'ordine parziale di Bier' $\mathrm{Bier}(P, I)$ da ogni insieme parzialmente ordinato (finito) P e da ogni suo ideale non banale $I \subseteq P$. Questi autori mostrano che $\mathrm{Bier}(P, I)$ è *shellable*[2] quando $P = B_n$, il reticolo booleano, e ottengono così 'many shellable spheres' come chiesto da Kalai [**68**].

Qui otteniamo 'more shellable spheres' mostrando che i *blowups* combinatorï (e quindi le suddivisioni stellari) conservano la proprietà di essere *shellable* (Teorema 10.2.5).

Per illustrare l'ultima applicazione, sia $\Pi^{(k)}_{(n-1)k+1}$ l'insieme parzialmente ordinato delle partizioni di $[(n-1)k+1]$ con blocchi di grandezza congruente a 1 modulo k. Mostriamo che il complesso d'ordine $\Delta(\Pi^{(k)}_{(n-1)k+1})$ suddivide il complesso dei k-alberi $\mathcal{T}_n^k$ (Teorema 11.2.12), rispondendo così ad una questione posta da Feichtner [**58**, 5.2].

Il risultato è ottenuto con una conveniente generalizzazione di alcuni concetti della teoria di Feichtner e Kozlov oltre il caso dei reticoli parziali.

[1]La teoria è piuttosto recente: l'uso non ha ancora stabilito i nomi italiani di questi oggetti, né ci sembra appropriato coniarli in questa sede. Qui e in seguito useremo il termine inglese.

[2]Come sopra.

Introduction

Background

Let V be a d-dimensional vector space. An *arrangement of hyperplanes* in V is a finite set $\mathcal{A} = \{H_i\}_{i=1,\dots,n}$ of affine or linear codimension 1 subspaces of V. From a combinatorial point of view, the union of the H_i determines a stratification of V: the *combinatorial data* of the arrangement is the partial order of the strata, ordered by inclusion. On the topological side, it is interesting to study the link $\bigcup H_i$ and the complement of the arrangement $\mathcal{M}(\mathcal{A}) := V \setminus \bigcup H_i$.

The general question in arrangement theory is to determine to what extent does the combinatorial data of the arrangement determine topological invariants of the complement or of the link, such as homotopy or homology groups, Betti numbers etc. We point the interested reader to [**95, 55, 56**] as general references on arrangements of hyperplanes and their topology.

We consider the case where V is a complex vector space, and state a famous open question, the so-called $K(\pi, 1)$-problem. An arrangement is said to be $K(\pi, 1)$ (or *aspherical*) if the higher homotopy groups of the complement $\pi_i(\mathcal{M}(\mathcal{A}))$ vanish for $i > 1$. The problem is whether this topological property is determined by the combinatorial data of the arrangement.

The study of this question was initiated by Brieskorn in 1971, who conjectured the asphericity of the complexification of finite real reflection arrangements [**23**]. The conjecture was proved in 1973 by Deligne, who parametrized the universal cover of these arrangements by the *galleries* in the Coxeter complex [**36**]. The fact that these galleries can be brought in a particular normal form (since then called *Deligne normal form*) allowed him to show contractibility of the universal cover of the arrangement complement, thereby proving asphericity. Deligne saw that the normal form comes from a geometric property of the arrangement which is much weaker than to be Coxeter: it suffices that the connected components of the real arrangement are cones over simplices. These arrangements are called *simplicial* and form one of the two major classes that are up to now known to be $K(\pi, 1)$.

The other large class of $K(\pi, 1)$-arrangements is given by the *supersolvable* arrangements. This was proved in the mid ‘80s by Terao [**110**] and Falk and Randell [**54**]: they went back to the original idea that Brieskorn used to solve some special cases and support his conjecture. Brieskorn remarked that in some cases it is possible to gradually dismantle the arrangement in such a way that each step preserves the higher homotopy groups of the complement. The trivial (void) arrangement is then clearly aspherical. Falk and Randell studied the general feasibility of this process, and Terao saw that this actually depends on a combinatorial property of the intersection lattice of the arrangement: namely, having a maximal chain of modular elements.

Apart from these two principal achievements, in the last 35 years much effort has been spent on the characterization of asphericity of arrangements. The limited setting of this introduction only allows us to sketch a partial and exemplifying picture of this subject.

A necessary and sufficient condition for asphericity in terms of the fundamental group was found by Randell [**100**]; *formality* of an arrangement was shown to follow from asphericity (Yuzvinsky [**120**], Falk and Randell [**56**]), and the idea of supersolvable arrangements was further exploited to show asphericity of *hypersolvable* arrangements (Jambu and Papadima, [**82, 83**]). Some conjectures were settled negatively: neither all *general position* arrangements (Hattori [**80**]) nor all *free* arrangements (Edelman and Reiner [**49**]) are aspherical. Other conjectures are still open, mainly those regarding *factored arrangements* [**84**] and *affine Coxeter arrangements* [**93, 28**]. The case of complex reflection arrangements was tackled by Nakamura [**92**], and Orlik and Solomon [**94**]. Recently, Bessis gave a general argument that shows asphericity of all complex reflection arrangements... except for one [**5**].

All classes of $K(\pi,1)$-arrangements mentioned above (except for the complex reflection arrangements) can be characterized combinatorially. However, the big old question remains: it is still not clear *whether being* $K(\pi,1)$ *is a combinatorial property.*

A possible strategy for the study of this question is to develop *combinatorial models for* $\mathcal{M}(\mathcal{A})$, i.e., cell complexes that are built from the combinatorial data of the arrangement and that model $\mathcal{M}(\mathcal{A})$ up to homotopy equivalence or even homeomorphism. Many combinatorial models for $\mathcal{M}(\mathcal{A})$ exist at present. Among them let us only mention the *Salvetti complex*, introduced by Mario Salvetti in [**102**], which models the homotopy type of the complement of *complexified* arrangements (i.e., complex arrangements where the defining equations of the H_i have real coefficients), and the model developed by Björner and Ziegler in [**17**] that generalizes the work of Salvetti to any complex arrangement.

These combinatorial models led to important achievements in various contexts, for instance the computation of cohomology algebras of complements of certain classes of subspace arrangements in purely combinatorial terms, see [**64**], [**37**], [**38**].

The idea of the *wonderful compactifications* of De Concini and Procesi [**32**] is slightly different: rather than constructing a simplicial complex 'from scratch' starting with the combinatorial data, one starts with the 'actual' stratification of V given by the arrangement $\mathcal{A}$ and modifies it, reaching a variety $Y_{\mathcal{A}}$ with a projection onto V that maps $\mathcal{M}(\mathcal{A})$ bijectively. The point is that $Y_{\mathcal{A}} \setminus \mathcal{M}(\mathcal{A})$ is a divisor with normal crossings (which is not necessarily the case for $V \setminus \mathcal{M}(\mathcal{A})$). Note that this construction works in both the real and the complex case.

The modifications we referred to make use of the algebraic geometric operation of *blowup*. Indeed, $Y_{\mathcal{A}}$ is constructed from V by 'blowing up' some of the intersections of the hyperplanes in some prescribed order.

Combinatorics enters the picture at this point, with the work of Feichtner and Kozlov [**60**]: indeed, in order to know 'which intersection has to be blown up and in which order', it is enough to know the intersection lattice. Moreover, the pattern of intersections of the variety after a blowup is obtained from the previous by the (entirely combinatorial) operation of *combinatorial blowup*. After those observations, Feichtner and Kozlov generalized this situation and defined *building sets* ('the irreducible divisors') and *nested sets* ('sets of intersecting divisors') for arbitrary meet-semilattices. This was the starting point to develop a theory of combinatorial building sets and nested set complexes.

These combinatorial structures quickly found application also outside the theory of hyperplane arrangements (see [**61, 63, 59, 58, 42, 62, 31, 41**]). For example, it proved to be a very useful framework to study the topology of various types of *complexes of trees*. Such complexes were recently put again under the spotlight in [**8**] as a geometric model for phylogenetic questions. Moreover, they were studied in different generalizations ([**79, 101, 1**]) because of their combinatorial and algebraic properties.

Overview

This work begins with a prologue, that is intended to give a picture of the fundament on which we build our considerations about arrangements, to fix notations and to quickly develop the results needed later.

Chapter 1. We review the basics on combinatorics of arrangements and introduce the Salvetti complex of a complexified arrangement. Moreover, we give a short introduction to oriented matroids with particular focus on the structure of covectors and of acyclic sets.

Chapter 2. This chapter is devoted to the groupoid of a complexified arrangement. This is an object that is used at least since the classical work of Deligne [**36**] and is nothing else than the fundamental groupoid of the arrangement complement, in a form that reflects the real structure of the arrangement. We give a short but rigourous categorical construction of this object and recall the basic facts of the well-known theory of coverings of groupoids. Later on, we will build the formulation and the study of our models on this formalism.

Chapter 3. A fundamental tool in the study of our models is the theory of homotopy colimits for combinatorial applications (developed by Welker, Ziegler and Živaljevič [**118**]), that we briefly present in chapter 3 giving the definitions and recalling some results that are needed in the subsequent work.

Part 1 reports on our study of the topology of the covers of a complexified arrangement. Indeed, in view of the $K(\pi, 1)$-problem, it is interesting to look at the universal covering space of the complement of an arrangement, and in particular to study its contractibility. Let us briefly mention that also the study of the topology (in particular homology) of certain 'cyclic' coverings of arrangements is of interest, mainly due to the work of Cohen and Orlik [**30**]. But our main concern here is the asphericity problem.

Chapter 4. We begin by introducing combinatorial models for the covers of a complexified arrangement. These spaces are expressed as homotopy colimits of certain diagrams of spaces over the face poset of the arrangement. The 'spaces' are sets of objects of covers of the arrangement groupoid. We prove that these homotopy colimits are indeed topological covers of the complement, and we describe their combinatorial structure explicitly: with an application of the simplicial model lemma for homotopy colimits we express any covering of the complement as the order complex of some (explicitly described) poset. Moreover, it is easy to see that this order complex is the barycentric subdivision of a regular CW-complex, whose cells are copies of the zonotope of the arrangement. In particular, in the case of the identical cover we retrieve the 'original' Salvetti complex.
The idea to use homotopy colimits to describe the Salvetti complex came as an attempt to generalize a 2-dimensional construction by Michael Falk (see section 4.2). Our results generalize the simplicial description of the universal cover of a linear arrangement by

Paris [**97**] to all covers of any affine, locally finite arrangement. The link to the work of Paris, who classifies the covers of a complexified arrangement by means of certain ad-hoc objects, the 'oriented systems', is given in section 4.6.

Chapter 5. This chapter originates from a discussion with professor Rade Živaljevič and mainly contains a selfcontained derivation of a spectral sequence for homotopy colimits. All literature known to us refers in this case to the paper of Segal [**104**], written in terms of bisimplicial sets. We give a direct derivation of cohomology and homology spectral sequences for homotopy colimits of diagrams over posets and show that, if the poset over which our diagram is defined is indeed the poset of cells of a regular CW-complex, the spectral sequence takes a particularly compact form. Then, we express the obtained spectral sequences in the language of sheaves over posets.
The results of chapter 5 do not only apply to our models, but to any diagram of spaces over a poset. Moreover, the application of the 'compact' form of the spectral sequence to the diagram models of chapter 4 yields the cellular (co)homology of the zonotopal complex we described above.

Chapter 6. The discussion of the role played by the posets of regions of a real arrangement in the topology of the complement of its complexification was initiated in [**12**], by noting that all classes of arrangements which are known to be $K(\pi, 1)$ have the property that, at least for one choice of base chamber, the order of regions is a lattice. Since also inductively factored arrangements [**84**] share this property, the discussion of it seemed interesting.

In a discussion during a hike, Graham Denham brought to my attention the work of McCammond on combinatorial Garside structures [**90**]. In particular, in that theory certain simplicial complexes are constructed from an edge-labelled poset, that generalizes the weak Bruhat order of Coxeter groups. These complexes are shown to be aspherical through contractibility of their universal cover, which is tiled by copies of the order complex of the poset. The connection between this complex and the universal cover of Coxeter arrangements was already noted by Bestvina [**6**]. In chapter 6 we use the toolkit about homotopy colimits that we prepared in chapter 3 in order to construct a simplicial (flag) complex that is tiled by copies of the order complexes of the posets of regions. For Coxeter arrangements, our model reduces to Bestvina's complex.
It is known (see [**12**]) that an arrangement is simplicial if and only if all its posets of regions are lattices. We rewrite Deligne's argument from this point of view and show that using our 'Garside-type models' the topological part becomes very easy, and the focus is really put on the decisive property on paths in the arrangement groupoid. The key theorem by Deligne is that if the arrangement is simplicial, then the property on paths holds (see 6.4.6). Our approach gives an easy proof that the implication is in fact double: if the property on paths is satisfied, then the arrangement must be simplicial.
We end chapter 6 with a couple of remarks about a possible link with the Varchenko-Gel'fand ring VG of a real arrangement. Indeed, the fact that we can express our models as diagrams over the poset of acyclic sets allows us to write the cohomology of the arrangement as the cohomology of a sheaf over the poset of ideals of VG, with stalks isomorphic to a certain type of principal ideals of VG. Hoping that our setting may bring some new light on the connection between this ring and the cohomology of the arrangement, we formulate some questions that may lead in this direction.

Chapter 7. We have seen that the attempt to prove the 'property on paths' with weaker assumptions than simpliciality is hopeless. Nevertheless, the fact that (by now) all known $K(\pi, 1)$ complexified arrangements have a chamber for which the poset of regions is a lattice remains. In chapter 7 we show that this condition is too weak to ensure asphericity: we give an arrangement that is not aspherical but has a chamber giving rise to a lattice of regions, answering to a question that was raised in a discussion with Michael Falk. After that, we nevertheless try to see what can be said about the topology of the universal cover under the above (weak) assumption. We propose to consider a stratification of the 'Garside-type' universal cover complex into 'big cells'. Under the assumption of our weak condition these big cells can be shown to be contractible. We then determine the incidence relations of the big cells and begin to study the topology of their intersection. We could not (yet?) show contractibility of the intersections in general, but we hope that what is done may be a step in a useful direction.

At this point an interlude follows, that introduces the character of the second part of our work.

Chapter 8. The only chapter of the interlude is a short reminder of some facts from the theory of combinatorial building sets and nested set complexes that was recently developed by Feichtner and Kozlov [**60**]. As in the prologue, all results are known, except for the theorem relating stellar subdivisions and combinatorial blowups, of which we give (and prove) a general version due to a joint work with Sonja Čukić [**31**].

Part 2 then uses the methods of this theory to study the topology of different combinatorial objects, namely two generalizations of the complex of phylogenetic trees and a combinatorial generalization of the Bier spheres. The material of this second part is a collection of [**41, 31, 42**].

Chapter 9. We fix a finite group G and a natural number n in order to study the structure of the complex of nested sets of the associated Dowling lattice $\mathcal{Q}_n(G)$ (see [**44**]) and of its subposet of the G-symmetric partitions $\mathcal{Q}_n^G$. The latter object was recently introduced by Axel Hultman in [**81**], together with the complex of G-symmetric phylogenetic trees $\mathcal{T}_n^G$. Hultman shows that they are homotopy equivalent and Cohen-Macaulay, and determines the rank of their top homology. As an application of the theory of building sets and nested set complexes, we show that in fact $\mathcal{T}_n^G$ is subdivided by the order complex of $\mathcal{Q}_n^G$. As naturally associated with the full Dowling lattice, we introduce a new object: the complex of Dowling trees $\mathcal{T}_n(G)$. We prove that it is subdivided by the order complex of $\mathcal{Q}_n(G)$ and contains $\mathcal{T}_n^G$ as a subcomplex. The last assertion is proved by showing that $\mathcal{T}_n(G)$ is obtained from $\mathcal{T}_n^G$ by successive coning over certain subcomplexes. It is well known that $\mathcal{Q}_n(G)$ is shellable, and of the same dimension as $\mathcal{T}_n^G$. We explicitly and independently calculate how many homology spheres are added in passing from $\mathcal{T}_n^G$ to $\mathcal{T}_n(G)$.

Chapter 10. This is a joint work with Sonja Čukič and studies a generalization of the construction of Bier spheres that was described in [**14**] by Björner, Paffenholz, Sjöstrand and Ziegler. The classical construction of the Bier sphere $\mathrm{Bier}(K)$ for a simplicial complex K is due to Bier ([**7**], [**89**]). The authors of [**14**] obtain a Bier poset $\mathrm{Bier}(P, I)$ from any bounded poset P and any proper ideal $I \subseteq P$. Moreover, they show shellability of $\mathrm{Bier}(P, I)$ for the case when P is the boolean lattice, and thereby obtain 'many shellable spheres' in the sense of Kalai [**68**].
We put the Bier construction into the general framework of the theory of nested set

complexes and obtain 'more shellable spheres' by proving the general statement that combinatorial blowups, hence stellar subdivisions, preserve shellability.

Chapter 11. The last chapter is again about a generalization of the phylogenetic trees of Billera, Holmes and Vogtmann [**8**]. Let $\Pi^{(k)}_{(n-1)k+1}$ denote the poset of partitions of $\{1, 2, \ldots, (n-1)k+1\}$ with block sizes congruent to 1 modulo k. We prove that the order complex $\Delta(\Pi^{(k)}_{(n-1)k+1})$ is a subdivision of the complex of k-trees $\mathcal{T}^k_n$, thereby answering a question posed by Feichtner [**58**, 5.2]. This result is obtained by an *ad-hoc* generalization of concepts from the theory of nested set complexes beyond semilattices.

Prologue

CHAPTER 1

The Salvetti complex and oriented matroids

We review the concepts and definitions that will be used in the first part of this work. In the last section we introduce in somewhat nonstandard way the poset of chambers of an arrangement, relating it to the simplicial complex of acyclic sets of the associated oriented matroid. This motivates then the approach of chapter 6.

We do not aim for completeness: extensive references on each topic will be given. For a comprehensive introduction to arrangement theory we refer to [**95**].

1.1. Basic Notions

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^d$, i.e., a set of linear or affine codimension 1 subspaces of $\mathbb{R}^d$.

For an affine subspace X of $\mathbb{R}^d$ it is standard to define the *contraction* of X in $\mathcal{A}$, given by the subarrangemente $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$ of $\mathcal{A}$, and the *restriction* of $\mathcal{A}$ to X, $\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}$. The set of connected components of $\mathbb{R}^d \setminus \mathcal{A}$ we will denote with $\mathcal{C}(\mathcal{A})$, the set of *chambers* of $\mathcal{A}$.

There are two posets associated with $\mathcal{A}$ that we will frequently use: the *intersection lattice* of $\mathcal{A}$,

$$\mathcal{L}(\mathcal{A}) := \Big\{ X := \bigcap_{H \in \mathcal{H}} H \,\Big|\, \mathcal{H} \subseteq \mathcal{A}, X \neq \emptyset \Big\}, \quad X \geq Y \Leftrightarrow X \subseteq Y,$$

i.e., intersections of hyperplanes ordered by reversed inclusion, and the *face poset* of $\mathcal{A}$,

$$\mathcal{F}(\mathcal{A}) := \bigcup_{X \in \mathcal{L}(\mathcal{A})} \big\{ F \mid F \in \mathcal{C}(\mathcal{A}^X) \big\}, \quad F_1 \geq F_2 \Leftrightarrow \overline{F_1} \subseteq \overline{F_2},$$

i.e., cells in $\mathbb{R}^d$ that are chambers of restricted arrangements, ordered by containment.

In the latter definition, $\overline{F}$ denotes the closure of F in the standard topology of $\mathbb{R}^d$. In fact, $\mathcal{F}(\mathcal{A})$ is the poset of strata for the stratification of $\mathbb{R}^d$ into relatively open polyhedral cells induced by the arrangement.

Given a poset $\mathcal{P}$, it is standard to write $\Delta(\mathcal{P})$ for the *order complex* of $\mathcal{P}$, i.e., the abstract simplicial complex whose vertices are the elements of $\mathcal{P}$ and whose simplices are the chains $p_0 > \cdots > p_k$ in $\mathcal{P}$.

Taking the same defining equations as for $\mathcal{A}$ in complex d-space, we obtain the complexification $\mathcal{A}_\mathbb{C}$ of $\mathcal{A}$. We will denote by $\mathcal{M}(\mathcal{A})$ (or simply $\mathcal{M}$) the complement of $\mathcal{A}_\mathbb{C}$ in $\mathbb{C}^d$:

$$\mathcal{M}(\mathcal{A}) := \mathbb{C}^d \setminus \mathcal{A}_\mathbb{C}.$$

1.2. The Salvetti Complex

A central tool in the study of the topology of $\mathcal{M}(\mathcal{A})$ is the simplicial complex $Sal(\mathcal{A})$, introduced by Salvetti in [**102**]. We will recall its definition and thereby essentially follow the original paper, although we will adapt the notation for our purposes.

1.2.1. Faces act on chambers. Before starting with the construction, we want to define a composition of faces. Fix a face $F \in \mathcal{F}(\mathcal{A})$ and let $|F|$ denote its affine hull in $\mathbb{R}^d$.

The canonical projection $\mathbb{R}^d \to \mathbb{R}^d/|F|$ induces a map

$$\pi_F : \mathcal{F}(\mathcal{A}) \mapsto \mathcal{F}(\mathcal{A}_{|F|}).$$

For $P \in \mathcal{F}(\mathcal{A})$, let $F \circ P$ be the face of smallest dimension in $\pi_F^{-1}(\pi_F(P))$ whose closure contains F.

It is not hard to see that this composition satisfies the following properties:

$$\begin{array}{ll} (1) & F_1 \circ F_2 = F_2 \quad \text{for } \; F_1 \geq F_2 \text{ in } \mathcal{F}(\mathcal{A}). \\ (2) & F \circ C \in \mathcal{C}(\mathcal{A}) \quad \text{ for any } F \in \mathcal{F}(\mathcal{A}),\; C \in \mathcal{C}(\mathcal{A}). \end{array}$$

Writing C_F for $F \circ C$ we can also state:

$$(3) \quad (C_{F_1})_{F_2} = C_{F_1 \circ F_2} = C_{F_2} \quad \text{for } F_1 \geq F_2 \text{ and } C \in \mathcal{C}(\mathcal{A}).$$

After these (perhaps at this point still obscure) remarks, we move on to define the Salvetti complex.

1.2.2. Definition of $Sal(\mathcal{A})$. We start with the real arrangement $\mathcal{A}$, and we choose a point $w(F)$ in the interior of every face $F \in \mathcal{F}(\mathcal{A})$. We write the complexification of $\mathbb{R}^n$ as $\mathbb{C}^n = \mathbb{R}^n \oplus \mathbb{R}^n$, and consider points

$$w(F, C) := w(F) \oplus [w(F) - w(C_F)],$$

for any $F \in \mathcal{F}(\mathcal{A})$ and $C \in \mathcal{C}(\mathcal{A})$.

Notice that, although indexed by F and C, the points $w(F, C)$ actually depend on F and C_F: only the relative position of C with respect to F is important.

These points will form the vertex set of the simplicial complex $Sal(\mathcal{A})_{simpl}$. The convex hull of $\{w(F_0, C_0), \ldots, w(F_d, C_d)\}$ is a simplex in $Sal(\mathcal{A})_{simpl}$ if and only if $F_0 > F_1 > \cdots > F_d$ is a chain in $\mathcal{F}$ and, for any i, $(C_0)_{F_i} = (C_i)_{F_i}$. In other words, each $(d+1)$-chain ϕ in $\mathcal{F}$ and every chamber $C \leq max(\phi)$ determine a d-simplex that we call $\Delta(\phi, C)$, and all simplexes are of this form.

There is another way to look at the Salvetti complex, which may be more suggestive for some purposes. Indeed, one may take a vertex for every chamber, and connect each pair of vertices corresponding to adjacent chambers by a pair of opposite oriented edges. Higher dimensional cells are attached to the resulting (oriented) graph in such a way that as many d-cells correspond to every $(n-d)$-face F of $\mathcal{A}_{\mathbb{R}}$ as there are chambers adjacent to F: The d-cell corresponding to the face F and the chamber C is denoted by $[F, C]$. To see how it is attached to the $(d-1)$-skeleton, consider the chamber $\tilde{C}$ opposite to C with respect to F and the positive directed minimal paths from C to $\tilde{C}$: these form the 1-skeleton $[F, C]^1$ of $[F, C]$, which consists of all cells of dimension $k \leq d$ whose 1-skeleton lies in $[F, C]^1$. This way one can construct a CW-complex $Sal(\mathcal{A})_{CW}$ by induction on the skeleta. It is easy to see that $Sal(\mathcal{A})_{simpl}$ is the barycentric subdivision of $Sal(\mathcal{A})_{CW}$: in fact, the simplices $\Delta(\phi, C)$ with $max(\phi) = F$ are simplices of the barycentric subdivision contained in the closed cell $[F, C]$. The two complexes are canonically homeomorphic, and we shall therefore identify them and use the name $Sal(\mathcal{A})$ for both constructions.

1.2.3. The fundamental group $\pi_1(Sal(\mathcal{A}))$. Salvetti gave different presentations of $\pi_1(Sal(\mathcal{A}))$ in part 2 of [**102**], and we refer there for explicit results. Here we only want to remind that we can write the fundamental group as

$$\pi_1(Sal(\mathcal{A})) = \pi_1(Sal(\mathcal{A})^1)/\sigma, \tag{2.1}$$

where σ denotes the smallest normal subgroup containing

$$\{\partial[F, C] \mid codim(F) = 2\},$$

the set of boundaries of 2-cells (see e.g. [**88**] ch. 7, Thm. 2.1). In fact, the fundamental group is determined by the 2-skeleton of the complex.

From the construction of $Sal(\mathcal{A})$ as a CW-complex one can easily see that σ is generated by the elements $\gamma_1\gamma_2^{-1}$, where the γ_i are directed, minimal paths with same beginning and end.

This shows how orientation of the edges carries the whole information about the attaching of higher cells, and in particular about the fundamental group.

1.3. Oriented matroids

The reader may have noticed that in the previous considerations the only important data are combinatorial. In fact, all one needs to know is the poset $\mathcal{F}$, that can be seen as the incidence poset of the stratification induced by $\mathcal{A}$ on the unit sphere, completed by adding a maximal element. The most natural combinatorial framework for this object is the theory of oriented matroids, that we introduce in this section.

Note: for the rest of this chapter we will assume that the arrangement we are considering is central, i.e., all hyperplanes contain the origin.

Since any hyperplane divides the real space in two connected components, we want to subdivide $\mathbb{R}^d$ into strata of points that have the same position with respect to the given arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ (i.e., lie on the same side of - or "in"- each hyperplane). This is achieved by choosing an orientation, i.e., by declaring a positive and a negative side of each hyperplane, and then assigning to every vector v of $\mathbb{R}^d$ a sign sequence $X(v) = (X_1(v), \dots, X_n(v))$, where $X_i(v)$ is defined to be 0, + or − according to whether v lies on the positive or negative side of H_i, or $v \in H_i$. Let $\mathcal{V}$ denote the set of sign sequences that arise in the stratification of $\mathbb{R}^n$ by our given arrangement $\mathcal{A}$.

Of course, in general not all 2^n sign patterns correspond to positions of vectors in the space. However, there are four things that one can see fairly quickly: first of all, since all hyperplanes contain the origin, we have that $(0, 0, \dots, 0) \in \mathcal{V}$. It is also clear that the negative of any vector v realizes the opposite sign configuration. Therefore, if $X \in \mathcal{V}$, also $-X \in \mathcal{V}$. Moreover, if a vector v is contained in the intersection of a set S of hyperplanes that do not contain the vector w then, moving a little bit from v towards w on the segment joining the two points, one finds a vector that is on the same side as w of all hyperplanes not containing w, but is not contained in any of the hyperplanes in S. Last, suppose that the hyperplane H separates v from w, but H' does not. Then the segment from v to w intersects H in a point z. It follows that $X_H(z) = 0$, $X_{H'}(z) \neq 0$, and that if $X_j(z) \neq 0$, then H_j cannot contain both v and w.

The idea now is to formalize these four properties, and set them up as axioms defining the so-called *set of vectors of an oriented matroid.* Before to state the precise definition, we have to make precise the sense of the above "moving a little bit". Indeed, given two sign vectors $X, Y \in \{0, +, -\}^n$, moving "a little bit from X towards Y" one encounters the sign configuration denoted by $X \circ Y$ and defined by

$$(X \circ Y)_i := \begin{cases} X_i & \text{if } X_i \neq 0, \\ Y_i & \text{else.} \end{cases}$$

Let us note here that if Y has no 0 components, and thus represents a chamber of the arrangement, $X \circ Y$ also represents a chamber. In fact this operation is a formalization of the geometric operation defined in 1.2.1.

DEFINITION 1.3.1. *Given a ground set E, a collection $\mathcal{V} \subset \{+, -, 0\}^E$ is the set of vectors of an oriented matroid if and only if following properties are satisfied:*

(1) $(0, 0, \ldots, 0) \in \mathcal{V}$,
(2) *if $X \in \mathcal{V}$, then $-X \in \mathcal{V}$,*
(3) *for all $X, Y \in \mathcal{V}$, $X \circ Y \in \mathcal{V}$,*
(4) *for all $X, Y \in \mathcal{V}$,*
given $e, f \in E$ such that $X_e = -Y_e$ and not both X_f, Y_f equal 0,
there is $Z \in \mathcal{V}$ such that $Z_e = 0$, $Z_f \neq 0$, and if $Z_i \neq 0$ then Z_i equals X_i or Y_i.

Let us point out that this is only one of the many ways to characterize oriented matroids. For a complete account of the many different possible axiomatizations we refer to chapter 5 of [**13**]. Moreover, the above axioms are satisfied also by the sign patterns of all possible linear combinations that hold in a set of n vectors. Indeed, if one chooses a normal vector for each hyperplane in $\mathcal{A}$ and considers this last point of view, one obtains an oriented matroid that is called *dual* to the one we defined above, and the sign patterns of the strata we considered form its set of *covectors*. We mentioned this for completeness of exposition and in order to prevent confusion in the following, as the face lattice of an oriented matroid is classically based on a partial ordering of the covectors of an oriented matroid.

Indeed, we can define a partial order on $\mathcal{V}$ by setting

$$X \leq Y \Leftrightarrow \forall e \in E : \begin{cases} Y_e = 0 \Rightarrow X_e = 0 \text{ and} \\ Y_e \neq 0 \Rightarrow X_e \neq -Y_e. \end{cases}$$

We call the set $\mathcal{V}$ endowed with this ordering the *face poset* of the oriented matroid. For the oriented matroid of the stratification of $\mathbb{R}^d$ by hyperplanes this corresponds to the above defined $\mathcal{F}(\mathcal{A})$. This poset is often completed by adjunction of an 'artificial' minimal element $\hat{0}$. This operation turns it into a lattice, and it is then called the *face lattice* of the oriented matroid.

It turns out that these axioms do *not* completely characterize the stratifications obtained by arrangements of linear hyperplanes in $\mathbb{R}^d$ or, equivalently, of equators on the sphere S^{d-1}. The fundamental *topological realization theorem* of Folkman and Lawrence shows that oriented matroids correspond rather to *arrangements of pseudospheres*, a pseudosphere being any centrally symmetric homeomorphic mapping of S^{d-2} into S^{d-1}. Given an arrangement $\mathscr{S}$ of pseudospheres on S^{d-1}, we write $\mathcal{F}(\mathscr{S})$ for the poset of cells of the induced cellularization of S^{d-1}, ordered by reverse inclusion. Thus, the situation of real arrangements corresponds to the case where the pseudospheres can be "stretched" and become equators of S^{d-1}. We state the precise theorem.

THEOREM 1.3.2 (Folkman and Lawrence, see [**65, 13**]). *Let $\mathcal{F}$ be the face lattice of an oriented matroid of rank d on n elements. Then there is an arrangement $\mathscr{S}$ of n pseudospheres on S^{d-1} such that $\mathcal{F} \setminus \{\hat{0}, \hat{1}\} = \mathcal{F}(\mathscr{S})$.*

1.4. Acyclic sets of oriented matroids

Paul Edelman studied in [**46**] the combinatorics of the simplicial complex of acyclic sets of an oriented matroid. We introduce it because of its relation with the so-called *tope poset* of the oriented matroid, that will be under the spotlight in chapters 6 and 7.

The idea is to look at a modified definition of an oriented matroid. Instead of looking at elements of $\{+, 0, -\}^E$, one looks at subsets of $\check{E} := E \cup \{e^* | e \in E\}$. The translation is that to a vector X correspond the set A such that $e \in A \Leftrightarrow X(e) = +$ and $e^* \in A \Leftrightarrow X(e) = -$. Note that the operation of 'changing side' can be described by an involution $* : \check{E} \to \check{E}$ defined by $*(e) = e^*$, $*(e^*) = e$.

The geometric interpretation is that one looks at *pseudohemispheres* rather than at the pseudospheres bounding them - and because of theorem 1.3.2 this interpretation is in fact exhaustive: we may state the following definition in terms of pseudohemispheres whitout loss of generality. So for $a \in \check{E}$ let s_a denote the related (open) pseudohemisphere.

The following definition can be found in [**46**] as lemma 3.2.

DEFINITION 1.4.1. *A subset $A \subseteq \check{E}$ is called* ***acyclic*** *if and only if*

$$\bigcap_{a \in A} s_a \neq \emptyset.$$

From now on we will denote by s_A the above intersection, and by $\widetilde{\mathcal{O}}$ the poset of acyclic sets ordered by reverse inclusion.

With theorem 1.3.2 and the above definition we have immediately the following lemma, that we will need (and prove) in a more explicit form in chapter 6.

LEMMA 1.4.2. *The acyclic sets of an oriented matroid of rank d give a simplicial complex that is isomorphic to the nerve of the covering of S^{d-1} by the s_A's (where A is acyclic), and thus is homotopy equivalent to the $(d-1)$-sphere. In particular, it is homotopy equivalent to the reduced (i.e., whitout maximal and minimal element) face lattice of the oriented matroid, that is the nerve of the stratification of S^{d-1} by covectors.*

One sees that, for the topology, what counts are the sets s_A. So a more essential definition of this complex is at hand.

DEFINITION 1.4.3. *The convex closure of an acyclic set A is defined as*

$$\overline{A} := \{a \in \check{E} \mid s_A = s_{A \cup \{a\}}\}.$$

An acyclic set A is called ***convex*** *if $\overline{A} = A$. The poset of convex sets ordered by reverse inclusion is called $\mathcal{O}$, and its minimal elements are called* ***acyclic orientations*** *of the oriented matroid.*

The following corollary sharpens the obvious statement of homotopy equivalence.

COROLLARY 1.4.4. *The order complex $\Delta(\mathcal{O})$ can be obtained from $\Delta(\widetilde{\mathcal{O}})$ by a sequence of elementary collapses.*

Proof. The convex closure map $\overline{\cdot} : \widetilde{\mathcal{O}} \to \widetilde{\mathcal{O}}$ is decreasing and maps $\widetilde{\mathcal{O}}$ onto $\mathcal{O}$. The claim now follows by standard tools of topological combinatorics (see e.g. [**10**]). $\square$

We conclude this short introduction by the following remark. In the case of a hyperplane arrangement, the minimal elements of $\mathcal{O}$ (and $\widetilde{\mathcal{O}}$) can be identified with the topes (i.e., the "chambers") of the arrangement. If we fix a chamber C_0 we can order the others by setting

$$C_1 >_{C_0} C_2 \Leftrightarrow C_1 \vee C_0 > C_2 \vee C_0$$

in $\mathcal{O}$. The right hand side is well-defined because the acyclic set corresponding to $C_0 \vee C_i$ is the complement in the acyclic orientation corresponding to C_0 of the pseudohemispheres that are bounded by pseudospheres separating C_0 from C_i. This set of pseudohemispheres is convex by e.g. [**12**, Prop. 5.4] and is clearly minimal above both C_0 and C_1. The poset one obtains this way is called *tope poset based at* C_0 and will be denoted as $\mathcal{P}_{C_0}(\mathcal{A})$ in chapter 6, specializing to the arrangement case.

Thus, any tope poset is a subposet of a maximal interval of $\mathcal{O}$. Indeed,

$$\mathcal{P}_{C_0} \subseteq J(C_0) := \big[\{a \mid C_0 \subseteq s_a\}, \emptyset\big] \subset \mathcal{O}.$$

The question of the topological relation between $\Delta(\mathcal{P}_{C_0} \setminus \{\hat{0}, \hat{1}\})$ and $\Delta(J(C_0) \setminus \{\hat{0}, \hat{1}\})$ lies now at hand. However, no special topological relation seems to hold. Figure 1.4.1 shows that the two posets are not homotopy equivalent for the Coxeter arrangement of type A_3.

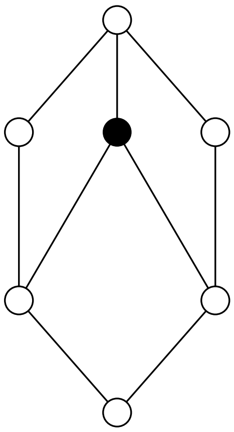


FIGURE 1. A maximal interval of the poset $\mathcal{O}$ for the Coxeter arrangement of type A_3. The black node is not an element of the corresponding poset of regions, depicted in white.

Figures 1.4.2 and 1.4.3 depict the posets of acyclic and convex sets for the same arrangement, as an example of such an object.

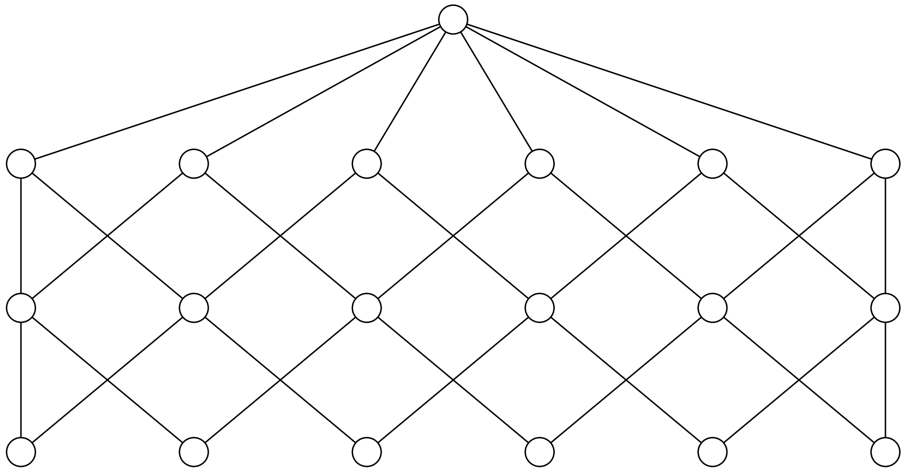


FIGURE 2. The poset $\mathcal{O}$ of convex sets of the reflection arrangement of Coxeter type A_3.

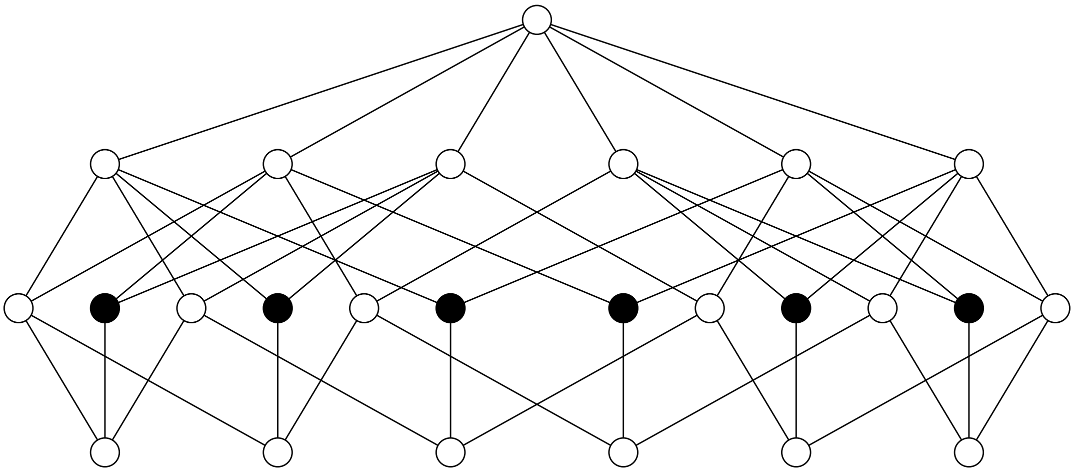


FIGURE 3. The poset $\tilde{\mathcal{O}}$ of acyclic sets of the arrangement of type A_3. Black dots are non-convex sets.

CHAPTER 2

The arrangement groupoid

The goal of this chapter is to establish Theorem 2.3.1 on the construction of the coverings of the arrangement groupoid, which will be a fundamental tool for the constructions of chapter 4 to 7. For this, we will present the notion of the groupoid associated to a complexified arrangement (section 2.1) and recall the general theory of coverings of groupoids (section 2.2) before applying it to our setting (section 2.3). In sections 2.1 and 2.3 we will also establish some notation that will be used later on when speaking about coverings of the arrangement groupoid.

It is a well-known fact that every group can be seen as a category with one object $*$ and with one arrow $* \xrightarrow{g} *$ for each element g of the group. Composition of arrows is of course defined by the group operation. Any category that is equivalent (in the categorical sense) to such a category is called a *groupoid*.

From the definition of equivalence of categories one sees immediately that in a groupoid every arrow must be invertible, and that the set of endomorphisms of any object has a natural group structure which is isomorphic to the 'original' group. In fact, these properties define classically the concept of groupoid.

DEFINITION 2.0.5. *A* ***groupoid*** *is a category $\mathcal{Q}$ where each arrow is invertible. This means that for any $A, B \in \mathrm{Ob}(\mathcal{Q})$ and any $a \in \mathrm{Mor}_{\mathcal{Q}}(A, B)$ there is $a' \in \mathrm{Mor}_{\mathcal{Q}}(B, A)$ such that $aa' = id \in \mathrm{End}_{\mathcal{Q}}(A)$.*

In particular, note that, for any $A \in \mathrm{Ob}(\mathcal{Q})$ the set $\mathcal{Q}(A) := \mathrm{End}(A)$ inherits a natural group structure.

If we start with the fundamental group $\pi_1(X)$ of a topological space X, there is a representative of the class of associated groupoids that was called *the Poincaré groupoid* of X by Gabriel and Zisman [**72**, II.7.3], and is customarily denoted by $\Pi_1(X)$. This category has the points of X as objects, and the set of morphisms $\mathrm{Mor}(P, Q)$ between two objects P and Q is the set of homotopy classes of continuous paths in X that start in P and end in Q.

The point in considering other categories than the one-object one is that, particularly in the case of topological spaces, one may gain some understanding of the structure of the group by giving a particularly suggestive associated groupoid.

Let us refer to the classical book by Gabriel and Zisman [**72**, Part II] for a careful and complete treatment of the concept of fundamental groupoid, and in particular for a description of the functorial relation between topological spaces, simplicial sets and the associated groupoid. Here we only want to point out that, for any choice of a set $S \subset X$ of base-points, one may define a groupoid $\Pi_S(X)$ given by

$$\mathrm{Ob}(\Pi_S(X)) := S$$
$$\mathrm{Mor}(s_1, s_2) := \{\text{homotopy classes of paths in } X \text{ from } s_1 \text{ to } s_2\}.$$

It is clear that the natural inclusion $\Pi_S(X) \hookrightarrow \Pi_X(X) = \Pi_1(X)$ induces an equivalence of categories.

In the following section we define the *arrangement groupoid* of a complexified arrangement as connected with a particular choice of base-points in the arrangement complement.

2.1. Definition of the arrangement groupoid

The notion of what we call *arrangement groupoid* is more or less standard and well-known. The goal of this section is to give a quick but rigorous definition from scratch in order to make this work self-contained and to fix ideas and notations.

Let $\mathcal{A}$ be a real arrangement, $\mathcal{C}(\mathcal{A})$ the set of its chambers. Naturally associated with $\mathcal{A}$ is a directed graph $G(\mathcal{A})$ (in what follows simply written G) with vertex set $V[G] = \mathcal{C}(\mathcal{A})$ and a directed edge (C_1, C_2) whenever the chambers C_1 and C_2 are adjacent.

Given this graph, we can consider $\texttt{Free}(G)$, the free category on G (see [**86**, section II.4] for the definition). Recall that the objects of $\texttt{Free}(G)$ are the vertices of G, and the morphisms are formally defined as directed paths in G. We therefore can associate a natural *length* to morphisms in this free category by defining a function $\lambda : \mathrm{Mor}(\texttt{Free}(G)) \to \mathbb{N}$ that maps every directed path to its edge-length.

For any two objects C_1, C_2 of $\texttt{Free}(G)$, we have a natural notion of 'distance', given by the length of a shortest (directed) path connecting them in G. Call this distance $\delta(C_1, C_2)$. We can therefore define an equivalence relation R_{C_1,C_2} in $\mathrm{Mor}(C_1, C_2)$: $R_{C_1,C_2}(\alpha, \beta)$ holds if $\lambda(\alpha) = \lambda(\beta) = \delta(C_1, C_2)$. We take the minimal extension of this relation that makes it compatible with composition of morphisms and call R the equivalence relation that we obtain on $\texttt{Free}(G)$.

Let us then define the category $\mathcal{G}^+(\mathcal{A})$ (sometimes simply written $\mathcal{G}^+$) as the quotient $\texttt{Free}(G)/R$ (again, see [**86**, section II.8] for the definition). It is clear that $\mathcal{G}^+$ is the category of positive paths considered by Salvetti [**103**].

We want now to add to this free category the inverse to every morphism, thus obtaining the category $\mathcal{G}(\mathcal{A}) := \mathcal{G}^+[(\mathrm{Mor}\mathcal{G}^+)^{-1}]$, which is then by definition a groupoid and will be often denoted just by $\mathcal{G}$, if no confusion can arise. It is connected, and therefore $B\mathcal{G}$ is a $K(\mathrm{End}_{\mathcal{G}}(C), 1)$ for any base chamber C.

It is easy to see (compare e.g.[**102**]) that the equivalence relation R identifies two morphisms if and only if the two corresponding paths in the Salvetti complex are homotopic. The groupoid $\mathcal{G}$ is therefore equivalent to the Poincaré groupoid (Gabriel-Zisman) of the Salvetti complex, and in particular $B\mathcal{G}(\mathcal{A})$ is a $K(\pi_1(\mathcal{A}), 1)$.

We call $\mathcal{G}(\mathcal{A})$ the ***arrangement groupoid***, while $\mathcal{G}^+(\mathcal{A})$ will be called ***positive category*** of the arrangement.

Notation: We will frequently have to deal with the arrangement groupoid. We therefore fix some notational convention. First of all, the objects of $\mathcal{G}$ are chambers, so we will refer to them speaking about adjacency, incidence and so on.

Second, after the completion of the positive category, a morphism in $\mathrm{Mor}_{\mathcal{G}}(C_1, C_2)$ can be described by a (not necessarily directed) path in G. We may write this path as a word with letters corresponding to edges. Every letter will have an exponent $+1$ or -1 according to whether the corresponding edge is traversed along or against its orientation.

A morphism that can be written by a path involving only positive exponents, and the path itself, will be called *positive*.

Moreover, to any edge of the arrangement graph there is naturally associated the hyperplane that it "crosses". A positive path is called *minimal* if it does not cross any hyperplane more than once. For fixed $C_1, C_2 \in \mathcal{C}(\mathcal{A})$ all positive minimal paths from C_1 to C_2 represent the same element of $\mathrm{Mor}_{\mathcal{G}}(C_1, C_2)$. This element will be denoted by $(C_1 \to C_2)$.

In general, we will denote paths by lowercase greek letters. If $\alpha_1 \in \mathrm{Mor}(C_1, C_2)$ and $\alpha_2 \in \mathrm{Mor}(C_2, C_3)$, the element of $\mathrm{Mor}(C_1, C_3)$ obtained by composition is represented by the concatenation of the paths, and denoted by $\alpha_1\alpha_2$.

Finally, if $\alpha \in \mathrm{Mor}(C_1, C_2)$, we will say that C_1 is the *beginning* and C_2 is the *end* of α, and write $C_1 = \mathit{begin}(\alpha)$, $C_2 = \mathit{end}(\alpha)$.

2.2. Coverings of groupoids

The point of groupoids here is that, as already remarked, the set of endomorphisms of any object inherits a natural group structure. This is used to develop a covering theory that parallels the topological one in a suggestive way. We follow essentially the exposition of Brown [**24**, Chapter 9], with slight changes of notations. We begin with the definition of covering morphisms of groupoids.

Definition 2.2.1 (see p. 339 of [**24**]). *Let $\mathcal{Q}$ be a groupoid, $X \in \mathrm{Ob}(\mathcal{Q})$. The **star** of $\mathcal{Q}$ at X is defined as*

$$St_{\mathcal{Q}}(X) := \bigcup_{Y \in \mathrm{Ob}(\mathcal{Q})} \mathrm{Mor}_{\mathcal{Q}}(X, Y).$$

*Consider now a functor $p : \mathcal{Q}' \to \mathcal{Q}$ (also called "morphism of groupoids"). We call p a **covering morphism** if the restriction of p*

$$St_{\mathcal{Q}'}(X') \to St_{\mathcal{Q}}(pX')$$

is bijective for every $X' \in \mathrm{Ob}(\mathcal{Q}')$.

*The **characteristic group** of this covering is defined as the subgroup $p(\mathcal{Q}'(X'))$ of $\mathcal{Q}(p(X'))$. Note that this definition is independent of the choice of $X' \in \mathrm{Ob}(\mathcal{Q}')$ and that the caracteristic group is isomorphic to the object group $\mathcal{Q}'(X')$.*

The next definition will be useful for the explicit construction of covering groupoids.

Definition 2.2.2 (9.4.1 of [**24**]). *An **action** of a groupoid $\mathcal{Q}$ on a set X is given by a pair of maps*

$$\begin{gathered} w : X \to \mathrm{Ob}(\mathcal{Q}) \\ [\cdot] : \mathrm{Mor}_{\mathcal{Q}}(A, B) \to \mathrm{Mor}_{Set}(w^{-1}(A), w^{-1}(B)), \end{gathered}$$

such that the following hold:

1. *For any $A \in \mathrm{Ob}(\mathcal{Q})$, $[id_A](x) = x$ for all $x \in w^{-1}(A)$,*
2. *For any $a \in \mathrm{Mor}_{\mathcal{Q}}(A, B)$, $b \in \mathrm{Mor}_{\mathcal{Q}}(B, C)$, $[ab](x) = [a]([b](x))$ for all $x \in w^{-1}(A)$.*

Note that with this definition we have that, for any $x \in X$,

$$Q_x := \{a \in \mathrm{End}_{\mathcal{Q}}(w(x)) | [a](x) = x\}$$

is a subgroup of $\mathcal{Q}(w(x))$. It is called the *group of stability* of x.

Furthermore, the action is called *transitive* if for any $A_1, A_2 \in \mathrm{Ob}(\mathcal{Q})$ and any $x_i \in w^{-1}(A_i)$ $(i = 1, 2)$, there is $a \in \mathrm{Mor}_{\mathcal{Q}}(A_1, A_2)$ such that $[a](x_1) = x_2$.

We now define the basic operation for the explicit construction of covers of groupoids.

Definition 2.2.3 (9.4.2 of [**24**]). *Given an action of a groupoid $\mathcal{Q}$ on a set X, the semi-direct product groupoid $X \rtimes \mathcal{Q}$ is the category defined by:*

$\mathrm{Ob}(X \rtimes \mathcal{Q}) := X,$
$\mathrm{Mor}_{X \rtimes \mathcal{Q}}(x, y) := \{(x, a) | a \in \mathrm{Mor}_{\mathcal{Q}}(w(x), w(y))$ *and* $[a](x) = y\}.$

The next theorem explains how exactly the covering theory of groupoids parallels the one for (connected) topological spaces.

Theorem 2.2.4 (see 9.4.3 of [**24**], p.348). *Let $\mathcal{Q}$ be a connected groupoid, $A \in \mathrm{Ob}(\mathcal{Q})$. Let H be a subgroup of the object group $\mathcal{Q}(A) = End_{\mathcal{Q}}(A)$. Let X be the set of right cosets*

$$Ha := \{ha | h \in H\}$$

for $a \in St_{\mathcal{Q}}(A)$. Then the map $w : X \to \mathrm{Ob}(\mathcal{Q})$ sending Ha to the target of a defines a transitive action of $\mathcal{Q}$ on X.
The corresponding connected covering morphism $p : X \rtimes \mathcal{Q} \to \mathcal{Q}$ has characteristic group H. Moreover, $p^{-1}(x)$ is the ***set*** *$\mathcal{Q}(x)/H$.*

This quick run through the principal definitions and facts of the covering theory of groupoids has no aim of exhaustiveness. It should rather provide a sketch of the general setup that we will enter in the particular case of the arrangement groupoid. As already pointed out, the interested reader will find all proofs, explanations and examples in the book of Brown [**24**, Chapter 6 and 9]. We conclude this section introducing some notation. To underline the topological flavour of our approach to groupoids let us denote by $\pi(\mathcal{Q}, X)$ the object group $\mathrm{End}(X)$ of an object X of a groupoid $\mathcal{Q}$ and, by abuse of terminology, call it the *fundamental group of the groupoid $\mathcal{Q}$ at X*. The change of base-object can be seen as a change of presentation for the group. If no specification will be needed, we will omit the reference to X and write simply $\pi\mathcal{Q}$.

2.3. Coverings of the arrangement groupoid $\mathcal{G}(\mathcal{A})$

We now turn to the arrangement groupoid. Recall that we have an isomorphism $\pi(\mathcal{G}(\mathcal{A})) \simeq \pi_1(\mathcal{A})$.

Therefore theorem 2.2.4 takes following form:

Theorem 2.3.1. *Let $\mathcal{A}$ be a real arrangement, and let $\mathcal{G} := \mathcal{G}(\mathcal{A})$. For any chamber C_0 and any subgroup $H < \pi(\mathcal{G}, C_0) \cong \pi(\mathcal{A})$ let X_{H,C_0} be the set of right cosets of H in $St_{\mathcal{G}}(C_0)$. Then the functor*

$$\rho : \mathcal{G}_\rho := (X_{H,C_0} \rtimes \mathcal{G}) \to \mathcal{G}$$

is a covering of groupoids with characteristic group H. In particular, $\pi\mathcal{G}_\rho \cong H$.

We will often write just $\mathcal{G}_\rho$ or ρ to indicate the corresponding covering of $\mathcal{G}$, if no specification of the full situation is needed.

Notation: Given a covering $\rho : \mathcal{G}_\rho \to \mathcal{G}$ of the arrangement groupoid, $\alpha \in \mathrm{Mor}(\mathcal{G})$ and $a \in \mathrm{Ob}(\mathcal{G}_\rho)$ such that $\rho(a) = begin(\alpha)$, we will denote by $\hat{\gamma}_a(\alpha)$ the unique lift at a of the path α. In general, we will write lowercase latin letters for objects of $\mathcal{G}_\rho$.

REMARK 2.3.2. As a concluding remark leading naturally to the next chapter, we wish to point to the work of Dror-Farjoun, Loday, Brown *et al.* [**45, 25, 26**]. Since the fundamental groupoid of a space behaves functorially under homotopy colimit of diagrams of topological spaces (see [**45**]), they searched for a "higher" analogue of the fundamental groupoid. Dror-Farjoun suggests that *crossed objects* (studied especially by Brown and Loday [**26**]) may lead to the desired generalization. The goal is to calculate the higher homotopy groups of the homotopy colimit of a diagram of spaces.

CHAPTER 3

Homotopy colimits of diagrams of spaces

The goal of this chapter is to introduce the notion of homotopy colimit of diagrams of spaces (Definition 3.1.1) and state some basic tools for handling them (Lemmas 3.2.1 - 3.2.5).

The theory of homotopy colimits of diagrams of spaces comes from homological algebra and category theory. It was developed by Quillen, Bousfield, Kan and others (see for example [**99**], [**19**]), and has now reached remarkable extension and depth. The literature is very ample, and it is impossible to give a complete account of it in this restricted setting. The reader may find some orientation (and a good introduction to the general subject) in the book by Goerss and Jardine [**76**] and its bibliography.

The work of Reiner M. Vogt [**114**] leaves the complete generality of diagrams over categories and focuses on what can be said if one makes more and more assumptions on the target category (never going beyond the properties satisfied by the category of topological spaces) and on the index category, requiring it to be *small* and, as a further restriction, *directed*. For example, Vogt derives the explicit form of definition 3.1.1.

In our work we will take the latter and more combinatorial point of view, which was adopted by Welker, Ziegler and Živaljević in [**118**]. They developed a useful toolkit for applications of homotopy colimits in discrete mathematics, which led e.g. to the construction of models for the link of arrangements, see [**124**], [**71**] for details. Here we will only recall the main definitions and some basic results, all of which can be derived in principle from [**118**]. Nevertheless, we will sometimes sketch some proofs in the more explicit style of the preprint [**117**] that was circulating before the final version of [**118**]. We refer to [**118**] for a comprehensive and detailed introduction to the subject.

3.1. Definitions

A ***diagram of spaces*** is a covariant functor

$$\mathscr{D} : \mathcal{I} \longrightarrow Top,$$

where $\mathcal{I}$ denotes some small category and Top is the category of topological spaces and continuous maps.

In our combinatorial setting, $\mathcal{I}$ will always be some poset $\mathcal{P}$. Indeed, a poset is a small category if we say that there is an arrow from $p \in \mathcal{P}$ to $q \in \mathcal{P}$ if and only if $p \geq q$ in $\mathcal{P}$.

In order to simplify notation, we will sometimes write D_p for the space $\mathscr{D}(p)$, and $f_{p,q}$ for the map $\mathscr{D}(p > q)$.

A *morphism* from a diagram $\mathscr{D}$ on the poset $\mathcal{P}$ to a diagram $\mathscr{E}$ on $\mathcal{Q}$ is a pair $(\mu, (\alpha_p)_{p\in\mathcal{P}})$, where $\mu : \mathcal{P} \to \mathcal{Q}$ is a morphism of posets, and $(\alpha_p)_{p\in\mathcal{P}}$ is a family of continuous maps $\alpha_p : \mathscr{D}(p) \to \mathscr{E}(\mu(p))$ (indexed by elements of $\mathcal{P}$) that commute with the diagram maps, i.e.,

$$\begin{array}{ccc}
\mathscr{D}(p) & \xrightarrow{\alpha_p} & \mathscr{E}(\mu(p)) \\
{\scriptstyle \mathscr{D}(p>q)}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{E}(\mu(p)\geq\mu(q))} \\
\mathscr{D}(q) & \xrightarrow{\alpha_q} & \mathscr{E}(\mu(q)).
\end{array}$$

Given a diagram of spaces, one can consider its homotopy colimit. Again, we will take here the 'constructive', topological definition given in [**118**].

DEFINITION 3.1.1. *Given a diagram of spaces $\mathscr{D} : \mathcal{P} \to Top$, the* ***homotopy colimit*** *of $\mathscr{D}$ is defined by*

$$\mathrm{hocolim}\mathscr{D} := \coprod_{p\in\mathcal{P}} \Delta(\mathcal{P}_{\leq p}) \times D_p \Big/ \sim$$

where the relation $\sim$ is given, for $p > q$, by the following identifications:

$$\Delta(\mathcal{P}_{\leq q}) \times D_p \hookrightarrow \Delta(\mathcal{P}_{\leq p}) \times D_p,$$

$$\Delta(\mathcal{P}_{\leq q}) \times D_p \overset{(id\times f_{p,q})}{\longrightarrow} \Delta(\mathcal{P}_{\leq q}) \times D_q.$$

As a very small example consider the poset with two elements p and q and the order relation given by $p > q$. Let a diagram over this poset be given that associates to p the space A, to q the space B, and to the order relation $p > q$ the map $f : A \to B$. Then the homotopy colimit is

$$[0,1] \times A \sqcup \{1\} \times B \Big/ \big\{\{1\} \times a \sim \{1\} \times f(a) \text{ for } a \in A\big\},$$

hence it is the mapping cylinder of f (see [**119**] I,5).

3.2. The toolkit

Explicit computations of homotopy colimits can become very long and cumbersome, even for small examples. Therefore it is convenient to apply the tools described in [**118**] for manipulating a diagram and bringing it into a more manageable form. In this section we list some results for later reference. In particular the following will be about:

3.2.1 *The homotopy lemma,*
3.2.2 *The inverse image lemma,*
3.2.3 *Two subdivision lemmas,*
3.2.4 *The simplicial model lemma,*
3.2.5 *The projection lemma.*

The proof of some of them can be found directly in [**118**], for the others we will give appropriate references or sketch a quick proof. All of them were present in some form in a preliminary version of [**118**] (see [**117**]).

We will conclude mentioning the question of spectral sequences for homotopy colimits, that will be treated extensively later on.

3.2.1. The homotopy lemma. We start with a lemma that can be regarded as *the* result about homotopy colimits. Namely, it says that one can change the diagram spaces up to homotopy without affecting the homotopy type of the homotopy colimit of the whole diagram. This is of course not true for 'usual' colimits, and is therefore a topological point of interest for looking at homotopy colimits.

LEMMA 3.2.1 (Proposition 3.7 of [**118**]). *Let $\phi := (id, (\alpha_p)_{p\in\mathcal{P}}) : \mathscr{D} \to \mathscr{E}$ be a morphism between two diagrams over the same poset $\mathcal{P}$.*

If every α_p is a (weak) homotopy equivalence, then the induced map

$$\hat{\phi} : \mathrm{hocolim}\mathscr{D} \to \mathrm{hocolim}\mathscr{E}$$

is a (weak) homotopy equivalence.

3.2.2. The inverse image lemma. The above homotopy lemma shows how one can change the spaces of a diagram maintaining the homotopy type of its homotopy colimit. The next result is concerned with modifying the poset underlying the diagram or, from another viewpoint, with "pulling back" a diagram along a poset morphism. The requirement to this morphism is that it induces homotopy equivalence between posets.

LEMMA 3.2.2 (Lemma 4.7 of [**118**]). *Let $\mu : \mathcal{P} \to \mathcal{Q}$ be a poset morphism and let $\mathscr{D}$ be a diagram over $\mathcal{Q}$. Define the diagram $\mu^*\mathscr{D}$ by*

$$\mu^*\mathscr{D}(p) := \mathscr{D}(\mu(p)), \quad \mu^*\mathscr{D}(p > q) := \mathscr{D}(\mu(p) \geq \mu(q)).$$

If the preimage $\mu^{-1}(\mathcal{Q}_{\leq q})$ is contractible for all $q \in \mathcal{Q}$, then μ induces a homotopy equivalence.

$$\mathrm{hocolim}\,\mu^*\mathscr{D} \to \mathrm{hocolim}\,\mathscr{D}.$$

The following section describes two consequences of this very useful lemma.

3.2.3. Two subdivision lemmas. We now present a technical tool that we will need in this exposition. It requires some preparatory definitions:

DEFINITION 3.2.3. *Given a poset $\mathcal{P}$ and an element $p \in \mathcal{P}$, let $\mathcal{P}_{\leq p}$ denote the subposet of $\mathcal{P}$ consisting of all elements smaller than or equal to p. Moreover, $\Sigma^\vee(\mathcal{P})$ will denote the set of all chains $\sigma = p_1 < \cdots < p_k$ in $\mathcal{P}$, ordered by reverse inclusion: $\sigma \geq \sigma' \Leftrightarrow \sigma \subseteq \sigma'$. Given a diagram of spaces $\mathscr{D}$ on $\mathcal{P}$, we will denote by $\mu^*\mathscr{D}$ the inverse image of $\mathscr{D}$ with respect to the map of posets $\mu : \Sigma^\vee(\mathcal{P}) \to \mathcal{P}$ that sends each chain to its minimal element. The spaces are defined as $(\mu^*\mathscr{D})(\sigma) := \mathscr{D}(\mu(\sigma))$, and since $\sigma > \sigma'$ implies $min(\sigma) \geq min(\sigma')$ in $\mathcal{P}$, the natural choice for the maps $(\mu^*\mathscr{D})(\sigma > \sigma')$ is to take the morphisms $\mathscr{D}(min(\sigma) \geq min(\sigma'))$.*

LEMMA 3.2.4. *Let $\mathscr{D}$ be a diagram of spaces. We have*

$$\mathrm{hocolim}\,\mathscr{D} \simeq \mathrm{hocolim}\,\mu^*\mathscr{D}.$$

Proof. Consider an element $P \in \mathcal{P}$. The preimage $\mu^{-1}(\mathcal{P}_{\leq p})$ is meet-contractible via $\{p\} \in \mu^{-1}(p)$ (i.e., the meet $\sigma \wedge \{p\}$ exists for all $\sigma \in \mu^{-1}(\mathcal{P}_{\leq p})$), hence it is contractible by corollary 10.13 of [**10**]. The claim now follows from the inverse image lemma 3.2.2. □

Analogously, for a poset $\mathcal{P}$ let us define $\Sigma(\mathcal{P})$ as the poset of chains of $\mathcal{P}$ ordered by inclusion. Then $\Sigma(\mathcal{P})$ is isomorphic to the face poset $\mathcal{F}(\Delta(\mathcal{P}))$ of the order complex of

$\mathcal{P}$. We immediately have following corollary of lemma 3.2.2. The proof is similar to the previous, but less involved.

COROLLARY 3.2.5. *Given a diagram of spaces $\mathscr{D}$ over a poset $\mathcal{P}$, define the diagram $\mathscr{D}^\Delta$ over $\Sigma(\mathcal{P})$ by*

$$\mathscr{D}^\Delta(\omega) := \mathscr{D}(max\,\omega)$$

for any chain ω of $\mathcal{P}$, with the obvious maps. We have

$$\text{hocolim}\mathscr{D} \simeq \text{hocolim}\mathscr{D}^\Delta.$$

Proof. Note that $\mathscr{D}^\Delta$ is the inverse image of $\mathscr{D}$ with respect to the map $\Sigma(P) \to \mathcal{P}$ that sends a chain σ to its maximal element. The inverse image lemma can be applied since the preimage of $\mathcal{P}_{\leq p}$ is join-contractible via p. □

3.2.4. The simplicial model lemma. We now consider the special case of diagram spaces $\mathscr{D}(p)$ being homotopy equivalent to order complexes of posets $\mathcal{Q}_p$. Then it is possible to write the homotopy colimit as an order complex of an explicitly given poset.

DEFINITION 3.2.6. *A **diagram of posets** is a diagram of spaces $\mathscr{D} : \mathcal{P} \to Top$ such that the spaces assigned to poset elements are order complexes, i.e., for every $p \in \mathcal{P}$ there is a poset $\mathcal{Q}_p$ such that $\mathscr{D}(p) = \Delta(\mathcal{Q}_p)$.*

*In this situation, we also can define the **poset limit** Plim$\mathscr{D}$ of the diagram of posets $\mathscr{D}$. This is a poset with set of elements*

$$\text{Plim}\mathscr{D} := \bigcup_{p \in \mathcal{P}} \{p\} \times \mathcal{Q}_p$$

and order relations defined by

$$(p_1, q_1) \geq (p_2, q_2) :\Leftrightarrow \begin{cases} p_1 \geq p_2 & and \\ f_{p_1,p_2}(q_1) \geq q_2. \end{cases}$$

We can now state the *simplicial model lemma:*

LEMMA 3.2.7. *Let $\mathscr{D}$ be a diagram of posets. Then the homotopy colimit of $\mathscr{D}$ is homotopy equivalent to the order complex of the poset limit of $\mathscr{D}$:*

$$\text{hocolim}\mathscr{D} \simeq \Delta(\text{Plim}\mathscr{D}).$$

Proof. See [**2**], note after Corollary 2.11. □

3.2.5. The projection lemma. Even if the spaces are not expressed as order complexes of posets, one can exploit the structure of the poset underlying a diagram to get a topological idea of its homotopy colimit. However, this is possible only if the diagram morphisms do not add 'extra' topological structure in the gluing of the spaces. We make this precise in the following.

LEMMA 3.2.8 (Lemma 4.5 of [**118**]). *Let $\mathscr{D}$ denote a diagram of spaces on a poset $\mathcal{P}$. If all diagram maps $\mathscr{D}(p > q)$ are closed cofibrations, then the natural map*

$$\text{hocolim}\mathscr{D} \to \text{colim}\mathscr{D}$$

induces a homotopy equivalence.

3.2.6. Spectral sequences for homotopy colimits.

Spectral sequences for homotopy colimits in this combinatorial setting are already used in [**118**] to compute the homology of the homotopy colimit of a diagram associated to toric varieties. In that occurrence, the authors refer to the paper of Segal [**104**] where a spectral sequence is described in the full generality of semi-simplicial spaces. In chapter 5 we will give a direct topological argument for the general case of diagrams of topological spaces over posets, and study the particularly nice forms of these sequences that arise with some assumption on the poset.

Part 1

Covers of complexified arrangements

CHAPTER 4

Salvetti-type diagram models

To each cover of the Salvetti complex of a complexified arrangement $\mathcal{A}$ we will associate a diagram of spaces whose homotopy colimit is homotopy equivalent to the given cover (Theorem 4.5.1). With the simplicial model lemma 3.2.7 we are then able to write each covering space as the order complex of a poset (Scholium 4.5.2). This order complex is shown to be the barycentric subdivision of a "Salvetti-type" cell structure on the covering space (Remark 4.4.1).

The structure of the diagram explicitly separates the purely topological information (encoded in the arrangement groupoid) and the combinatorics of the stratification (carried by the index category - in our case the face poset of the arrangement).

Luis Paris already worked on the covers of a complexified arrangement in [**96, 97, 98**]. Here the combinatorial characterization that is implicit in [**97**] for the universal cover of a linear arrangement is extended to any cover of any locally finite arrangement. We want also to point out that our construction uses the classical notion of fundamental groupoid and the established theory of groupoids instead of the oriented systems introduced by Paris to encode essentially the same information.

The precise relation with Paris' models is explained in section 4.6 (Proposition 4.6.9).

We want to point out that the whole construction can actually be carried out for arbitrary oriented matroids, constructing the groupoid from the tope graph (note that by [**13**, Theorem 4.4.7] the equivalence relation on minimal paths still makes sense) and the index poset from the face lattice of the oriented matroid. The cell complexes one obtains this way are *MH-complexes* in the sense of Salvetti [**103**], who carefully studied their homotopical properties as determined by the groupoid.

4.1. The main character

As all mathematical stories, also ours begins with a definition.

Recall that we always have in mind a locally finite, possibly affine real arrangement $\mathcal{A}$ that we want to study. In particular we consider its face poset $\mathcal{F}$, its arrangement graph G and the associated arrangement groupoid $\mathcal{G}$. We refer to the corresponding chapters of the prologue for definitions and specific notations about these objects.

Definition 4.1.1. *Given a cover of the arrangement groupoid $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$, we define a diagram of spaces*

$$\mathscr{D}_\rho : \mathcal{F} \longrightarrow Top$$

with discrete spaces

$$\mathscr{D}_\rho(F) := \{v \in \mathrm{Ob}(\mathcal{G}_\rho) | \rho(v) < F\},$$

and maps being inclusions

$$\begin{array}{rccl} f^{\rho}_{F_1,F_2} := \mathscr{D}_\rho(F_1 > F_2): & \mathscr{D}_\rho(F_1) & \longrightarrow & \mathscr{D}_\rho(F_2) \\ & v & \longmapsto & end\big(\hat{\gamma}_v(\rho(v) \to \rho(v)_{F_2})\big) \end{array}$$

(see section 2.3 for notations).

Let us first make the following observation.

Proposition 4.1.2. *For the identical cover $id : \mathcal{G}_{id} \to \mathcal{G}(\mathcal{A})$, the homotopy colimit of $\mathscr{D}_{id}$ is homotopy equivalent to the Salvetti complex of $\mathcal{A}$:*

$$\mathrm{hocolim}\mathscr{D}_{id} \simeq Sal(\mathcal{A}),$$

in fact, the order complex of the poset limit $\mathrm{Plim}\mathscr{D}_{id}$ *is identical with the Salvetti complex as a simplicial complex.*

Proof. Let us first see, what spaces and maps are in this case:

$$\mathscr{D}_{id}(F) = \{C \in \mathcal{C}(\mathcal{A}) | C < F\}$$

$$\begin{array}{rccl} f^{id}_{F_1,F_2} := \mathscr{D}_{id}(F_1 > F_2): & \mathscr{D}_{id}(F_1) & \longrightarrow & \mathscr{D}_{id}(F_2) \\ & C & \longmapsto & C_{F_2} \end{array}$$

The "spaces" are in fact finite sets of points. We can consider them as order complexes of posets $\mathcal{Q}_F$ where $\mathcal{Q}_F = \{C \in \mathcal{C}(\mathcal{A}) |\, C < F\}$ as a set, and the order relation is the empty one: $C_1 \geq C_2$ if and only if $C_1 = C_2$.

In this situation we can apply the *simplicial model lemma (3.2.7)*: $\mathrm{hocolim}\mathscr{D}_{id}$ has the same homotopy type as $\Delta(\mathrm{Plim}\mathscr{D}_{id})$, which we want to study now. In the following we set $W := \Delta(\mathrm{Plim}\mathscr{D}_{id})$. The vertex set of the simplicial complex W is clearly

$$\mathscr{V} := \{(F, C) \in \mathcal{F} \times \mathcal{C} |\, F > C\}.$$

For the higher-dimensional simplices we have to look at the chains in $\mathrm{Plim}\mathscr{D}_{id}$. There we have

$$(F_1, C_1) \geq (F_2, C_2) \Leftrightarrow \begin{cases} 1) & F_1 \geq F_2, \text{ and} \\ 2) & (C_1)_{F_2} \leq C_2. \end{cases}$$

Note that in a chain $(F_1, C_1) > (F_2, C_2) > \cdots > (F_j, C_j)$ we have (see 1.2.2) $C_i = (C_{i-1})_{F_i} = \cdots = (\ldots((C_1)_{F_2})_{F_3} \ldots)_{F_i} = (C_1)_{F_2 \circ F_3 \ldots F_i} = (C_1)_{F_i}$. A chain in $\mathrm{Plim}\mathscr{D}_{id}$ is then given by a chain in $\mathcal{F}$ and a chamber adjacent to the maximal element of this chain. The rest can be reconstructed as above. Since all chains are of this form, we can encode each simplex of W by $\Delta(\phi, C)$, with which we will mean the d-simplex given by the $d+1$-chain ϕ and the chamber $C > max(\phi)$. A quick look at definition 1.2.2 shows that the evident bijection $\mathscr{V} \to V(Sal(\mathcal{A}))$ can be linearly extended to all simplices to get an isomorphism of simplicial complexes. □

4.2. The original motivation

This construction was motivated by a question asked by Michael Falk at the end of section 2 in [**53**], where he presented a model $\hat{S}$ for the complement of complexified line arrangements. It was constructed starting from the local data of the arrangements $\mathcal{A}_X$ for $X \in \mathcal{L}$ and attaching the complexes $Sal(\mathcal{A}_X)$ "in the right way". By contraction of an appropriate subcomplex of $\hat{S}$ he obtained a minimal 2-dimensional complex carrying the homotopy type of the arrangement complement.

The following theorem yields a natural generalization of the construction of $\hat{S}$ for affine arrangements of any dimension. Indeed, one can easily see that the homotopy colimit of the diagram $\mathcal{E}$ defined below is stratified by the face structure of the arrangement, and over any face F we have a copy of $Sal(\mathcal{A}_{|F|}) \times F$.

THEOREM 4.2.1. *Given a real arrangement $\mathcal{A}$, define a diagram of spaces $\mathscr{E}$ on the poset $\mathcal{F}^{op}$ by*

$$\begin{aligned} \mathscr{E}(F) &:= Sal(\mathcal{A}_{|F|}) \\ \mathscr{E}(F_1 > F_2): &\ \textit{the natural map of complexes } Sal(\mathcal{A}_{|F_1|}) \hookrightarrow Sal(\mathcal{A}_{|F_2|}), \\ &\ \textit{induced by the inclusion } \mathcal{A}_{|F_1|} \hookrightarrow \mathcal{A}_{|F_2|}. \end{aligned}$$

Then we have

$$\mathrm{hocolim}\mathscr{E} \simeq \mathrm{hocolim}\mathscr{D}_{id} \simeq \mathcal{M}(\mathcal{A}).$$

Proof. In the following, given a diagram $\mathcal{D}$ over a poset $\mathcal{P}$ and an element $P \in \mathcal{P}$, let $\mathcal{D}[\mathcal{P}_{\leq P}]$ denote the diagram over $\mathcal{P}_{\leq P}$ that is naturally induced by $\mathcal{D}$. Moreover, $\mathcal{F}$ will always denote $\mathcal{F}(\mathcal{A})$, the face poset of the whole arrangement.

Consider a face $F \in \mathcal{F}$. It is easy to see that $\mathcal{F}(\mathcal{A}_{|F|}) = \mathcal{F}_{\leq F}$. We have

$$\mathcal{E}(F) = Sal(\mathcal{A}_F) \simeq \mathrm{hocolim}\mathscr{D}_{id}[\mathcal{F}_{\leq F}].$$

We now consider the diagram $\mu^*\mathscr{D}_{id}$ as defined in 3.2.3. The subdivision lemma 3.2.4 guarantees

$$\mathrm{hocolim}\mu^*\mathscr{D}_{id} \simeq \mathrm{hocolim}\mathscr{D}_{id}.$$

Consider now the map of posets $f : \Sigma^{\vee}(\mathcal{F}) \to \mathcal{F}^{op}$ defined by $f(\sigma) = max(\sigma)$. The homotopy colimit of a diagram over a poset $\mathcal{P}$ with a unique minimal element $\hat{0}$ is homotopy equivalent to the space attached to $\hat{0}$ (indeed, consider the inclusion $\iota : \{\hat{0}\} \to \mathcal{P}$: since $\hat{0}$ is the unique minimal element, the preimage $\iota^{-1}(\mathcal{P}_{\leq P})$ is contractible for all $P \in \mathcal{P}$ and we can apply lemma 4.7 of [**118**]). Hence by lemma 4.8 of [**118**], there is an homotopy equivalence

$$\mathrm{hocolim}\mu^*\mathscr{D}_{id} \simeq \mathrm{hocolim} f_\sharp(\mu^*\mathscr{D}_{id}),$$

where $f_\sharp(\mu^*\mathscr{D}_{id})$ is defined over the poset $\mathcal{F}^{op}$ by

$$f_\sharp(\mu^*\mathscr{D}_{id})(F) := \mathrm{hocolim}(\mu^*\mathscr{D}_{id})[f^{-1}(\mathcal{F}^{op}_{\geq F})],$$

the maps $f_\sharp(\mu^*\mathscr{D}_{id})(F > F')$ being the natural inclusions of homotopy colimits.

Now we leave it to the reader to check that $(\mu^*\mathscr{D}_{id})[f^{-1}(\mathcal{F}^{op}_{\geq F})] = \mu^*(\mathscr{D}_{id}[\mathcal{F}_{\leq F}])$. For any $F \in \mathcal{F}^{op}$ we then have an homotopy equivalence by lemma 3.2.4:

$$f_\sharp(\mu^*\mathscr{D}_{id})(F) \simeq \mathrm{hocolim}\mathscr{D}_{id}[\mathcal{F}_{\leq F}] \to \mathcal{E}(F).$$

These maps commute with the morphisms of the diagrams and hence induce an homotopy equivalence between the homotopy colimits by lemma 4.6 of [**118**].

Putting everything together we have then

$$\mathcal{M}(\mathcal{A}) \simeq \mathrm{hocolim}\mathscr{D}_{id} \simeq \mathrm{hocolim}\mu^*\mathscr{D}_{id} \simeq \mathrm{hocolim} f_\sharp(\mu^*\mathscr{D}_{id}) \simeq \mathrm{hocolim}\mathcal{E}, \qquad \square$$

as required. $\square$

4.3. Covering maps

We now turn to the general case of a cover $\rho : \mathcal{G}_\rho \to \mathcal{G}$. In fact, all diagrams $\mathscr{D}_\rho$ are diagrams of posets (see the proof of 4.1.2), and an application of the *simplicial model lemma* 3.2.7 gives

$$\mathrm{hocolim}\mathscr{D}_\rho \simeq \Delta(\mathrm{Plim}\mathscr{D}_\rho).$$

For simplicity let us from now on write $W_\rho := \Delta(\mathrm{Plim}\mathscr{D}_\rho)$.

The simplicial complex W_ρ has vertex set

$$\mathscr{V}_\rho = \{(F, v) \in \mathcal{F} \times \mathrm{Ob}(\mathcal{G}_\rho) | \, \rho(v) < F\}$$

and the simplices are chains with respect to the partial order

$$(F_1, v_1) \geq (F_2, v_2) \Leftrightarrow \begin{cases} 1) & F_1 \geq F_2, \\ 2) & v_2 = end\big(\hat{\gamma}_{v_1}(\rho(v_1) \to \rho(v_1)_{F_2})\big). \end{cases}$$

A technical lemma has to be worked out here in order to make the following easier to read.

LEMMA 4.3.1. *For the vertex set $\mathscr{V}_\rho$ of $W_\rho := \mathrm{Plim}\mathscr{D}_\rho$ the following holds:*

$$\mathscr{V}_\rho = \mathcal{F} \times \mathrm{Ob}(\mathcal{G}_\rho)/\sim$$

where $\sim$ is the smallest equivalence relation generated by

$$(F, v) \sim \big(F, end\big(\hat{\gamma}_v(\rho(v) \to \rho(v)_F)\big)\big).$$

Proof. Consider the map $j : \mathscr{V}_\rho \to \mathcal{F} \times \mathrm{Ob}(\mathcal{G}_\rho)/\sim$. It is surjective, since in the equivalence class of each pair (F, v) there is some $(F, \tilde{v}) \in \mathscr{V}_\rho$, for example the one with $\tilde{v} = end(\hat{\gamma}_v(\rho(v) \to \rho(v)_F))$.

The map j would fail to be injective if we could find $(F, v_1), (F, v_2) \in \mathscr{V}_\rho$ with $v_1 \neq v_2$ such that there is $v \in \mathrm{Ob}(\mathcal{G}_\rho)$ with

$$v_1 = end\big(\hat{\gamma}_v(\rho(v) \to \rho(v)_F)\big) = v_2.$$

This is impossible by the uniqueness of lifting paths. $\square$

The functor $\rho : \mathcal{G}_\rho \to \mathcal{G}$ naturally induces a morphism of diagrams that we call

$$\lambda : \mathscr{D}_\rho \longrightarrow \mathscr{D}_{id}.$$

On diagram spaces of $\mathscr{D}_\rho$ it is given by a map

$$\begin{array}{rccc} \lambda_F : & \mathscr{D}_\rho(F) & \to & \mathscr{D}_{id}(F) \\ & v & \mapsto & \rho(v) \end{array}$$

for every $F \in \mathcal{F}$.

Remark: To check compatibility with the morphisms, consider that for each pair $F_1 > F_2$ and $v \in \mathscr{D}_\rho(F_1)$, we have

$$\begin{aligned} \lambda_{F_2}(f^{\rho}_{F_1,F_2}(v)) &= \rho(end(\hat{\gamma}_v(\rho(v) \to \rho(v)_{F_2}))) \\ &= \rho(v)_{F_2} = f^{id}_{F_1,F_2}(\rho(v)) = f^{id}_{F_1,F_2}(\lambda_{F_1}(v)). \end{aligned}$$

The morphism λ induces a map between the homotopy colimits by functoriality, and thus a map

$$\Lambda_\rho : W_\rho \longrightarrow W.$$

In fact, Λ_ρ is a simplicial extension of λ. The simplex $\Delta(\phi, v)$ of W_ρ is mapped to $\Delta(\phi, \rho(v))$ in W.

Note that the previous considerations can be followed step by step to see that a morphism $\eta : \mathcal{G}_{\rho_1} \to \mathcal{G}_{\rho_2}$ between two covers $\rho_i : \mathcal{G}_{\rho_i} \to \mathcal{G}$ induces a map $\Lambda_\eta : W_{\rho_1} \to W_{\rho_2}$.

The goal of this section is to prove

PROPOSITION 4.3.2. *The map $\Lambda_\rho : W_\rho \to W$, induced by a cover of groupoids $\rho : \mathcal{G}_\rho \to \mathcal{G}$, is a topological cover of W.*

Proof.[1] The first thing one should ensure when speaking of covering spaces is that the base space is connected and locally arcwise connected. In our case this follows from the fact that W is finite dimensional and locally finite.

Now take $P \in W$ and $\epsilon > 0$ such that $B_\epsilon^N(P)$ does not contain any vertex of W (except P if P happens to be a vertex). Let σ be the smallest dimensional simplex of W containing P, and let U be the star of σ. We have to show that each component of ${\Lambda_\rho}^{-1}(B_\epsilon^N(P) \cap W)$ is mapped homeomorphically to $B_\epsilon^N(P) \cap W$. For this, we will show that ${\Lambda_\rho}^{-1}(U)$ is a disjoint union of copies of U, each of which is mapped identically to U by Λ_ρ.

There is a chain $\tilde{\phi} \subset \mathcal{F}$ and a chamber $\widetilde{C} < max(\tilde{\phi})$ such that $\sigma = \Delta(\tilde{\phi}, \widetilde{C})$. Defining $\tilde{F} := max(\tilde{\phi})$, we can write U as

$$U = \bigcup_{\substack{\phi \supseteq \tilde{\phi} \\ C \in R(\phi)}} \Delta(\phi, C),$$

where we write $R(\phi)$ for the set of all chambers $C < max(\phi)$ such that $C_{\tilde{F}} = \widetilde{C}$. Analogously,

$${\Lambda_\rho}^{-1}(U) = \bigcup_{\substack{\phi \supseteq \tilde{\phi} \\ v\,:\,\rho(v) \in R(\phi)}} \Delta(\phi, v).$$

The last notation we now introduce is $v(C, w)$ denoting the object of $\mathcal{G}_\rho$ where the lift of the morphism $(C \to \rho(w))$ ending in w starts. More precisely, for $w \in \mathrm{Ob}(\mathcal{G}_\rho)$ and $C \in \mathrm{Ob}(\mathcal{G})$ we denote by $v(C, w)$ the (unique) $v \in \mathrm{Ob}(\mathcal{G}_\rho)$ such that

$$w = end\big(\hat{\gamma}_v(C \to \rho(w))\big).$$

For a fixed $v' \in \rho^{-1}(\widetilde{C})$ define

$$W_{v'} := \bigcup_{\substack{\phi \supseteq \tilde{\phi} \\ C \in R(\phi)}} \Delta(\phi, v(C, v')).$$

Claim 1:

$$\bigcup_{\tilde{v} \in \rho^{-1}(\widetilde{C})} W_{\tilde{v}} = {\Lambda_\rho}^{-1}(U).$$

Proof: Let us first show inclusion from left to right. Take a simplex on the left hand side, say $\Delta(\phi, v(C, \tilde{v}))$. Then $\Lambda_\rho(\Delta(\phi, v(C, \tilde{v}))) = \Delta(\phi, \rho(v(C, \tilde{v}))) = \Delta(\phi, C)$, which is a simplex of U since by assumption $\phi \supseteq \tilde{\phi}$ and $C \in R(\phi)$.

For the other inclusion, take $\Delta(\phi, v)$ with $\phi \supseteq \tilde{\phi}$ and $\rho(v) \in R(\phi)$. Define

[1] This proof does not make use of the properties of the groupoid: the arguments only involve positive paths. This makes the proof a little bit longer, but shows that the arguments are valid also restricting to the "positive complex" (see 6.4.1).

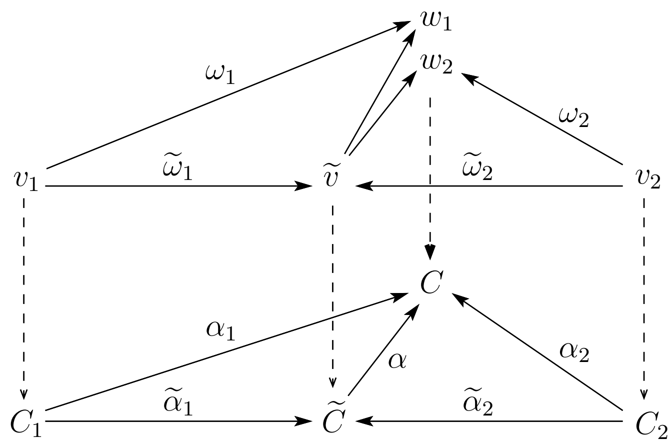


FIGURE 1. Sketch for the case $\tilde{F} > F$.

$\tilde{v} := end(\hat{\gamma}_v(\rho(v) \to \widetilde{C}))$. By construction, $\rho(\tilde{v}) = \widetilde{C}$ and $\Delta(\phi, v)$ is a simplex of $W_{\tilde{v}}$.

$$\begin{tikzcd} v(C,\tilde{v}) = v \arrow[r] \arrow[d, "\rho"] & \tilde{v} = end(\hat{\gamma}_v(C \to \widetilde{C})) \arrow[d, "\rho"] \\ C \arrow[r] & \widetilde{C} \end{tikzcd}$$

Claim 2: *Fix $\tilde{v} \in \rho^{-1}(\widetilde{C})$, then $\Lambda_\rho : W_{\tilde{v}} \to U$ is a homeomorphism.*

Proof: We have already seen that each simplex of $W_{\tilde{v}}$ maps to a simplex of U. Surjectivity is clear since, given a simplex $\Delta(\phi, C)$ of U, $\sigma := \Delta(\phi, v(C, \tilde{v}))$ is a simplex of $W_{\tilde{v}}$ with $\Lambda_\rho(\sigma) = \Delta(\phi, C)$.

Now it will suffice to show injectivity on the vertex sets, since linear extension on each simplex will then give a PL-map. Take two different vertices of $W_{\tilde{v}}$, say $w_1 := (F_1, v(C_1, \tilde{v}))$ and $w_2 := (F_2, v(C_2, \tilde{v}))$, and suppose $\Lambda_\rho(w_1) = \Lambda_\rho(w_2)$. This means, first of all, that $F_1 = F_2 =: F$ and $(C_1)_F = (C_2)_F = C$. In order to simplify notation we will write $v_1 := v(C_1, \tilde{v})$, $v_2 := v(C_2, \tilde{v})$. Recalling that $\mathscr{V}_\rho = \mathcal{F} \times \mathrm{Ob}(\mathcal{G}_\rho)/\sim$, injectivity is then equivalent to

$$end(\hat{\gamma}_{v_1}(C_1 \to C)) = end(\hat{\gamma}_{v_2}(C_2 \to C)).$$

By assumption, the face F must be comparable to each element of $\tilde{\phi}$, but technical reasons suggest a different treatment depending on whether F is bigger or smaller than $\tilde{F} = max(\tilde{\phi})$. In any case, one important remark is that C_1 and C_2 are by construction in $R(\phi)$ for some ϕ (say, the ϕ giving the smallest simplex of $W_{\tilde{v}}$ containing w_1, resp. w_2). This implies then in particular $(C_1)_{\tilde{F}} = (C_2)_{\tilde{F}} = \widetilde{C}$. Before going into further details, it is useful to translate lemma n. 3 of [**102**] to our language:

LEMMA 4.3.3. *Let $\bar{C} \in \mathrm{Ob}(\mathcal{G})$, and consider $C \in \mathrm{Ob}(\mathcal{G})$ such that $C < F$ for an $F \in \mathcal{F}$. Then one positive minimal path from $\bar{C}$ to C is the composition of a positive minimal path from $\bar{C}$ to $\bar{C}_F$ with a positive minimal path from $\bar{C}_F$ to C.*

We conclude the proof by separating the two cases:

Case $\tilde{F} > F$: Since by construction $F > C$, this means $\tilde{F} > C$.

Define $\omega_i = \hat{\gamma}_{v_i}(C_i \to C)$, $\tilde{\omega}_i = \hat{\gamma}_{v_i}(C_i \to \widetilde{C})$, $w_i = end(\omega_i)$, $\tilde{\alpha}_i = \rho(\tilde{\omega}_i)$, for $i = 1, 2$.

Since $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are positive minimal paths and $\widetilde{C} = (C_1)_{\widetilde{F}} = (C_2)_{\widetilde{F}}$, with lemma 4.3.3 we obtain a positive minimal path $\alpha_i : C_i \to C$ by composing $\tilde{\alpha}_i$ with a positive minimal path $\alpha : \widetilde{C} \to C$. So for $i = 1, 2$:

$$\tilde{\alpha}_i \alpha \sim C_i \to C.$$

Now we know that $\omega_i = \hat{\gamma}_{v_i}(\alpha \tilde{\alpha}_i)$, and by construction (namely, since $w_1, w_2 \in W_{\tilde{v}}$) we have $end(\tilde{\omega}_1) = \tilde{v} = end(\tilde{\omega}_2)$. With this we can write for $i = 1, 2$:

$$end(\omega_i) = end(\hat{\gamma}_{v_i}(\tilde{\alpha}_i \alpha)) = end(\hat{\gamma}_{end(\tilde{\omega}_i)}(\alpha)) = end(\hat{\gamma}_{\tilde{v}}(\alpha)).$$

This clearly does not depend on i, and proves the claim in the case $\tilde{F} > F$.
Case $\boldsymbol{F \leq \tilde{F}}$: In this case both $\widetilde{C}$ and C are adjacent to F. Moreover, recall that, by construction, $C = (C_1)_F = (C_2)_F$.

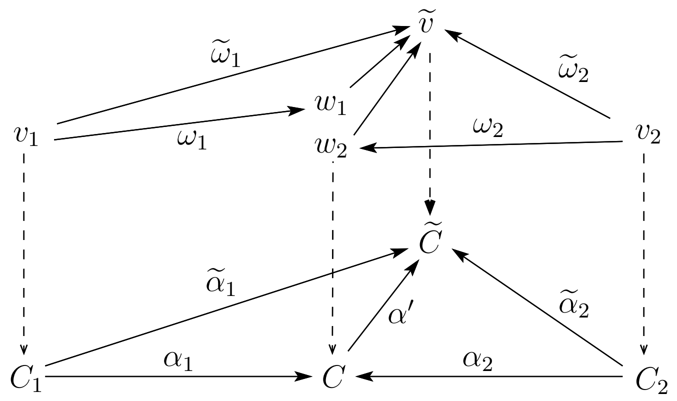


FIGURE 2. Sketch for the case $F \leq \tilde{F}$.

Let ω_i, w_i, $\tilde{\omega}_i$,$\tilde{\alpha}_i$ be defined as above and α' be a positive minimal path $C \to \widetilde{C}$. Let $\alpha_i := \rho(\omega_i) : C_i \to C$.

Then by 4.3.3 we know that $\alpha_i \alpha' : C_i \to \widetilde{C}$ is a positive minimal path, and therefore equivalent to $\tilde{\alpha}_i$. This implies $\hat{\gamma}_{v_i}(\alpha_i \alpha') \sim \hat{\gamma}_{v_i}(\tilde{\alpha}_i) = \tilde{\omega}_i$. Consider now $\omega_i^{-1}\tilde{\omega}_i$ for $i = 1, 2$.
It is easy to see that $\rho(\omega_i^{-1}\tilde{\omega}_i) = \alpha_i^{-1}\tilde{\alpha}_i \sim \alpha'$, and because of $end(\omega_i^{-1}\tilde{\omega}_i) = \tilde{v}$,

$$end(\omega_1) = begin(\omega_1^{-1}\tilde{\omega}_1) = v(C, \tilde{v}) = begin(\omega_2^{-1}\tilde{\omega}_2) = end(\omega_2).$$

This shows the result in the case $\tilde{F} \geq F$ and concludes the proof of Claim 2.

Claim 3: For $v_1 \neq v_2 \in \rho^{-1}(\widetilde{C})$ we have

$$W_{v_1} \cap W_{v_2} = \emptyset$$

Proof: Suppose that there is a simplex $\sigma := \Delta(F, v) \in W_{v_1} \cap W_{v_2}$. The fact that σ belongs to both W_{v_1} and W_{v_2} implies that we can write v in two ways:

$$v = v(C, v_1) = v(C, v_2),$$

where C denotes $\rho(v)$. This is the same as to say

$$v_1 = end(\hat{\gamma}_v(C \to \rho(v_1)))$$
$$v_2 = end(\hat{\gamma}_v(C \to \rho(v_2))),$$

which implies $v_1 = v_2$ since $\rho(v_1) = \rho(v_2) = \widetilde{C}$. Such a shared simplex σ can therefore not exist: this concludes the proof of Claim 3.

These considerations on the map Λ_ρ show that Proposition 4.3.2 holds. $\square$

4.4. The fundamental group $\pi_1(\mathbf{hocolim}\mathscr{D}_\rho)$

The goal of this section is to obtain more insight into the structure of the complexes W_ρ. In the end we will be able to state and prove Proposition 4.4.3, which relates the fundamental groups of a diagram model and of the corresponding groupoid.

Preliminary remark: At this point it is convenient to recall the construction of the arrangement groupoid $\mathcal{G}$ as a quotient of the free category on the arrangement graph G (chapter 2 of the prologue). We are going to describe an analogous description of any cover of the arrangement groupoid. In fact, one might consider the set M of all lifts of morphisms corresponding to edges of the arrangement graph, and consider the oriented graph G_ρ defined by $V(G_\rho) = \mathrm{Ob}(\mathcal{G}_\rho)$, $E(G_\rho) = M$. It is now easy to see that $\mathcal{G}_\rho$ is a quotient of the free category on G_ρ. Therefore we will allow ourselves to freely switch between the 'graph' viewpoint (vertices, edges, paths) and the categorical setting (objects, morphisms) as the situation will require it.

In the following the graphs G_ρ will also come out naturally as geometric objects (1-dimensional complexes). We will then write $\pi(G_\rho)$ for the fundamental group of such spaces.

The following general observation is at the core of this section: for any covering $\rho : \mathcal{G}_\rho \to \mathcal{G}$, the space $W_\rho = \Delta(\mathrm{Plim}\mathscr{D}_\rho)$ is the *barycentric subdivision of a CW-complex* having $\mathrm{Ob}(\mathcal{G}_\rho)$ as vertices, and a d-cell $[F, v]$ for each pair $F \in \mathcal{F}$, $F \geq \rho(v)$ with $codim(F) = d$.

In fact, if $codim(F) = d$,

$$[F, v] := \bigcup_{\phi\colon max(\phi)=F} \Delta(\phi, v)$$

is the barycentric subdivision of a closed d-ball.

The cell $[F, v]$ is attached to those vertices v' that can be written as

$$v' = end\big(\hat{\gamma}_v(\rho(v) \to \rho(v)_{F'})\big)$$

with $F' < F$.

4.4.1. 1-cells. In particular, between two vertices v_1, v_2 we have one 1-cell for each codimension 1 face F with $F > \rho(v_1)$, $F > \rho(v_2)$, and for each $v \in \mathrm{Ob}(\mathcal{G}_\rho)$ such that $F > \rho(v)$ and $v_i = end(\hat{\gamma}_v(\rho(v) \to \rho(v)_{\rho(v_i)}))$ for $i = 1, 2$.

$$\begin{tikzcd}
& (F, v) \arrow[dl, dash] \arrow[dr, dash] & \\
(\rho(v_1), v_1) & & (\rho(v_2), v_2)
\end{tikzcd}$$

As candidates for v we have only v_1 and v_2: first of all, since $codim(F) = 1$ and $\rho(v) < F$ we only can have have either $\rho(v) = \rho(v_1)$ or $\rho(v) = \rho(v_2)$. Now, if we suppose $\rho(v) = \rho(v_1)$, the condition imposes that

$$v_1 = end(\hat{\gamma}_v(\rho(v) \to \rho(v)_{\rho(v_1)} = \rho(v_1))),$$

which is v. Therefore $v_1 = v$.

For v_2 we must then similarly have $v_2 = end(\hat{\gamma}_{v_1}(\rho(v_1) \to \rho(v_2)))$, which is true if and only if there is a directed edge $v_1 \to v_2$ in G_ρ.

Taking $\rho(v) = \rho(v_2)$ one comes up with a directed edge $v_2 \to v_1$ and, as we can see, there is no third choice.

Summarizing, we can say that we have a 1-cell between $[\rho(v_1), v_1]$ and $[\rho(v_2), v_2]$ in W_ρ for each edge connecting v_1 and v_2 in G_ρ. This amounts to saying that the 1-skeleton of W_ρ as a CW-complex is G_ρ.

4.4.2. 2-cells. Recall that the fundamental group of a CW-complex is determined by the 2-skeleton. So we have to take a closer look at the 2-cells of W_ρ. In the following we will denote by ${W_\rho}^i$ the i-skeleton of W_ρ.

Fix $v \in \mathrm{Ob}(\mathcal{G}_\rho)$ and $F \in \mathcal{F}$ with $codim(F) = 2$ (w.l.o.g. $\rho(v) < F$). The vertices in the boundary $\partial[F, v]$ are those of the form

$$\big[C, end\big(\hat{\gamma}_v(\rho(v) \to C)\big)\big],$$

where C is any chamber adjacent to F. We label the points $end(\hat{\gamma}_v(\rho(v) \to C))$ as v_i, $i = 0, \dots, 2k-1$, assuming w.l.o.g. $v = v_0$, $C_0 := \rho(v_0)$. So the vertices in $\partial[F, v]$ are now the $[\rho(v_i), v_i]_{i=0,\dots,2k-1}$.

Now consider $v_i \neq v_j$, and suppose that in $\partial[F, v]$ an edge between them exists. This means that $\rho(v_i)$ is adjacent to $\rho(v_j)$, and that there is an F_1 with $F > F_1 > \rho(v_i)$ and $F > F_1 > \rho(v_j)$, such that this edge can be written as $[F_1, \tilde{v}]$. To determine whether $\tilde{v} = v_i$ or v_j (which gives the direction of the edge!) recall that we must have the following adjacencies in $\mathrm{Plim}\mathscr{D}_\rho$:

$$\begin{tikzcd}[arrows=dash]
 & (F, v_0) \arrow[d] & \\
 & (F_1, \tilde{v}) \arrow[dl] \arrow[dr] & \\
(\rho(v_i), v_i) & & (\rho(v_j), v_j)
\end{tikzcd}$$

In particular, $\rho(\tilde{v}) = \rho(v_0)_{F_1} = (C_0)_{F_1}$.

In $\partial[F, v]$ we have then one edge for each codimension 1 facet F_1 incident to F, and this edge is oriented away from the vertex that projects to the chamber on the same side as C_0 w.r.t. $|F_1|$. This is how one can naturally read the structure of G_ρ out of the diagram.

We may also view $\partial[F, v]$ as a subgraph of ${W_\rho}^1$ with an orientation; namely, a subdivision of S^1 with only one source (C_0) and one sink (the chamber opposite to C_0, say C_k).

Corresponding to it, we have a subcategory in $\mathcal{G}_\rho$ consisting of two positive paths from v_0 to v_k. These are mapped by ρ to two positive paths $C_0 \to \cdots C_i \cdots \to C_k$ with $C_i < F$ for all i. They are both minimal, hence equivalent.

Remark 4.4.1. *At this point it should be clear that, if $\mathcal{A}$ is central and P is the maximal element of $\mathcal{F}$, then any $[P, \gamma]$ is the barycentric subdivision of the zonotope of the arrangement.*

4.4.3. The fundamental group. We recall the definition of the fundamental group of a groupoid (see end of section 2.2) and take a closer look at the one of $\mathcal{G}_\rho$, a cover of the arrangement groupoid $\mathcal{G}(\mathcal{A})$. A loop $\alpha \in \pi(G_\rho)$ is equivalent to the identity in $\pi\mathcal{G}_\rho$ if and only if there are two paths γ_1, γ_2 with $\gamma_1 = \gamma_2$ and $\alpha = \gamma_1\gamma_2^{-1}$. This is precisely the case if $\rho(\gamma_1) = \rho(\gamma_2)$ in $\mathcal{G}(\mathcal{A})$, since the equivalence relation $\sim$ of $\mathcal{G}_\rho$ on paths in G_ρ is induced through ρ from $\mathcal{G}(\mathcal{A})$ as the smallest relation such that $\alpha \sim \beta$ if $begin(\alpha) = begin(\beta)$ and $\rho(\alpha) \sim \rho(\beta)$. We can state this precisely by saying

$$\pi\mathcal{G}_\rho = \pi(G_\rho)\big/\,\Sigma_\rho,$$

where Σ_ρ is the subgroup generated by the loops of the form $\gamma_1\gamma_2^{-1}$ with $\rho(\gamma_1) = \rho(\gamma_2)$.

As a final remark, note that $\rho(\gamma_1) \sim \rho(\gamma_2)$ in $\mathcal{G}(\mathcal{A})$ if and only if the corresponding paths are homotopic in $Sal(\mathcal{A})$.

Consider $\pi_1(W_\rho)$, the fundamental group of hocolim$\mathscr{D}_\rho$. By standard tools of algebraic topology (see for ex. [**88**] ch. 7 Thm. 2.1) we can define

$$\begin{aligned}\sigma_\rho := \; & \textit{the smallest normal subgroup of } \pi(W_\rho) \textit{ containing} \\ & \{\partial[F,v] \,|\, codim(F) = 2, F > \rho(v)\},\end{aligned}$$

and say

$$\pi_1(W_\rho) = \pi({W_\rho}^1)\big/\,\sigma_\rho.$$

Now consider the natural inclusion $i : G_\rho \to W_\rho$. We are ready to prove

Lemma 4.4.2. *The induced map*

$$i_\star : \Sigma_\rho \to \sigma_\rho$$

is an isomorphism.

Proof. First note that Σ_ρ is a normal subgroup of $\pi(G_\rho)$: for each $\alpha \in \pi(G_\rho)$ and $\gamma_1\gamma_2^{-1} \in \Sigma_\rho$, we have $\alpha\gamma_1\gamma_2^{-1}\alpha^{-1} = \alpha\gamma_1\alpha^{-1}\alpha\gamma_2^{-1}\alpha^{-1} \in \Sigma_\rho$ since

$$\rho(\alpha\gamma_1\alpha^{-1}) = \rho(\alpha)\rho(\gamma_1)\rho(\alpha)^{-1} \sim \rho(\alpha)\rho(\gamma_2)\rho(\alpha)^{-1} = \rho(\alpha\gamma_2\alpha^{-1}).$$

Therefore, Σ_ρ is the smallest normal subgroup containing $\gamma_1\gamma_2^{-1}$ for each pair (γ_1, γ_2) with $\rho(\gamma_1) = \rho(\gamma_2)$.

We have already seen that each $i_\star^{-1}(\partial[F,v])$ is always of the form $\gamma_1\gamma_2^{-1}$ with $\rho(\gamma_1) \sim \rho(\gamma_2)$. Let $\omega_i := \rho(\gamma_i)$: by definition of $\omega_1 \sim \omega_2$, they are constructed by concatenation and conjugation from positive, minimal paths. Since Σ_ρ is a normal subgroup, it remains to show that *for each pair* (γ_1, γ_2) *with* $\omega_1 \sim \omega_2$ *and* ω_i *positive and minimal, there are* $\tilde{\omega}_1$, $\tilde{\omega}_2$, α, F, v *with* $\rho(v) = end(\alpha)$ *and such that*

$$\omega_i \sim \alpha\tilde{\omega}_i \textit{ for } i = 1, 2$$

and

$$i\big(\hat{\gamma}_v(\tilde{\omega}_1)\hat{\gamma}_v(\tilde{\omega}_2)^{-1}\big) = \partial[F, v].$$

First notice that given $\omega_1 \sim \omega_2$ positive and minimal in G and $F > end(\omega_1) = end(\omega_2)$ with $codim(F) = 2$, by 4.3.3 there are a chamber $C > F$ and positive minimal paths $\tilde{\omega}_1$, $\tilde{\omega}_2$, α such that $\omega_i \sim \alpha\tilde{\omega}_i$ for $i = 1, 2$ and $end(\alpha) = C$. Moreover, each vertex of $\tilde{\omega}_i$ is adjacent to F, but no vertex of α other than C is adjacent to F.

So given a pair $\gamma_1 \sim \gamma_2$ in G_ρ with ω_1, ω_2 positive and minimal as above, let $v' := begin(\gamma_1) = begin(\gamma_2)$. Applying the previous considerations we can write $\rho(\gamma_1\gamma_2^{-1}) = \omega_1\omega_2^{-1} = \alpha\tilde{\omega}_1\alpha^{-1}\tilde{\omega}_2^{-1} \sim \tilde{\omega}_1\tilde{\omega}_2^{-1}$.
Consider then $\tilde{\gamma}_i := \hat{\gamma}_v(\tilde{\omega}_1)$, where $v := end(\hat{\gamma}_{v'}(\alpha))$. We have shown that $\gamma_1\gamma_2^{-1} \sim \tilde{\gamma}_1\tilde{\gamma}_2^{-1}$:

it is now straightforward to check $i_\star(\tilde\gamma_1\tilde\gamma_2^{-1}) = \partial[F, v]$. This concludes the proof. □

The following proposition is now immediate:

PROPOSITION 4.4.3. *For any cover of groupoids $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$, the fundamental group of the homotopy colimit of the diagram model $\mathscr{D}_\rho$ is isomorphic to the fundamental group of the groupoid $\mathcal{G}_\rho$:*

$$\pi_1(\mathrm{hocolim}\mathscr{D}_\rho) \cong \pi\mathcal{G}_\rho.$$

4.5. Results and examples

4.5.1. The main result. Combining the results of our studies we are now able to formulate and prove the main theorem.

THEOREM 4.5.1. *For any topological cover $r : S' \to Sal(\mathcal{A})$ of the Salvetti complex of an arrangement $\mathcal{A}$, there exists a cover of the arrangement groupoid $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ such that the homotopy colimit of the associated diagram of spaces $\mathscr{D}_\rho$ is homotopy equivalent to S'. Moreover, the poset limit of $\mathscr{D}_\rho$ is isomorphic to S' as a covering space of $Sal(\mathcal{A})$.*

Proof. By 4.4.3 we have an isomorphism $\varphi : \pi\mathcal{G}(\mathcal{A}) \to \pi_1(\mathrm{hocolim}\mathscr{D}_{id})$. Since $\mathrm{hocolim}\mathscr{D}_{id} \simeq Sal(\mathcal{A})$ by 4.1.2, we can consider the preimage $U := \varphi^{-1}(r_\star(\pi_1(S')))$ of the fundamental group of S' in $\pi\mathcal{G}(\mathcal{A})$.

Theorem 2.2.4 gives a cover $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ with $\rho_\star(\pi\mathcal{G}_\rho) = U$. Again by 4.4.3 we have an isomorphism $\varphi_\rho : \pi\mathcal{G}_\rho \to \pi_1(\mathrm{hocolim}\mathscr{D}_\rho)$. These isomorphisms come naturally from the inclusion ι of the graphs G_ρ as 1-skeletons of the CW-version of the homotopy colimits. Therefore the following diagrams commute

$$\begin{tikzcd} \mathcal{G}_\rho \arrow[r, "\iota"] \arrow[d, "\rho"] & W_\rho \arrow[d, "\Lambda_\rho"] \\ \mathcal{G} \arrow[r, "\iota"] & W \end{tikzcd} \qquad \begin{tikzcd} \pi\mathcal{G}_\rho \arrow[r, "\varphi_\rho"] \arrow[d, "\rho_\star"] & \pi_1(W_\rho) \arrow[d, "(\Lambda_\rho)_\star"] \\ \pi\mathcal{G} \arrow[r, "\varphi"] & \pi_1(W) \end{tikzcd}$$

and $r_\star(\pi_1(S')) = \varphi\rho_\star(\pi\mathcal{G}_\rho) = (\Lambda_\rho)_\star\varphi_\rho(\pi_1(W_\rho)) \cong (\Lambda_\rho)_\star(\pi_1(W_\rho))$. Therefore the cover $\Lambda_\rho : W_\rho \to Sal(\mathcal{A})$ is isomorphic to $r : S' \to Sal(\mathcal{A})$. □

As already remarked, application of the simplicial model lemma 3.2.7 yields naturally

SCHOLIUM 4.5.2. *Any cover $\rho : X \to Sal(\mathcal{A})$ of the Salvetti complex can be written as the order complex of a poset $\mathcal{S}_\rho$ (defined as the poset limit of the diagram $\mathscr{D}_\rho$). The poset $\mathcal{S}_{id}$ is naturally isomorphic to the poset of cells of the Salvetti complex.*

An immediate consequence of the theorem is

COROLLARY 4.5.3. *Let $\hat\rho : \hat{\mathcal{G}} \to \mathcal{G}(\mathcal{A})$ denote the universal cover of $\mathcal{G}(\mathcal{A})$. Then $W_{\hat\rho} = \Delta(\mathrm{Plim}\mathcal{D}_{\hat\rho})$ is the universal cover of $Sal(\mathcal{A})$.*

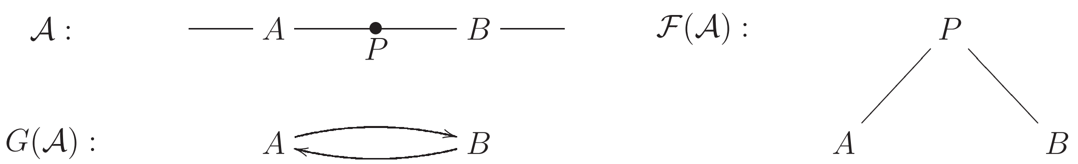
Proof. We prove universality. Take any cover $r : S' \to Sal(\mathcal{A})$; we have to show that there is a morphism of covers $m : W_{\hat\rho} \to S'$. By the theorem, we know that there is a cover $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ with $W_\rho \cong S'$ as a cover. Universality of $\hat{\mathcal{G}}$ implies the existence of a morphism $\mu : \hat{\mathcal{G}} \to \mathcal{G}_\rho$, and this induces a morphism of diagrams $\lambda_\mu : \mathscr{D}_{\hat\rho} \to \mathscr{D}_\rho$. By functoriality, we have $\Lambda_\mu : W_{\hat\rho} \to W_\rho$, which gives the required morphism, as in the following diagram.

$$
\begin{tikzcd}
\hat{\mathcal{G}} \arrow[rr, dashed] \arrow[dr, "\mu"] \arrow[dd, "\hat{\rho}"'] & & \mathscr{D}_{\hat{\rho}} \arrow[rr, dashed] \arrow[dr, "\lambda_\mu"] \arrow[dd, "\lambda_{\hat{\rho}}"'] & & W_{\hat{\rho}} \arrow[dr, "\Lambda_\mu"] \arrow[rr, bend left, "m"] \arrow[dd, "\Lambda_{\hat{\rho}}"'] & & \\
& \mathcal{G}_\rho \arrow[rr, dashed] \arrow[dl, "\rho"] & & \mathscr{D}_\rho \arrow[rr, dashed] \arrow[dl, "\lambda_\rho"] & & W_\rho \arrow[r, "\simeq"] \arrow[dl, "\Lambda_\rho"] & S' \arrow[dll, "r"] \\
\mathcal{G}(\mathcal{A}) \arrow[rr, dashed] & & \mathscr{D}_{id} \arrow[rr, dashed] & & W \arrow[r, equal] & Sal(\mathcal{A}) \simeq \mathcal{M}(\mathcal{A}) &
\end{tikzcd}
$$

□

As an illustration, we want to work out an easy but instructive example.

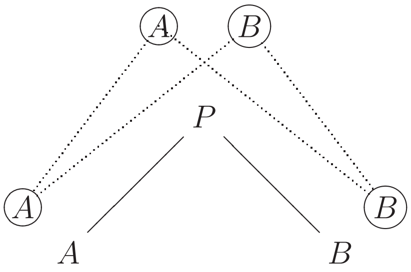
4.5.2. An example. Consider the arrangement given by one point $P \in \mathbb{R}$. The space $\mathbb{R}$ is divided by P in two chambers A and B. It is easy to write down the face poset $\mathcal{F}(\mathcal{A})$ and the arrangement graph $G(\mathcal{A})$ as follows:



The complexification of $\mathcal{A}$ is the arrangement given by a point in the complex plane. The complement $\mathcal{M}(\mathcal{A})$ is then homotopy equivalent to S^1, hence its universal cover is $\mathbb{R}$. We will now see how the diagram models come to this conclusions.

First consider $\mathscr{D}_{id}$, a diagram on the poset $\mathcal{F}$. The 'space' associated to A and B is one point: $\mathscr{D}_{id}(A) = \{A\}$, $\mathscr{D}_{id}(B) = \{B\}$. To the element P we attach a space consisting of two points, one for each chamber adjacent to P. To keep track of which point is associated to which chamber, we will write $\mathscr{D}_{id}(P) = \{A, B\}$. The maps between spaces are in this case trivial, but let us explain where they come from:

$$
\begin{aligned}
\mathscr{D}_{id}(P > A): \quad & A \mapsto A_A = A \\
& B \mapsto B_A = A \\
\mathscr{D}_{id}(P > B): \quad & A \mapsto A_B = B \\
& B \mapsto B_B = B
\end{aligned}
$$



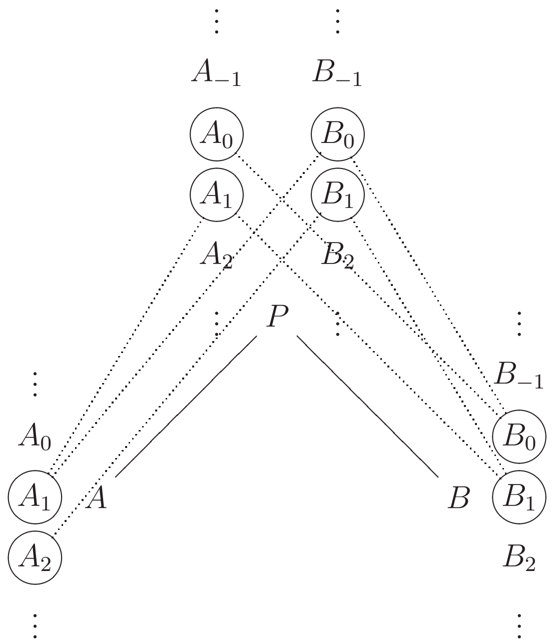
We now have to look at the universal cover of $\mathcal{G}(\mathcal{A})$. Since $Sal(\mathcal{A})$ has no 2-cells, the identification on $\texttt{Free}(G(\mathcal{A}))$ is empty. Let us then write $\hat{\mathcal{G}}$ as follows:

$$
\hat{G}: \qquad \dashrightarrow A_{-1} \longrightarrow B_{-1} \longrightarrow A_0 \longrightarrow B_0 \longrightarrow A_1 \dashrightarrow
$$

where $\hat{\rho} : \hat{\mathcal{G}} \to \mathcal{G}(\mathcal{A})$ is clearly defined by $\hat{\rho}(A_i) = A$, $\hat{\rho}(B_i) = B$ for all i.
Writing down the diagram $\mathscr{D}_{\hat{\rho}}$ we have to keep in mind that the space associated to an element $F \in \mathcal{F}$ has as many disconnected points as there are objects in $\hat{\mathcal{G}}$ that project to a chamber adjacent to F. So we have $\mathscr{D}_{\hat{\rho}}(A) = \{A_i | \, i \in \mathbb{Z}\}$, $\mathscr{D}_{\hat{\rho}}(B) = \{B_i | \, i \in \mathbb{Z}\}$, $\mathscr{D}_{\hat{\rho}}(P) = \mathscr{D}_{\hat{\rho}}(A) \cup \mathscr{D}_{\hat{\rho}}(B)$. For the maps one has to take care of how paths are lifted. Let us work out some special case and write down the diagram in the same fashion as above:

$\mathscr{D}_{\hat{\rho}}(P > B)$:
$$\begin{aligned} A_i &\mapsto end(\hat{\gamma}_{A_i}(A \to B)) \\ &\qquad = B_i \\ B_i &\mapsto end(\hat{\gamma}_{B_i}(B \to B)) \\ &\qquad = B_i \end{aligned}$$

$\mathscr{D}_{\hat{\rho}}(P > A)$:
$$\begin{aligned} A_i &\mapsto end(\hat{\gamma}_{A_i}(A \to A)) \\ &\qquad = A_i \\ B_i &\mapsto end(\hat{\gamma}_{B_i}(B \to A)) \\ &\qquad = A_{i+1} \end{aligned}$$



By lemma 3.2.7, we now only have to write down the poset $\mathrm{Plim}\mathscr{D}_{\hat{\rho}}$. The order relation is such that for $F_i \in \mathcal{F}$ and $v_i \in \mathrm{Ob}(\hat{\mathcal{G}})$ we have $(F_1, v_1) \geq (F_2, v_2)$ if and only if $F_1 \geq F_2$ and $v_2 = end(\hat{\gamma}_{v_1}(\rho(v_1) \to \rho(v_1)_{F_2}))$.

In our case, this means that the dotted lines in the above picture are yet a piece of the Hasse diagram of $\mathrm{Plim}\mathscr{D}_{\hat{\rho}}$, which we can redraw in a more readable way as

$$\begin{tikzcd}[column sep=tiny]
 & (P,B_{-1}) \arrow[dl, dash, dashed] \arrow[dr, dash] & & (P,A_0) \arrow[dl, dash] \arrow[dr, dash] & & (P,B_0) \arrow[dl, dash] \arrow[dr, dash] & & (P,A_1) \arrow[dl, dash] \arrow[dr, dash] & & \\
 & & (A,A_0) & & (B,B_0) & & (A,A_1) & & (B,B_1) \arrow[ur, dash, dashed] &
\end{tikzcd}$$

It is now clear that $\mathrm{hocolim}\mathscr{D}_\rho \cong \Delta(\mathrm{Plim}\mathscr{D}_{\hat{\rho}}) \simeq \mathbb{R}$, as required.

4.6. Comparison with Paris' models

We want to draw the connection to the models that Luis Paris constructed for the covers of (central) complexified arrangements [**96**]. Indeed, the models coincide for the case of the universal cover of a central arrangement, but our method describes any cover of linear and affine arrangements. In this section we give an open cover of Paris' spaces which nerve is the poset limit of the corresponding diagram.

4.6.1. Oriented systems and their covers. Paris introduced the notion of *oriented system*, that we briefly recall. It is based on an oriented graph, i.e., a set of vertices V

and a set of edges E consisting of some ordered pairs $e = (v_1, v_2)$ of elements $v_i \in V$. As usual, we write a path α in an oriented graph as a sequence of edges $e_1 e_2 \ldots e_n$, each with an exponent $\epsilon_i = \pm 1$, meaning that there is a sequence of vertices $(x_i)_{i=0,\ldots,n}$ with $e_i = (x_{i-1}, x_i)$ if $\epsilon_i = 1$, and $e_i = (x_i, x_{i-1})$ if $\epsilon_i = -1$. The inverse of a path α is then defined as $\alpha^{-1} = e_n^{-\epsilon_n} e_{n-1}^{-\epsilon_{n-1}} \ldots e_1^{-\epsilon_1}$. We also define $begin(\alpha) := x_0$, $end(\alpha) := x_n$.

Definition 4.6.1. *An Oriented System is a pair* $(\Gamma, \sim)$ *where* Γ *is an oriented graph and* $\sim$ *is an identification between paths of* Γ *such that*

(1) $\alpha \sim \beta$ *implies* $begin(\alpha) = begin(\beta)$ *and* $end(\alpha) = end(\beta)$.
(2) $\alpha\alpha^{-1} \sim begin(\alpha)$ *for every* α.
(3) $\alpha \sim \beta$ *implies* $\alpha^{-1} \sim \beta^{-1}$.
(4) $\alpha \sim \beta$ *implies* $\gamma_1 \alpha \gamma_2 \sim \gamma_1 \beta \gamma_2$ *for any* γ_1, γ_2 *with* $end(\gamma_1) = begin(\alpha)$ *and* $begin(\gamma_2) = end(\alpha)$.

Given a real arrangement of hyperplanes $\mathcal{A}$, the natural way to associate to it an oriented system $(\Gamma(\mathcal{A}), \sim)$ is of course to take $\Gamma(\mathcal{A}) = G(\mathcal{A})$, the arrangement graph, and identify two paths if they are both positive (i.e., $\epsilon_i = 1$ for all i), they start at the same point v and end at the same point w, and they both have minimal length among all positive paths connecting v to w (compare section 2.1).

Forgetting orientation of edges, one can view the oriented graph Γ as a 1-complex, and therefore consider its fundamental group $\pi_1(\Gamma)$ (here we always deal with connected graphs, so we do not need to specify a base point). The conditions that were required in the definition of the equivalence relation on paths ensure that $\sim$ induces an equivalence relation on $\pi_1(\Gamma)$; therefore we can consider the quotient

$$\pi(\Gamma, \sim) := \pi_1(\Gamma)/\sim,$$

which is called by Paris the *fundamental group* of the oriented system $(\Gamma, \sim)$.

For oriented systems, Paris [**96**] introduced the following concept of a cover:

Definition 4.6.2. *Given two oriented systems* $(\Theta, \sim_\Theta)$ *and* $(\Psi, \sim_\Psi)$, *a morphism of oriented graphs* $\rho : \Theta \to \Psi$ *is said to be a* ***cover*** *of* $(\Psi, \sim_\Psi)$ *if*

(1) for every vertex v *of* Θ *and every path* α *in* Ψ *with* $begin(\alpha) = \rho(v)$ *there is a unique path* $\hat{\alpha}_v$ *in* Θ *with* $\rho(\hat{\alpha}_v) = \alpha$ *and* $begin(\hat{\alpha}_v) = v$. *This path is called the lift of* α *at* v.
(2) for any two paths α, β *in* Ψ *with* $begin(\alpha) = begin(\beta) = \rho(v)$, *if* $\alpha \sim \beta$ *then* $\hat{\alpha}_v \sim \hat{\beta}_v$.

At this point, the similarity with the theory of groupoids as sketched in the prologue is clear, and we summarize it.

Scholium 4.6.3. Let $\mathcal{G}(\mathcal{A})$ denote the arrangement groupoid, i.e, the completion of a quotient of the free groupoid on the oriented graph $\Gamma(A)$ (compare chapter 2). We have immediately $\mathrm{Ob}(\mathcal{G}(\mathcal{A})) = V(\Gamma(\mathcal{A}))$. Any path γ on Γ identifies an equivalence class of morphisms $[\gamma]$ of $\mathcal{G}$; moreover,

$$\gamma_1 \sim \gamma_2 \text{ in } (\Gamma, \sim) \text{ if and only if } [\gamma_1] = [\gamma_2] \in \mathrm{Mor}_{\mathcal{G}}(begin(\gamma_1), end(\gamma_2)).$$

In particular, observe that $\pi(\Gamma(\mathcal{A}), \sim) \cong \pi\mathcal{G}(\mathcal{A})$.

Moreover, the requirement of definition 2.2.1 of covering of a groupoid translates naturally into condition (1) of the above definition 4.6.2, where condition (2) takes account of the quotient to which we pass in defining the arrangement groupoid. Therefore, to any choice of a subgroup of $\pi(\Gamma(\mathcal{A}),\sim) \simeq \pi\mathcal{G}(\mathcal{A})$ we obtain a covering $(\Theta,\sim)$ (by theorem 4.6.4) and a covering $\mathcal{G}_\rho$ (by theorem 2.3.1) that may be compared as follows:

$$V(\Theta) = \mathrm{Ob}(\mathcal{G}_\rho)$$
$$(v,w) \in V(\Theta) \Leftrightarrow \text{there is } \alpha \in \mathrm{Mor}_{\mathcal{G}_\rho}(v,w) \text{ with } \lambda(\alpha) = 1.$$

In view of the similarity with the classical theory of groupoids, the following theorem is natural.

THEOREM 4.6.4. *Consider an oriented system $(\Psi,\sim)$. For each subgroup G of $\pi(\Psi,\sim)$ there exists a cover $\rho : (\Theta,\sim) \to (\Psi,\sim)$ with $\rho_\star(\pi(\Theta,\sim)) = G$.*

Proof. See [**96**] □

4.6.2. Topological covers associated to oriented systems. We fix from now on a cover of oriented systems $\rho : (\Theta,\sim) \to (\Gamma,\sim)$, the latter being the oriented system naturally associated to the arrangement graph, and describe the construction developed by Paris in [**96**] that associates a topological space to $(\Theta,\sim)$.

The key objects out of which the space is built are pieces of the form

$$M(C) := \bigcup_{F\in\mathcal{F}} (F + iC_F) \subset \mathbb{R}^d \oplus i\mathbb{R}^d = \mathbb{C}^d.$$

REMARK 4.6.5. *Here the faces F have to be considered as convex, relatively open subsets of $\mathbb{R}^d$, and C_F denotes the unique chamber of $\mathcal{A}_{|F|}$ containing C. Thus, $C \subseteq C_F$ for any chamber C and any face F.*

To every vertex v of Θ we associate a copy $N(v)$ of an $M(C)$:

$$N(v) := M(\rho(v)),$$

and the topological space associated to $(\Theta,\sim)$ is defined by

$$N_\Theta := \coprod_{v\in V(\Theta)} N(v) \Big/ \approx \ .$$

The relation $\approx$ is defined pointwise by identifying two points $z \in N(v)$, $z' \in N(v')$ whenever (v,v') is an edge of Θ and they correspond to a unique point $z = z'$ in $M(\rho(v)) \cap M(\rho(v'))$ which real part lies on the same side as $\rho(w)$ with respect to the hyperplane separating $\rho(v)$ from $\rho(v')$.

Paris defines for any two vertices v, w of Θ a topological space $Z(v,w)$ as the interior of $\bigcup \overline{\rho(u)}$, where the overline denotes topological closure in $\mathbb{R}^d$ and the union is over all $u \in V(\Theta)$ that are reachable from both v and w by lifts of positive minimal paths of Γ. If (v,w) is an edge of Θ, then $Z(v,w)$ contains the real part of the set that is identified between $N(v)$ and $N(w)$. For general v and w, Paris shows the following fact:

LEMMA 4.6.6 (see lemma 3.5 of [**96**]). *If $N(v) \cap N(w) \neq \emptyset$ in N_Θ, then $Z(v,w)$ is not empty.*
Proof. *Ibidem.* □

4.6.3. Combinatorial stratifications and diagram models. We now describe a stratification of N_Θ by contractible subsets with contractible intersections, and show that its nerve is isomorphic to $\mathrm{Plim}\mathscr{D}_\rho$, thereby proving directly $\mathrm{hocolim}\mathscr{D}_\rho \simeq N_\Theta$.

Let us define, for any $F \in \mathcal{F}$ and any $C \in \mathcal{C}$, a set

$$M_C^F := \bigcup_{F' \leq F} F' + iC_{F'}.$$

Accordingly, for any $v \in V(\Theta)$ let $N_v^F := M_{\rho(v)}^F \subset N(v)$.

It is then clear that

$$M(C) = \bigcup_{F \in \mathcal{F}} M_C^F, \qquad N(v) = \bigcup_{F \in \mathcal{F}} N_v^F.$$

LEMMA 4.6.7. *For any $F \in \mathcal{F}(\mathcal{A})$ and any $C \in \mathcal{C}(\mathcal{A})$, the space M_C^F is contractible. This implies contractibility of N_v^F for all $v \in V(\Theta)$, $F \in \mathcal{F}$.*

Proof. Fix $F \in \mathcal{F}$, $C \in \mathcal{C}$ and $z \in F + iC_F$. We show that the segment connecting z to any $x \in M_C^F$ lies fully in M_C^F.

If $x \in F + iC_F$ the claim is trivial by convexity. Choose therefore $F' \leq F \in \mathcal{F}$ such that $x \in F' + iC_{F'}$, and consider the segment

$$\gamma(t) := tz + (1-t)x, \qquad 0 \leq t \leq 1.$$

It is clear that $\Re(\gamma(t)) \in F'$ for $0 \leq t < 1$, and by assumption $\Re(\gamma(1)) \in F$.

For the imaginary part, note that $\Im(x) \in C_F'$ and $\Im(z) \in C_F \subset C_F'$ (see remark 4.6.5), so $\Im(\gamma(t))$ is a straight line between two points of the convex set iC_F.

This concludes the proof. $\square$

LEMMA 4.6.8. *Given $v_1, v_2 \in V(\Theta)$ and $F_1, F_2 \in \mathcal{F}$, $N_{v_1}^{F_1} \subset N_{v_2}^{F_2}$ if and only if $(F_1, v_1) < (F_2, v_2)$ in $\mathrm{Plim}\mathscr{D}_\rho$.*

Proof. For the "only if"-part, suppose that $N_{v_1}^{F_1} \subset N_{v_2}^{F_2}$. In particular, this means $N(v_1) \cap N(v_2) \neq \emptyset$, which is only possible if $Z(v, w) \neq \emptyset$. This implies in particular:

$$\text{(i) } v_1 = v_2 \hat{\gamma}_{v_2}(\rho(v_2) \to \rho(v_1))$$

(recall that the v_i, correspond to homotopy classes of paths, and as such may be concatenated with other paths). On the other hand, $N_{v_1}^{F_1} \subset N_{v_2}^{F_2}$ implies $M_{\rho(v_1)}^{F_1} \subset M_{\rho(v_2)}^{F_2}$, which is clearly equivalent to:

$$\text{(ii) } F_1 < F_2 \text{ and } \rho(v_1) = \rho(v_2)_{F_1}.$$

The sentences (i) and (ii) above constitute the definition of the order relation $(F_1, v_1) < (F_2, v_2)$ in $\mathrm{Plim}\mathscr{D}_\rho$.

For the "if"-part, suppose $(F_1, v_1) < (F_2, v_2)$ in $\mathrm{Plim}\mathscr{D}_\rho$, that means

$$\text{(a) } F_1 < F_2, \quad \text{(b) } \rho(v_1) = \rho(v_2)_{F_1}, \quad \text{(c) } v_1 = v_2(\rho(v_2) \to \rho(v_1)).$$

By (c), $N(v_1)$ and $N(v_2)$ intersect. After [**96**, Lemma 3.5], the intersection can be written as

$$N(v_1) \cap N(v_2) = M(\rho(v_1)) \cap M(\rho(v_2)) \cap (Z(v_1, v_2) + iV). \tag{6.1}$$

Since the (a) and (b) above ensure that $M_{\rho(v_1)}^{F_1} \subseteq M_{\rho(v_2)}^{F_2}$, to prove the inclusion $N_{v_1}^{F_1} \subset N_{v_2}^{F_2}$ we only have to show that $Z(v_1, v_2)$ contains all faces $F < F_1$.

The latter assertion means that no chamber C such that $C < F$ (i.e., $F \subset \overline{C}$) is separated from $\rho(v_1)$ by any hyperplane that separates $\rho(v_1)$ from $\rho(v_2)$. But $C < F$ implies $C < F_1$, and so C can be separated from $\rho(v_1)$ only by hyperplanes $H \in \mathcal{A}_{|F_1|}$ and, by construction, the set of hyperplanes separating $\rho(v_1)$ from $\rho(v_2)$ is contained in $\mathcal{A}_{|F_2|} \setminus \mathcal{A}_{|F_1|}$. This concludes the proof. □

We end by proving the announced proposition.

PROPOSITION 4.6.9. *Let $\Theta \to \Gamma(\mathcal{A})$ be a cover of oriented systems, $\rho : \mathcal{G}_\rho \to \mathcal{G}$ the associated cover of the arrangement groupoid, and $\lambda_\rho : \mathscr{D}_\rho \to \mathscr{D}_{id}$ the corresponding diagram map. Then*

$$N_\Theta = \bigcup_{\substack{F \in \max(\mathcal{F}) \\ v \in \mathrm{Ob}(\mathcal{G}_\rho)}} N_v^F$$

is a covering by open, contractible subsets with (empty or) contractible intersections. Moreover, the nerve of this covering is the poset $\mathrm{Plim}\mathscr{D}_\rho$.

Proof. After the above preparations, we only have to show that $N_{v_1}^{F_1} \cap N_{v_2}^{F_2}$ is not empty if and only if it equals $N_{\tilde{v}}^{F_1 \wedge F_2}$, where $\tilde{v} := v_1 \hat{\gamma}_{v_1}(\rho(v_1) \to \rho(v_1)_{F_1 \wedge F_2})$ and the wedge is taken in $\mathcal{F}$.

For this, look at equality 6.1, from which it follows that if $z \in N(v_1) \cap N(v_2)$ and, say, $\Re(z) \in F$, then all pieces of $N(v_1)$ of the form $F' + iC'_F$ with $F' < F$ are also in the intersection. Therefore, if an intersection is nonempty, it contains some $N_{v'}^{F'}$. Now, by lemma 4.6.8, this is equivalent with $(F', v') < (F_i, v_i)$ for $i = 1, 2$.

Again by lemma 4.6.8, it follows that $N_{\tilde{v}}^{F_1 \wedge F_2} \subset N_{v_1}^{F_1} \cap N_{v_2}^{F_2}$ because $(F_1 \wedge F_2, \tilde{v}) = (F_1, v_1) \wedge (F_2, v_2)$. The reverse inclusion follows because with some $z' \in N_{v_1}^{F_1} \cap N_{v_2}^{F_2} \setminus N_{\tilde{v}}^{F_1 \wedge F_2}$ we would have a whole $N_{v'}^{F'}$ included in both sets, but for which $(F', v') \not< (F_1, v_1) \wedge (F_2, v_2)$, contradicting lemma 4.6.8. □

CHAPTER 5

Spectral sequences for homotopy colimits

This chapter is devoted to some homological considerations about homotopy colimits of diagrams of spaces, at least in the version *for combinatorial applications* studied in [**118**], i.e., diagrams over posets. We assume that the indexing poset of the diagram of spaces is graded and we study the spectral sequence of the filtration obtained considering only the summands indexed by poset elements of rank at most s in the explicit topological construction of homotopy colimits given in [**118**].

In theorem 5.4.1 we recover a statement given in [**118**] with a reference to Segal's work [**104**] and to a filtration by the skeleta of the order complex of the indexing poset $\mathcal{P}$. Our arguments are of purely topological nature and do not rely on Segal's work. We first suppose that $\mathcal{P}$ is the face poset of some regular CW-complex and derive in this case a more compact form of the spectral sequence (Theorem 5.2.4). Afterwards, we prove a statement of full generality by translating a diagram over any poset into a diagram over the face poset of the order complex of $\mathcal{P}$ (Theorem 5.2.6).

We then express the obtained results in the language of sheaves on posets (see section 5.3 and Theorem 5.3.1), which appears very naturally tailored to this scope (see e.g. Theorem 5.3.3). In section 5.5 we apply the cohomology spectral sequence to the diagrams constructed in chapter 4.

5.1. Filtration by rank

Recall the definition of homotopy colimit of a diagram of spaces $\mathscr{D}$ over a poset $\mathcal{P}$:

Definition 5.1.1. *Given a diagram of spaces $\mathscr{D} : \mathcal{P} \to Top$, the* ***homotopy colimit*** *of $\mathscr{D}$ is defined by*

$$\text{hocolim}\mathscr{D} := \coprod_{p\in\mathcal{P}} \Delta(\mathcal{P}_{\leq p}) \times \mathscr{D}(p) \Big/ \sim \qquad (1.1)$$

where the relation $\sim$ is given, for $p > q$, by the following identifications:

$$\Delta(\mathcal{P}_{\leq q}) \times \mathscr{D}(p) \hookrightarrow \Delta(\mathcal{P}_{\leq p}) \times \mathscr{D}(p)$$
$$\Delta(\mathcal{P}_{\leq q}) \times \mathscr{D}(p) \overset{(id\times f_{p>q})}{\longrightarrow} \Delta(\mathcal{P}_{\leq q}) \times \mathscr{D}(p)$$

Suppose that $\mathcal{P}$ is a *ranked* poset, i.e., that in any interval of $\mathcal{P}$ all maximal chains have the same length. This length is also called the *rank of* $\mathcal{P}$ and is denoted $rk(\mathcal{P})$. In this case the subposet $\mathcal{P}_{\leq p}$ is also ranked, and the rank of p is defined by $rk(p) := rk(\mathcal{P}_{\leq p})$.

For any s, $0 \leq s < rk(\mathcal{P})$, set

$$X_s := \coprod_{rk(p)\leq s} \Delta(\mathcal{P}_{\leq p}) \times \mathscr{D}(p) \Big/ \sim .$$

This defines a filtration

$$\emptyset = X_{-1} \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_{rk(\mathcal{P})} = \text{hocolim}\mathscr{D}.$$

We are going to consider the Leray spectral sequence of this filtration, and therefore some preliminary observations on the spaces X_s/X_{s-1} are in order.

Since the identifications $\sim$ are the same, we can write

$$X_s/X_{s-1} = \Big[\coprod_{rk(p)\leq s} \Delta(\mathcal{P}_{\leq p}) \times \mathscr{D}(p) \Big/ \coprod_{rk(p)\leq s-1} \Delta(\mathcal{P}_{\leq p}) \times \mathscr{D}(p) \Big] \Big/ \sim$$

and note that the identifications $\sim$ actually take place only on the space X_{s-1}. Then our quotient is made out of pieces of the form

$$Y_p := [susp(\Delta(\mathcal{P}_{<p})) \times \mathscr{D}(p)]/\approx$$

for $p \in \mathcal{P}$, where the identification $\approx$ is made by choosing a point $*$ among the two 'suspension points' $*_1, *_2$ and contracting the space $\{*\} \times \mathscr{D}(p)$ to a point. All those pieces are glued together at $*$ and build the quotient

$$Q_s := X_s/X_{s-1} = \bigvee_{rk(p)=s} Y_p.$$

5.2. Cohomology spectral sequence

We are going to follow some standard ideas of algebraic topology, in particular of the theory of spectral sequences. The nowadays standard, very comprehensive reference to spectral sequences is [**91**]. A more topologically-tailored and quite readable reference is [**75**]. Let us also mention the old book by Fomenko, Fuchs and Gutenmacher [**66**], that conveys a lot of topological intuition behind (or beyond) the technicalities.

The cohomological Leray theorem (see e.g. [**66**, ch.18, p.134]) says that there is a spectral sequence with

$$E_0^{s,t} := C^{s+t}(X_s, X_{s-1}) = C^{s+t}(Q_s)$$

and differentials

$$d_0^{s,t} : C^{s+t}(Q_s) \longrightarrow C^{s+(t+1)}(Q_s)$$

given by the usual chain differential, that converges toward the cohomology of the homotopy colimit.

The first sheet of this spectral sequence is given by:

$$E_1^{s,t} = H^{s+t}(Q_s) = \bigoplus_{rk(p)=s} H^{s+t}(Y_p),$$

and differentials

$$d_1^{s,t} : H^{s+t}(Q_s) \longrightarrow H^{s+1+t}(Q_{s+1})$$

given by the connecting homomorphism of the exact sequence of cochain complexes

$$0 \to C^*(X_{s+1}, X_s) \to C^*(X_{s+1}, X_{s-1}) \to C^*(X_s, X_{s-1}) \to 0, \tag{2.1}$$

coming from

$$X_s/X_{s-1} \hookrightarrow X_{s+1}/X_{s-1} \to X_s/X_{s-1}.$$

For this to look at least a bit more friendly, we try to compute the cohomology of the spaces Y_p. We let $*$ denote one of the two suspension points of the first component and write $Y_p = (susp(\Delta(\mathcal{P}_{<p})) \times \mathscr{D}(p))/(\{*\} \times \mathscr{D}(p))$.

Transitory assumption: we shall for a moment suppose that the poset $\mathcal{P}$ is *the face poset of some regular CW-complex K*. Such a poset is ranked by the dimension of the cells indexed by the elements. As we will see at the end, this will not prevent us from giving a general statement about homotopy colimits over any ranked poset.

The assumption that $\mathcal{P}$ is the incidence poset of the cells of some regular CW-complex allows us to conclude that $\Delta(\mathcal{P}_{<p})$ is homotopy equivalent to the sphere S^{s-1}, where s is the rank of p (we set formally $S^{-1} := \emptyset$). Thus, $susp(\Delta(\mathcal{P}_{<p}))$ is homotopy equivalent to the s-sphere S^s. Therefore we may write

$$Y_p \simeq \big(S^s \times \mathscr{D}(p)\big)/\big(\{*\} \times \mathscr{D}(p)\big).$$

We first need a general lemma:

LEMMA 5.2.1. *Let X be a topological space, n a natural number. Then*

$$H^j(S^n \times X) = \begin{cases} H^j(X) & \text{if } j < n, \\ H^j(X) \oplus H^{j-n}(X) & \text{if } j \geq n. \end{cases}$$

Proof. Apply the Künneth theorem, or simply recall that the chain complex $C^*(S^n \times X)$ is the tensor product $C^*(S^n) \otimes_{\mathbb{Z}} C^*(X)$. Since $C^j(S^n) = \mathbb{Z}$ if j equals 0 or n, and $C^j(S^n) = \{0\}$ else, we see that $C^j(S^n \times X) = C^j(X)$ for $j < n$, and $C^j(S^n \times X) = C^j(X) \oplus C^{j-n}(X)$ for $j \geq n$, with the natural differentials. □

With this, we can determine the cohomology of Y_p as an application of the following theorem.

THEOREM 5.2.2. *Let X be a topological space, $*$ a distinguished point of the n-sphere S^n. Then*

$$H^j((S^n \times X); (\{*\} \times X)) = \begin{cases} \{0\} & \text{if } j < n, \\ H^{j-n}(X) & \text{if } j \geq n. \end{cases}$$

Proof. We consider the long exact sequence of the CW-pair $(S^n \times X, \{*\} \times X)$ and keep in mind that trivially $H^*(\{*\} \times X) = H^*(X)$.

• *Case $j < n$:* We have

$$\ldots \to H^j(X) \xrightarrow{\alpha^j} H^j(S^n \times X) \xrightarrow{\beta^j} H^j((S^n \times X)/(\{*\} \times X)) \xrightarrow{\gamma^j} H^{j+1}(X) \to \ldots,$$

and since α is bijective by lemma 5.2.1, β and γ are zero morphisms. By exactness, this forces $H^j((S^n \times X)/(\{*\} \times X)) = \{0\}$.

• *Case $j \geq n$:* Here the corresponding portion of the long exact sequence is

$$\ldots \to H^j(X) \xrightarrow{\alpha^j} H^j(X) \oplus H^{j-n}(X) \xrightarrow{\beta^j} H^j((S^n \times X)/(\{*\} \times X)) \xrightarrow{\gamma^j} H^{j+1}(X) \to \ldots,$$

and α^n is no more surjective, but still injective. Therefore the kernel of β^j is $H^n(X)$, and γ^j is the zero map. Hence $H^j((S^n \times X)/(\{*\} \times X)) = H^{j-n}(X)$, as we had to show. □

COROLLARY 5.2.3. *let $\mathscr{D}$ be a diagram of spaces over a poset $\mathcal{P}$ that is isomorphic to the poset of cells of some regular CW-complex, and consider the filtration of its homotopy colimit given by the X_i as above. Then for any $0 \leq s \leq rk(\mathcal{P})$ and any $j \geq 0$ we have*

$$H^{j+s}(X_s/X_{s-1}) = \bigvee_{\substack{p \in \mathcal{P} \\ rk(\mathcal{P}) = s}} H^j(\mathscr{D}(p))$$

To gain some insight in the differentials of the spectral sequence, it is convenient to take a more careful look to the proofs of the two preceding statements. In particular, one should not forget that in the proof of lemma 5.2.1 we used the canonical isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}} A \xrightarrow{\simeq} A$ for abelian groups A, that actually maps $m \otimes a \mapsto 1 \otimes (m \cdot a)$.

As a next step, consider the cellularization of the whole space induced by the CW-cell structure of K and by the choice of any cell structure on the $\mathscr{D}(p)$'s. The cells are products $p \times \gamma$, where p ($\in \mathcal{P}$) is a cell of K and γ is a cell of $\mathscr{D}(p)$ (*important remark:* a quick look to definition 5.1.1 ensures that for any point (x, y) of hocolim$\mathscr{D}$, if x is in the interior of the cell p, then $y \in \mathscr{D}(p)$ - this is because the order complex is the baricentric subdivision of K, and the barycenter of a cell p is the top element of any chain indexing a simplex that intersects the interior of p).

Considering the CW-cell structure along the topological operations performed above, one sees that the sphere in the first component of any Y_p has the minimal cell structure: p and one vertex. Therefore $\partial_K p = 0$ in the chain complex of the sphere. Recall that, by the Eilenberg-Zilber theorem, products of cells are translated into tensor products of chains, and the boundary map is defined by $\partial(p \otimes \gamma) = \partial_K p \otimes \gamma + (-1)^{dim(p)} p \otimes \partial_{\mathscr{D}(p)}\gamma$ (see e.g. [**106**, section 5.3]). Therefore, if a chain $p \otimes \gamma$ is in the kernel of the coboundary of $C^*(X_s/X_{s-1})$, then $p \otimes \partial_{\mathscr{D}(p)}\gamma = 0$, implying $\partial_{\mathscr{D}(p)}\gamma = 0$.

Now we perform diagram chase to recover the connecting homomorphism of the short exact sequence 2.1. Take a cochain $p \otimes \gamma$ of X_s/X_{s-1} as above, and consider it in the cochain complex of X_{s+1}/X_{s-1}. The boundary map of this bigger complex sends $p \otimes \gamma$ to

$$\partial(p \otimes \gamma) = \partial_K p \otimes \gamma' + (-1)^{dim(p)} p \otimes \partial_{\mathscr{D}(p)}\gamma, \tag{2.2}$$

where we write γ' to signify that we have to take care of the identification of $\gamma \in \mathscr{D}(p)$ with $f_{pp'}(\gamma) \in \mathscr{D}(p')$ for every cell p' in the boundary of p (as prescribed in the definition of homotopy colimit). Therefore, passing to the boundary of p implicitly involves applying $f_{pp'}$ to the second component.

With the previous considerations, the second summand of the right side of equation 2.2 is zero. Therefore, we are left with $\partial_K p \otimes \gamma'$ (recall that the space X_{s+1}/X_{s-1} is bigger and $\partial_K p$ is no more necessarily 0).

With the CW-structure, for any $q \succ p \in \mathcal{P}$ there is a well-defined incidence index $[p : q]$ of the cell q in the cell p such that the boundary of the CW-cochain complex for K is

$$\partial_K : \bigoplus_{rk(p)=s} z_p \mapsto \bigoplus_{rk(q)=s+1} \sum_{q \succ p} [p : q] z_p.$$

With this, and taking care of the identifications in definition 5.1.1, we may write

$$\partial_K p = \sum_{q \succ p} [p : q] q \otimes f^*_{qp}(\gamma),$$

where f^*_{qp} denotes the morphism induced in cohomology by the map $f_{qp} = \mathscr{D}(q \succ p)$.

The right side is a cycle in $H^*(X_{s+1}/X_s)$, which we usually wrote as a direct sum over the elements q of rank $s + 1$. In each of the summands we then apply the isomorphism implicit in lemma 5.2.1: the above right side becomes then

$$\bigoplus_{rk(q)=s+1} [p : q] f^*(\gamma)$$

(recall that $[p : q] = 0$ if $p \not< q$).

Now everything is prepared for the next theorem, which follows from the above considerations.

THEOREM 5.2.4. *Let $\mathscr{D} : \mathcal{P} \to Top$ be a diagram of spaces and suppose $\mathcal{P}$ isomorphic to the poset of cells of some regular CW-complex K. Then there is a spectral sequence converging to the integral cohomology of the homotopy colimit of $\mathscr{D}$ with E_1-term given by*

$$E_1^{s,t} := \bigoplus_{\substack{p\in\mathcal{P} \\ rk(p)=s}} H^t(\mathscr{D}(p))$$

and the differentials

$$d_1^{s,t} := \bigoplus_{rk(p)=s} \gamma_p \longmapsto \bigoplus_{rk(q)=s+1} \sum_{p<q} [p:q] f_{qp}^*(\gamma_q),$$

where $[p : q]$ is the incidence index of p and q as cells of the CW complex K and f_{qp}^* is induced in cohomology by $\mathscr{D}(p > q) : \mathscr{D}(p) \to \mathscr{D}(q)$.

This is the result we get under the assumption that $\mathcal{P}$ is the poset of cells of a regular CW-complex. We chose to do this case separately first of all because it eliminates some notational heaviness that may have obscured the argument, and secondly because it gives a more compact formula when one has such a situation.

To deal with the general case, i.e., without assumptions on $\mathcal{P}$, we have to recall the following fact - that we state as an easy lemma.

LEMMA 5.2.5. *Let $\mathcal{P}$ be a poset. To any diagram of spaces $\mathscr{D}$ over $\mathcal{P}$, we associate a diagram $\mathscr{D}^\Delta$ over $\mathcal{F}(\Delta(\mathcal{P}))$, the face poset of the order complex of $\mathcal{P}$. This diagram is defined by $\mathscr{D}^\Delta(\omega) := \mathscr{D}(max\ \omega)$ for any chain ω of $\mathcal{P}$, with the obvious maps. We have*

$$\operatorname{hocolim}\mathscr{D} \simeq \operatorname{hocolim}\mathscr{D}^\Delta.$$

Proof. See lemma 3.2.5. □

Since $\Delta(\mathcal{P})$ is a simplicial complex, we can apply theorem 5.2.4 to the diagram $\mathscr{D}^\Delta$ and obtain the following proposition.

THEOREM 5.2.6. *Let $\mathcal{P}$ be any poset, and let $\mathscr{D}$ be a diagram of spaces over $\mathcal{P}$. There is a spectral sequence converging to the cohomology of* $\operatorname{hocolim}\mathscr{D}$ *with E_1-term given by*

$$E_1^{s,t} := \bigoplus_{\substack{p_0<p_1<\ldots<p_s \\ p_i\in\mathcal{P}}} H^t(\mathscr{D}(p_s))$$

and differentials $d_1^{s,t} : E_1^{s,t} \to E_1^{s+1,t}$ defined as

$$d_1^{s,t} : \bigoplus_{\sigma:=p_0<\ldots<p_s} w_\sigma \mapsto \bigoplus_{\tau:=q_0<\ldots<q_{s+1}} \left(\sum_{\substack{\sigma=\tau\setminus q_j \\ j\neq s+1}} (-1)^j w_\sigma \right) + (-1)^{s+1} f_{q_s q_{s+1}}^*(w_{q_0<\ldots<q_s}).$$

Proof. By theorem 5.2.4, $E_1^{s,t} \Rightarrow H^*(\operatorname{hocolim}\mathscr{D}^\Delta)$. The latter is isomorphic to $H^*(\operatorname{hocolim}\mathscr{D})$ by lemma 5.2.5. □

Acknowledgement: The idea to consider a filtration by the rank of the poset came from a discussion with P. Rade Živaljević.

5.3. Sheaves over posets

We would like to relate the above considerations to the context of sheaves over posets.

The idea of considering *sheaves over posets* is pretty natural if one considers Čech cohomology. The idea of the Čech resolution of a sheaf on a topological space with a given covering by contractible subsets is that the necessary homological data of the space are contained in the *nerve* of the covering, and it suffices to know the stalks of the sheaf corresponding to the intersections of the subsets. One may encode these data by considering the face poset of the nerve (which is a simplicial complex) and associating to each face the stalk of the sheaf at the subset corresponding to the face. In fact, this gives the first example of a sheaf over a poset.

The first appearance of this concept that is known to us is in the work of Deheuvels [**33**], who searched for an appropriate concept of 'Čech homology' and, in general, *"définir de façon satisfaisante l'homologie et la cohomologie à coefficients dans un préfaisceau ou un antifaisceau sur une catégorie abelienne"*.[1] The relation of this work with 'ordinary' sheaf homology is best seen by consulting the book of Godement [**75**] on algebraic topology and sheaf theory. To these references [**33, 75**] we point for a general and comprehensive introduction.

Combinatorics entered the picture with the paper of Bacławski [**3**], who began to exploit the combinatorics of partial orderings in the calculation of the homology of sheaves over posets. His goal was to develop an homology theory for which the Betti numbers of a poset correspond to its Whitney numbers, as conjectured by Rota. He calls this object the *Whitney homology* of the poset. Later on, it turned out that the Whitney homology of the intersection lattice of a complex arrangement of hyperplanes gives the cohomology of the complement of the arrangement (see theorem 3.3 of [**121**]).

This should provide enough motivation to the following considerations.

Note: For easier comparison with the existing literature easier, we keep the notations and conventions introduced by Deheuvels [**33**], even though this produces the effect that we sometimes have to consider the opposite of the posets we have in mind.

5.3.1. Definitions. Let X be any set and let a partial order relation $<$ be defined on X. Recall that, following [**118**], we do not distinguish between the following three objects, all denoted by $\mathcal{P}$: (i) A category $\mathcal{P}$ defined by $Ob(\mathcal{P}) := X$ and

$$Mor_{\mathcal{P}}(p,q) = \begin{cases} \{*\} & \text{if } p \leq q, \\ \emptyset & \text{if } p < q, \end{cases}$$

(ii) the partially ordered set $\mathcal{P} := (X, <)$, (iii) the set of elements (objects) of $\mathcal{P}$.

A *sheaf of abelian groups over the poset* $\mathcal{P}$ is a contravariant functor $\mathscr{A} : \mathcal{P} \to Ab$ from $\mathcal{P}$ into the category of abelian groups.

The *cohomology of* $\mathcal{P}$ *with coefficients in* $\mathscr{A}$ is the homology $H^*(\mathcal{P}, \mathscr{A})$ of the cochain complex given by

$$C^j(\mathcal{P}, \mathscr{A}) := \bigoplus_{\substack{p_i \in \mathcal{P} \\ p_0 < \cdots < p_s}} \mathscr{A}(p_s),$$

[1] In fact, the key concept for the construction of the resolutions is the notion of the nerve of a category, which is nowadays established. Comparing the definition of homotopy colimits given in [**118**] one might guess the form of the spectral sequence for the case of a diagram of spaces over a general small category.

$$\partial^j : C^j \to C^{j+1},$$

$$\bigoplus_{\substack{p_i \in \mathcal{P} \\ p_0 < \cdots < p_j}} a_{p_0,\ldots,p_j} \mapsto \bigoplus_{\substack{q_i \in \mathcal{P} \\ q_0 < \cdots < q_{j+1}}} \sum_{i=0}^{j} (-1)^i a_{q_0 < \ldots < \hat{q}_i < \ldots < q_j} + (-1)^{j+1} \mathscr{A}_{q_j, q_{j+1}}(a_{q_0 < \ldots < q_j}).$$

5.3.2. Cohomology of homotopy colimits through cohomology of sheaves on posets. Given a diagram of spaces $\mathscr{D} : \mathcal{P} \to Top$, it is natural to consider, for any $t \geq 0$, the following sheaf on $\mathcal{P}$:

$\mathcal{H}^t\mathscr{D} : \mathcal{P} \to Ab,$
$\mathcal{H}^t\mathscr{D}(p) := H^t(\mathscr{D}(p)),$
$\mathcal{H}^t\mathscr{D}(p \leq q) := f^t_{qp} : H^t(\mathscr{D}(p)) \to H^t(\mathscr{D}(q)).$

In fact we have

THEOREM 5.3.1. *Let $\mathscr{D}$ be a diagram of spaces over a poset $\mathcal{P}$. Then there is a spectral sequence converging to $H^*(\mathrm{hocolim}\mathscr{D})$ given by*

$$E_2^{s,t} = H^s(\mathcal{P}, \mathcal{H}^t\mathscr{D}).$$

Proof. After the previous definitions we have immediately that the E_1-term of the spectral sequence of the homotopy colimit of $\mathscr{D}^\Delta$ is

$$E_1^{s,t} = \bigoplus_{\substack{p_0 < p_1 < \ldots < p_s \\ p_i \in \mathcal{P}}} H^t(\mathscr{D}(p_s)) = C^s(\mathcal{P}, \mathcal{H}^t\mathscr{D})$$

and the differentials are the boundary maps of the cochain complex, as expressed above in this section. Taking the step to the next sheet shows the claim. ☐

REMARK 5.3.2. We would like to point out that, once one has realized the connection with homotopy colimits, many of the results of [**33**] become a precise geometric meaning. As an example we translate in our language proposition 9.3 of [**33**] that -in the case of the cohomology sheaves of diagrams- is easily seen to be a consequence of the direct image lemma 3.2.2, in view of which the contractibility requirement acquires a geometrical meaning.

PROPOSITION 5.3.3 (Proposition 9.3 of [**33**]). *Let $\mu : \mathcal{P} \to Q$ be a poset morphism, $\mathscr{A}$ a sheaf over the poset Q. Let the stalks of the sheaf $\mu^{-1}\mathscr{A}$ on $\mathcal{P}$ be defined as $\mu^{-1}\mathscr{A}(p) := \mathscr{A}(\mu(p))$ for all $p \in \mathcal{P}$, and the morphisms be naturally induced from $\mathscr{A}$. Then, if the subposet $\mu^{-1}(Q_{\leq q})$ is contractible for any $q \in Q$,*

$$H^n(\mathcal{P}, \mu^{-1}\mathscr{A}) = H^n(Q, \mathscr{A})$$

for all $n \in \mathbb{N}$.

Only the definition of $\mu^{-1}\mathscr{A}$ needs some explanation: the *objet des sections de la restriction á $Q_{\geq \mu(p)}$* defined at page 217 of [**33**] can be identified with $\mathcal{A}(\mu(p))$ since the poset has a minimal element. ☐

5.4. Homology spectral sequence

The reasoning of the preceding two sections goes of course through also for homology. We recover thereby the spectral sequence used in [**118**, sect. 5.3].

THEOREM 5.4.1. *Let $\mathcal{P}$ be any poset, and let $\mathscr{D}$ be a diagram of spaces over $\mathcal{P}$. There is a spectral sequence converging to the homology of* hocolim$\mathscr{D}$ *with E_1-term given by*

$$E^1_{s,t} := \bigoplus_{\substack{p_0<p_1<\ldots<p_s\\ p_i\in\mathcal{P}}} H_t(\mathscr{D}(p_s))$$

and differentials $d^1_{s,t} : E^1_{s,t} \to E^1_{s+1,t}$ defined as

$$d^1_{s,t} : \bigoplus_{\substack{\sigma:=\{p_0,\ldots,p_s\}\\ p_0<\ldots<p_s}} w_\sigma \mapsto \bigoplus_{\substack{\tau:=\{q_1,\ldots,q_s\}\\ q_1<\ldots<q_s}} \left(\sum_{\substack{q_{j-1}<q'<q_j\\ j>0}} (-1)^j w_{\sigma\cup\{q'\}} \right) + \sum_{q'<q_1} f^*_{q_1q'}(w_{\sigma\cup\{q'\}}).$$

We may also give a reformulation in terms of sheaves on posets, analog to theorem 5.3.1. This time we have to consider the dual of the poset $\mathcal{P}$ (defined as the poset $\mathcal{P}^{op}$ with the same elements as $\mathcal{P}$ but such that $p<q$ in $\mathcal{P}$ if and only if $p>q$ in $\mathcal{P}^{op}$) in order to make the situation fit into the formalism of Deheuvels [**33**]. We obtain the following statement.

THEOREM 5.4.2. *Let $\mathscr{D}$ be a diagram of spaces over a poset $\mathcal{P}$. Then there is a spectral sequence converging to H_*(hocolim$\mathscr{D}$) given by*

$$E^2_{s,t} = H_s(\mathcal{P}^{op}, \mathcal{H}_t\mathscr{D}).$$

5.5. Application: cohomology of $\mathscr{D}_\rho$

Given a real arrangement of hyperplanes $\mathcal{A}$, its poset of faces $\mathcal{F}(\mathcal{A})$ is naturally the poset of faces of a regular CW-complex: namely, it is the face poset of the stratification of $\mathbb{R}^d$ obtained by coning over the cellularization of S^{d-1} induced by the covectors of the oriented matroid associated to the arrangement (compare chapter 1).

Thus, given a topological covering $\rho : X \to \mathcal{M}(\mathcal{A})$ of the arrangement complement, we can apply theorem 5.2.4 to the diagram $\mathscr{D}_\rho$ defined in chapter 4. For any F the space $\mathscr{D}_\rho(F)$ has as many contractible components as there are chambers adjacent to F in $\mathcal{A}$. The first sheet of the spectral sequence reduces then to the first row and is given by

$$E_1^{s,0}(\mathrm{hocolim}\mathscr{D}_\rho) = \bigoplus_{codim(F)=s} \bigoplus_{\substack{\gamma\in Ob(\mathcal{G}_\rho)\\ end(\gamma)\in\mathcal{C}(\mathcal{A}_F)}} \mathbb{Z}_C$$

with differentials

$$d_1^{s,0} := \bigoplus_{rk(F)=s} \bigoplus_{\substack{\gamma\in Ob(\mathcal{G}_\rho)\\ end(\gamma)\in\mathcal{C}(\mathcal{A}_F)}} m_p \longmapsto$$

$$\bigoplus_{rk(F')=s+1} \bigoplus_{\substack{\gamma'\in Ob(\mathcal{G}_\rho)\\ end(\gamma')=:C'\\ C'\in\mathcal{C}(\mathcal{A}'_F)}} \sum_{\substack{F<F'\\ \gamma'=\gamma(end(\gamma)\to C')}} \big[[F,\gamma] : [F',\gamma']\big] m_{[F,\gamma]}$$

(note the unfortunate clash of notations: outer square bracket denote incidence indexes, while inner square bracket denote cells of the CW-complex, see section 4.4).

It is a pretty easy fact that the incidence indices of the CW-complexes $K(\mathcal{F})$ and $K(\mathrm{Plim}\mathscr{D}_\rho)$ of which $\mathcal{F}$, respectively $\mathrm{Plim}\mathscr{D}_\rho$ are the face posets (see chapter 4, in particular section 4.4) are related by

$$\big[[F_1, \rho_1] : [F_2, \rho_2]\big]_{K(\mathrm{Plim}\mathscr{D}_\rho)} = \big[F_1 : F_2\big]_{K(\mathcal{F})},$$

where subfix specifies the CW-complex to which the incidence index refer.

Thus, we see that the CW-version of the spectral sequence for $\mathscr{D}_\rho$ gives nothing other that the cellular homology of the CW-version of the covering complexes of chapter 4.

For the same reasons (but without having to bother with incidence indices), it is also clear that the more general sequence of theorem 5.2.6 also reduces to the first row, and boils down to the computation of the simplicial homology of the simplicial complex $\Delta(\mathrm{Plim}\mathscr{D}_\rho)$. This fact can be explained also by viewing the corresponding sheaf on $\mathcal{F}$ as the "pushforward" of the constant sheaf of integers on $\mathrm{Plim}\mathscr{D}_\rho$ along the canonical projection $\mathrm{Plim}\mathscr{D}_\rho \to \mathcal{F}$, as an application of theorem 9.3 of [**33**].

Although this remark in some sense "closes the circle", it shows also that one does not make substantial progress by considering this spectral sequence applied to the previously defined diagrams. Thus, in the next chapters we will take a different viewpoint.

CHAPTER 6

Garside-type diagram models

This chapter is motivated by the quest for a better understanding of the role of the poset of regions of a complexified hyperplane arrangement for the topology of the arrangement complement. The theory of Garside structures, mainly its recent topological issues, appears to give an interesting framework to study this question, as we explain in section 6.1. Indeed, this leads to the formulation of what we would like to call *Garside-type models* for the covers of linear complexified arrangements (the complexes $U_\rho(\mathcal{A})$ of definition 6.2.9). The proof that these flag complexes indeed model the homotopy type of all covers of any linear complexified arrangement involves the tools of chapter 3. In fact, any U_ρ can be understood as the colimit of a certain diagram of spaces over the poset of convex sets $\mathcal{O}(\mathcal{A})$ (Proposition 6.2.12), and this colimit has the same homotopy type as the homotopy colimit by the projection lemma 3.2.5. Moreover, the diagram can be compared with those of chapter 4 to show homotopy equivalence of the homotopy colimits (Proposition 6.2.7). The link with Garside structures is given by the observation that, for the universal cover complex of a Coxeter arrangement, we recover the 'Garside complex' of the associated Artin group (Definition 6.1.2, Remark 6.2.10).

In section 6.3, we closely examine the complex we obtain for universal coverings and we state some questions that arise comparing some features of the theory of Garside groups and Randell's work [**100**].

An important point is that this new type of models allows us a precise combinatorial understanding of the topological significance of the lattice property of the poset of regions: along the lines of Deligne's proof, we show asphericity of simplicial arrangements starting from the assumption of the lattice property with respect to every chamber (Theorem 6.4.6). The approach via the lattice property also answers a question posed by Paris in [**97**]: in section 6.5 we show that the key property of paths on which Deligne's proof is based ("property D") indeed implies simpliciality of the arrangement (Theorem 6.5.3), thereby answering a question posed by Paris in [**97**, Section 4]. In section 6.6 we will see that the complexes arising from the application of the spectral sequence of chapter 5 can be expressed very naturally in the context of the Varchenko-Gel'fand ring of the arrangement (see e.g. Theorem 6.6.4).

Let us mention for completeness that recent work by David Bessis on so-called *dual Garside structures* brought some improvements on the $K(\pi, 1)$ problem for complex reflection arrangements [**4, 5**].

During the final revision of this thesis we remarked that Deligne himself acknowledged inspiration from the earlier work of Garside *...que je suis souvent de très près* (see [**36**, last line of the first page]).

Acknowledgment: I would like to thank Graham Denham for pointing out to me the work of Jon McCammond [**90**], which brought me in contact with Garside structures.

6.1. Garside structures

6.1.1. Generalities. Garside groups are generalizations of Artin groups (see [**35**]). The rough idea is to generalize the way an Artin group can be described as a group of words on the alphabet given by the corresponding Coxeter group, in order for example to study when a *Deligne normal form* (see [**29**, thm 2.7]) is defined. In particular, it turns out that (finite) $K(\pi, 1)$'s can be explicitly constructed for those groups.

A good first reading on Garside groups is the survey by McCammond [**90**], a concise but complete account of the associated $K(\pi, 1)$s is [**29**]. A comprehensive and articulated study of all aspects of the theory of Garside groups can be found in [**34, 35**].

Let us here give the definition of a Garside group, in the form that is most natural in our context.

Definition 6.1.1. *Let $\mathcal{P}$ be a bounded, graded lattice of finite height. Suppose that a labeling λ of the edges of the Hasse diagram of $\mathcal{P}$ is defined in some alphabet S (i.e., we assign a 'letter' $s \in S$ to every pair $p \gtrdot q$ of elements of $\mathcal{P}$ - the dot meaning that there is no $z \in \mathcal{P}$ such that $p > z > q$). Assume further that this labeling is* group-like *and* balanced *(see* [**90**]*).*

These data give rise to the ***Garside group*** $G(\mathcal{P}, \lambda)$*, obtained as the group of fractions of the monoid $M(\mathcal{P}, \lambda)$ which is generated by S with relations that identify any two words that can be read along maximal chains of $\mathcal{P}$ with same begin- and endpoint.*

Let us call *atom* any word that can be read (bottom-to-top) along some saturated chain in $\mathcal{P}$. We may then give the following definition of a simplicial (flag) complex that was defined by Brady for braid groups [**20**], by Bestvina for finite type Artin groups [**6**] and was extended by Charney, Meier and Whittlesey to the more general context of Garside groups [**29**].

Definition 6.1.2. *Given a Garside group $G = G(\mathcal{P}, \lambda)$, let $X(G)$ (or $X(\mathcal{P}, \lambda)$, if specification is needed) denote the simplicial complex on the vertex set G obtained by declaring a subset $\{g_0, \dots, g_d\} \subset G$ to be a simplex if for any $0 \leq i < j \leq d$ there is an atom a_{ij} with $g_j = g_i a_{ij}$.*

The following fact is proved as theorem 3.1 of [**29**], but see also theorem 3.6 of [**6**] and theorem 6.9 of [**20**]. Note the analogy with Deligne's work [**36**]!

Theorem 6.1.3 ([**29**]). *For any Garside group G, the complex $X(G)$ is contractible.*

Note that $X(\mathcal{P}, \lambda)$ is tiled by copies of the order complex $\Delta(\mathcal{P})$ (the simplicial complex of totally ordered subsets of $\mathcal{P}$). The connection with the universal covering of the Salvetti complex in the Coxeter case was already sketched by Bestvina [**6**, p. 281].

6.1.2. Coxeter systems and Artin groups. It is not difficult to see that, given a finite type Coxeter system (W, S), the common way one thinks of the corresponding Artin group actually is as Garside group $G(W, \lambda_S)$: here the poset is the *weak ordering* of W (defined by $w_1 < w_2$ if $w_1 s_1 \cdots s_k = w_2$ and $\ell(w_1) + k = \ell(w_2)$, where $s_i \in S$ and $\ell(w)$ is the minimal length of a word in S representing w, compare [**11**, def. 3.1.1]), and the label of $w_1 \lessdot w_2$ is the unique $s \in S$ such that $w_1 s = w_2$. This is called the *classical Garside structure* for Artin groups in [**90, 4**]. We point to [**11**] as a general reference on Coxeter groups and their combinatorics.

The weak ordering of a finite, irreducible Coxeter group has a striking geometric interpretation in terms of the real reflection arrangement associated to W.

To see this, let us define a poset that was introduced by Edelman in [**47**].

Fix a chamber C_0 of the real arrangement $\mathcal{A}$, and call it the *base chamber* of our construction. The ordering we will define depends on the choice of C_0.

DEFINITION 6.1.4. *Let $\mathcal{A}$ be a real arrangement of hyperplanes. For any two chambers $C_1, C_2 \in \mathcal{C}(\mathcal{A})$ let*

$$S(C_1, C_2) := \{H \in \mathcal{A} \mid H \text{ separates } C_1 \text{ from } C_2\}.$$

Fix a base chamber $C_0 \in \mathcal{C}(\mathcal{A})$ and define the ***partial order*** *$\mathcal{P}_{C_0}(\mathcal{A})$* ***with base chamber*** *C_0 on the set $\mathcal{C}(\mathcal{A})$ by setting*

$$C_1 \leq_{C_0} C_2 \text{ if and only if } S(C_0, C_1) \subseteq S(C_0, C_2).$$

When no confusion on the arrangement will be possible, we will write $\mathcal{P}_{C_0}$ for $\mathcal{P}_{C_0}(\mathcal{A})$.

Edelman shows that the poset of regions of a real reflection arrangement (by symmetry, the choice of base chamber does not matter) is isomorphic to the weak Bruhat ordering of the corresponding reflection group.

6.2. Diagram models over acyclic sets

The goal of this section is to exploit some of the techniques from the "toolkit" of Welker, Ziegler and Živaljević [**118**] in order to construct diagram models for covers of complexified arrangements. We call these models *of Garside type* because they give rise to a generalization of the Garside complex $X(P, \lambda)$ for Artin groups described in the previous section.

Recall from section 1.4 the notion of convex and acyclic sets in oriented matroids. Intuitively, an acyclic set can be thought of as a set of open halfspaces (bounded by the hyperplanes of the arrangement) with nonempty intersection. It is called convex if it has maximal size among those with the given intersection. We already saw how strongly the poset $\mathcal{O}$ of convex sets is related to the order of regions defined above. It is natural to take this object as a stepping stone for our construction.

DEFINITION 6.2.1. *Given an acyclic set O, we define its* ***support*** *$\underline{O}$ to be the set of elements of the underlying matroid that correspond to the elements of O (i.e., the set of hyperplanes bounding the hemispheres in O) and consider the set O^+ of all covectors of $\mathcal{M}$ that have positive sign in all entries corresponding to the elements of $\underline{O}$.*

With the notations of section 1.4, we see that O^+ contains all the faces of the cellularization of the sphere that are contained in the intersection of the hemispheres of O.

Of course, our interest is mainly devoted to the case of an oriented matroid that describes a real arrangement of hyperplanes. We will therefore from now switch to a terminology that is inspired to that situation, calling the elements of $\check{E}$ *halfspaces* and the topes (maximal covectors) *chambers*, hoping that the geometric intuition behind this will help the reader to understand the arguments. However, it is important to point out that this restriction is not essential: the arguments can be easily rephrased in terms of general oriented matroids, without realizability assumption. We did not (yet) tackle the study of topological invariants of general oriented matroids in relation to the combinatorial properties, although we consider this an intriguing question.

Let an acyclic set O be fixed. We define a relation R_O on $\mathcal{C}(\mathcal{A})$ by "forgetting" the hyperplanes in $\underline{O}$, and setting

$$R_O(C, C') \Leftrightarrow S(C, C') \subset \underline{O}. \tag{2.1}$$

To any acyclic set O and any chamber in O^+ we associate a poset $\mathcal{Q}_O(C)$ obtained by applying the ordering $<_C$ (see definition 6.1.4) to the chambers that are in relation with C by R_O.

We define the corresponding notion in terms of the arrangement groupoid. Recall that we identify paths on the arrangement graph with the corresponding morphisms in the arrangement groupoid, and abuse terminology speaking of morphisms as of paths: for instance, given a morphism γ we let *begin*(γ) and *end*(γ) denote the chambers (objects of the groupoid) where the associated path begins and ends. Recall also that a morphism is called *positive* if it can be represented by a path that traverses all its edges in the positive sense, and *minimal* if it does not traverse two edges that cross the same hyperplane: the equivalence class of such a path is uniquely determined by its beginning- and ending chambers C_b, C_e and often denoted $(C_b \to C_e)$. For more notation, terminology and background we refer to the prologue, especially to chapter 2.

DEFINITION 6.2.2. *Let γ denote a path in the arrangement groupoid $\mathcal{G}(\mathcal{A})$ and suppose $end(\gamma) \in O^+$. Then we define a natural generalization of the above posets by setting*

$$\mathcal{Q}_O(\gamma) := \big\{\gamma(C \to end(\gamma))^{-1} \mid R_O(C, end(\gamma)) \text{ holds}\big\}.$$

Given a cover $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ and an acyclic set O, let

$$\mathcal{Q}_O^\rho := \coprod_{\substack{\gamma \in Mor\mathcal{G}_\rho \\ \rho(end(\gamma)) \in O^+}} \mathcal{Q}_O(\gamma).$$

REMARK 6.2.3. *If the chosen acyclic orientation has maximal size (i.e., its support is the whole arrangement), $\mathcal{Q}_O(\gamma)$ is naturally isomorphic to the poset $\mathcal{P}_{end(\gamma)}(\mathcal{A})$ of chambers of the arrangement, with $end(\gamma)$ as base chamber.*

Now one would naturally set up a diagram of spaces over $\mathcal{O}$ that associates to every O the order complex of the corresponding $\mathcal{Q}_O^\rho$. This turns out to be too naive. Indeed, consider the following situation: let $h_1, \ldots, h_4$ denote the external elements of O, so that $O^+ = \bigcap_{i=1,\ldots 4} h_i$. Suppose further that $\mathcal{A} \setminus \underline{O} = \{H', H'', H'''\}$, and let h' denote one of the halfspheres bounded by H', such that the geometry depicted in figure 6.2.1 arises.

The problem now is that $O' := O \cup \{h'\}$ is a valid acyclic set, but it is not clear how to map the component of $\mathcal{Q}_O$ corresponding to the depicted chamber C into $\mathcal{Q}_{O'}$. Namely, C is separated from any chamber of $(O')^+$ by more than only elements of $\underline{O'}$. Hence there is no natural mapping of it, as R_O does not relate it to any chamber in O^+.

To get around this difficulty, we have to look for the acyclic sets that naturally correspond to elements of $\mathcal{F}$. It is then easy to see that there is only one such which is minimal above O in $\mathcal{O}$. Let us make this precise by recalling the definition of the natural map $f : \mathcal{F} \mapsto \mathcal{O}$.

REMARK 6.2.4. *The map $f : \mathcal{F} \to \mathcal{O}$ with $F \mapsto \{e | e(F) = +\}$ is a poset map, and the lower fibers $f^{-1}(\mathcal{O}_{\leq O}) = \bigcup_{F \subset O^+} \mathcal{F}_{\leq F}$ are nonempty, hence contractible by* [**47**, 3.2].

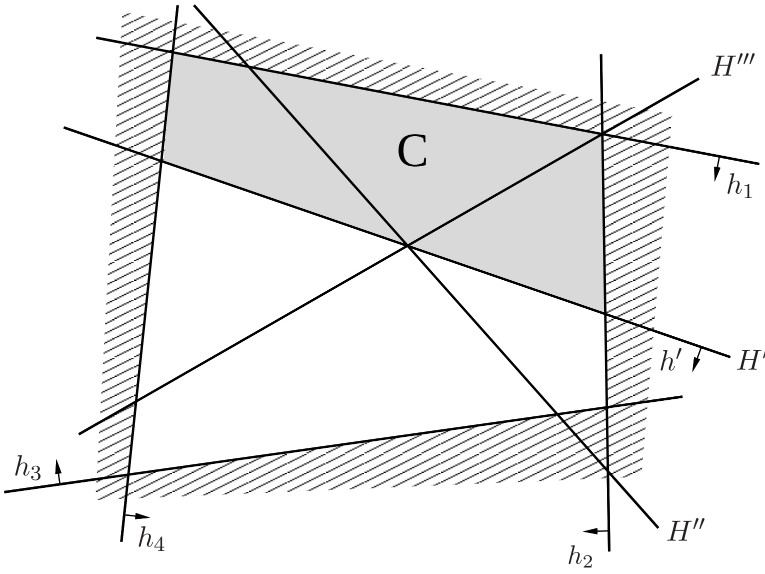


FIGURE 1

SCHOLIUM 6.2.5. *Let $O \in \mathcal{O}$. There is a unique minimal O' contained in O such that $O' = f(F)$ for some $F \in \mathcal{F}$. Let us call this face $F(O)$ as associated to O. It is clear that the map $O \mapsto F(O)$ is a left-inverse to f and induces homotopy equivalence between $\mathcal{F} \setminus \{\hat{1}\}$ and $\mathcal{O} \setminus \{\hat{1}\}$. For any $O \in \mathcal{O}$, we define $O^* := f(F(O))$.*

Proof. Take $F(O) = \bigvee f^{-1}(O)$. □

We are led to the following definition:

DEFINITION 6.2.6. *We consider the oriented matroid of the arrangement and call $\mathcal{O}$ its poset of (convex) acyclic sets (see 1.4). For any covering of groupoids $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ we define a diagram $\mathscr{G}_\rho : \mathcal{O} \to Top$ by setting*

$$\mathscr{G}_\rho(O) := \Delta(\mathcal{Q}^\rho_{O^*}),$$

$$\mathscr{G}_\rho(O_1 > O_2) : \qquad \begin{array}{rcl} \mathscr{G}_\rho(O_1) & \hookrightarrow & \mathscr{G}_\rho(O_2) \\ \gamma' \in \mathcal{Q}_{O^*_1}(\gamma) & \mapsto & \gamma' \in \mathcal{Q}_{O^*_2}(\gamma\alpha) \end{array}$$

where α is defined as the positive minimal path from $end(\gamma)$ to the unique chamber in O_2^+ that is related to $end(\gamma)$ by R_{O_2}. The right side is well defined because if $\gamma' = \gamma(C' \to C)^{-1}$, where $C = end(\gamma)$ and $C' = end(\gamma')$, then $\gamma' = (\gamma\alpha)((C' \to C)\alpha)^{-1}$.

Although this definition may seem a little bit involved, it turns out to be very natural with respect to the Salvetti-type diagram models that we illustrated in chapter 4.

Remark 6.2.4 allows us to apply the inverse image lemma 3.2.2 along the map f, which we will do in the following proposition.

PROPOSITION 6.2.7. *For any covering of groupoids $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$, the map f induces a homotopy equivalence*

$$\mathrm{hocolim}\mathscr{D}_\rho \simeq \mathrm{hocolim}\mathscr{G}_\rho.$$

Proof. Following remark 6.2.4, we can lift the diagram $\mathscr{G}_\rho$ along f by the inverse image lemma 3.2.2 to a diagram $\overline{\mathscr{D}} : \mathcal{F} \to Top$ with

$$\overline{\mathscr{D}}(F) = \coprod_{end(\gamma)<F} \Delta(\mathcal{Q}_{O(F)}(\gamma)),$$

where $O(F)$ is the acyclic set obtained by choosing for every $H \in \mathcal{A}\backslash\mathcal{A}_{|F|}$ the halfsphere containing F, and morphisms $\overline{\mathscr{D}}(F_1 > F_2)$ induced by the maps

$$\gamma \mapsto \gamma(end(\gamma) \to end(\gamma)_{F_2}).$$

Since $\Delta(\mathcal{Q}_O(C))$ is contractible for any $O \in O$ and $C \in O^+ \subset \mathcal{C}$, an application of the homotopy lemma 3.2.1 yields the homotopy equivalence $\mathrm{hocolim}\overline{\mathscr{D}}_\rho \simeq \mathrm{hocolim}\mathscr{D}_\rho$. □

We summarize the result, with a statement that is the counterpart of theorem 4.5.1.

Corollary 6.2.8. *For any topological covering $X \to \mathcal{M}(\mathcal{A})$, if $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ denotes the corresponding covering of groupoids, we have a homotopy equivalence*

$$\mathrm{hocolim}\mathscr{G}_\rho \simeq X.$$

In particular, for the identical cover we have $\mathrm{hocolim}\mathscr{G}_{id} \simeq \mathcal{M}$*, and if $\hat{\mathscr{G}}$ denotes the diagram associated to the universal cover,* $\mathrm{hocolim}\hat{\mathscr{G}} \simeq \widehat{\mathcal{M}}$*.*

Now, at least after the proof of the above proposition it is clear that the reason of considering the posets $\mathcal{Q}_O$ is not topological: in fact, for the sake of the homotopy type one may contract them to points. The reason to have the diagrams $\mathscr{G}_\rho$ defined as above is combinatorial: as we will see below, this is a step towards the formulation of a complex that generalizes the Garside complex of Artin groups $X(W)$ to the case where the arrangement is not the complexification of a real reflection arrangement.

Definition 6.2.9. *Let $\mathcal{A}$ be a complexified arrangement, and consider a covering $\mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ of the arrangement groupoid. We define a simplicial complex $U_\rho(\mathcal{A})$ (that we will sometimes denote only by U_ρ, if no confusion can arise) on the vertex set $V(U_\rho) := Ob(\mathcal{G}_\rho)$. This simplicial complex is defined by declaring a set $\{\gamma_1, \dots, \gamma_{d+1}\}$ to be a simplex if and only if there are positive minimal paths $\alpha_1, \dots, \alpha_d$ with*

(i) *$\gamma_{i+1} = \gamma_i\alpha_i$ for all $i = 1, \dots, d$, and*
(ii) *$\alpha_1 ... \alpha_d$ positive minimal.*

It is evident that this is a flag complex and that it is tiled by copies of all $\Delta(\mathcal{P}_C(\mathcal{A}))$'s (as C goes through $\mathcal{C}$). To be more precise, given $\gamma \in Ob(\mathcal{G}_\rho)$, the subcomplex spanned by all paths of the form $\gamma\alpha$ for α positive and minimal is naturally isomorphic to $\Delta(\mathcal{P}_{end(\gamma)})(\mathcal{A})$.

Remark 6.2.10. *For the universal cover diagram of Coxeter arrangements the translation to the description of the complex $X(W, \lambda)$ is straightforward: minimal positive paths are minimal galleries, i.e.,* atoms *in the terminology of Bestvina* [**6**].

However, our complexes model all covers of *any* complexified arrangement, as we show in the next proposition.

Definition 6.2.11. *Given $\gamma \in Ob(\mathcal{G}_\rho)$, let us denote by $\mathcal{P}_\gamma(\mathcal{A})$ the poset with elements all $\gamma\alpha$ (with α positive minimal) and the order defined according to the order of $\mathcal{P}_{end(\gamma)}(\mathcal{A})$. For $C \in \mathcal{C}$ let O_C denote the acyclic set consisting of all halfspaces (pseudohemispheres) that contain C, and for $\gamma \in Ob(\mathcal{G}_\rho)$ let $O_\gamma = O_{end(\gamma)}$.*

Note that from this definition we get, for any path γ ending in the chamber C, the equality $\mathcal{P}_\gamma(\mathcal{A}) = \mathcal{Q}_{O_C}(\gamma\tau)$, where τ stands for the longest positive minimal path that can be appended to γ (i.e., the path $(C \to -C)$). This notation will be used also for the proof of the next proposition.

PROPOSITION 6.2.12. *Let $\mathcal{A}$ be a real arrangement of linear hyperplanes, and let $\rho : \mathcal{M}_\rho \to \mathcal{M}$ be any topological covering of the complement of the complexification of $\mathcal{A}$. If $U_\rho(\mathcal{A})$ is defined as in 6.2.9 there is a homotopy equivalence*

$$U_\rho(\mathcal{A}) \simeq \mathcal{M}_\rho.$$

Proof. After proposition 6.2.7 it suffices to show that $\mathrm{hocolim}\mathscr{G}_\rho \simeq U_\rho$. For this, note that for any morphism $m = \mathscr{G}(O_1 > O_2)$, the image $m(\mathscr{G}_\rho(O_1))$ is a simplicial subcomplex of $\mathscr{G}_\rho(O_2)$ (compare definition 6.2.6). In particular $(\mathscr{G}_\rho(O_2), \mathscr{G}_\rho(O_1))$ is a NDR-pair and m is a closed cofibration.

So we are in the situation to apply the projection lemma 3.2.8, obtaining a homotopy equivalence

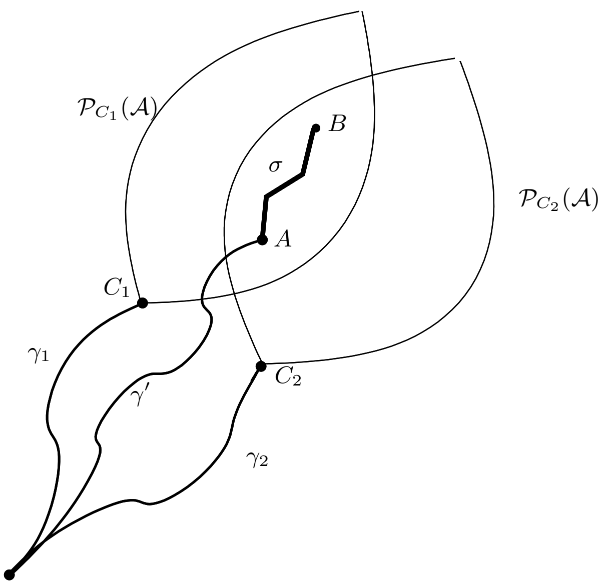
$$\mathrm{hocolim}\mathscr{G}_\rho \to \mathrm{colim}\mathscr{G}_\rho.$$

We are left with showing that the right hand side is the complex U_ρ. Indeed, every simplex is contained in (maybe more than) a $\Delta(\mathcal{P}_\gamma(\mathcal{A}))$. The maps of the diagram are inclusions, so

$$\mathrm{colim}\mathscr{G}_\rho = \Big(\coprod_{\gamma \in Ob(\mathcal{G}_\rho)} \Delta(\mathcal{P}_\gamma(\mathcal{A})) \Big) / \sim$$

and we only have to check the identifications.

Of course $\omega_1 \in \Delta(\mathcal{P}_{\gamma_1}(\mathcal{A}))$ and $\omega_2 \in \Delta(\mathcal{P}_{\gamma_2}(\mathcal{A}))$ are identified if and only if $\omega_1 = \omega_2$ in $\mathcal{G}_\rho$. Given a chain $\sigma \in \Delta(\mathcal{P}_{\gamma_1}(\mathcal{A})) \cap \Delta(\mathcal{P}_{\gamma_2}(\mathcal{A}))$ (which is then automatically of length $< n$), let us write $C_1 := end(\gamma_1)$, $C_2 := end(\gamma_2)$, $\gamma' := \min\sigma$, $A := end(\gamma')$, $B := end(\max(\sigma))$.



The case $C_1 = A = C_2$ is trivial, and if both $C_1 \neq A$ and $C_2 \neq B$ one may consider the poset $\mathcal{P}_{\gamma'}(\mathcal{A})$ that contains the chain σ. Then it is enough to show the claim for the case $C_1 = A$, $C_2 \neq A$.

In this case we may suppose that $\gamma_1 = \gamma_2\alpha$ with $\alpha = (A \to C_2)$ (all equalities of paths here are in fact equivalences in $\mathcal{G}_\rho$). We will argue by induction on the length of α, the case where $\ell(\alpha) = 0$ being trivial. If $\ell(\alpha) > 0$, let F denote the first face that is crossed by α and let $\epsilon = (C_2 \to C)$ denote the edge of α crossing F. Then clearly $O(F) > O(C_2)$, and by definition the hyperplane $H := |F|$ supporting F does not separate A from B. Since $\underline{O} = \mathcal{A} \setminus H$, we have that

$$\sigma \subset \mathcal{Q}_{O(F)}(\gamma_2\epsilon\tau),$$

and σ is mapped identically to $\sigma' \subset \mathcal{Q}_{O(C)}(\gamma_2\epsilon\tau)$, the latter being equal to $\mathcal{P}_{\gamma_2'}(\mathcal{A})$, where $\gamma_2' = \gamma_2\epsilon = \gamma_1(C \to A)^{-1}$.

Therefore σ is identified with σ' in the colimit, and it remains to show that $\sigma' \subset \mathcal{P}_{\gamma_2'}(\mathcal{A})$ is identified with $\sigma \subset \mathcal{P}_{\gamma_1}(\mathcal{A})$: but this follows now by induction, since $\ell((C \to A)) = \ell(\alpha) - 1$. This concludes the proof. $\square$

An important point here is that there is no unique poset playing the role of the lattice of the Garside structure. This role is played by the collection of all posets that can be obtained as posets of regions of $\mathcal{A}$ for different choices of the base-chamber. This fact suggests to consider a system of representatives of these posets, rather than 'the' poset, as the building block for these complexes. We will expand on this idea in the next chapter.

6.3. The universal cover complex

We specialize now to the case where $\hat{\rho} : \widehat{\mathcal{G}} \to \mathcal{G}$ is the universal cover morphism, and write $\widehat{U}$ for $U_{\hat{\rho}}$. This particular case gives rise to many interesting questions and remarks. In this short section we collect some of them and place them into context.

There is a natural embedding $\iota : \widehat{U} \to B\widehat{\mathcal{G}}$ induced by the inclusion of the vertices. Since $B\widehat{\mathcal{G}}$ is contractible, we can formulate the following version of the $K(\pi, 1)$ property:

SCHOLIUM 6.3.1. *A complexified arrangement $\mathcal{A}$ is $K(\pi, 1)$ if and only if the inclusion ι is a homotopy equivalence.*

Since U is included as a subcomplex, the following question is at hand.

QUESTION I. *Find a (combinatorial) condition for when $B\widehat{\mathcal{G}}$ retracts onto $\iota(\widehat{U})$.*

Another characterization of the asphericity of arrangements only in terms of the fundamental group was given (for general complex arrangements) by Randell, and rises an interesting question. Recall that a topological space is called *minimal* if it has the homotopy type of a CW-complex with torsion free homology, surjective cup-product, and a number of k-cells corresponding to the rank of the k-th homology (see [**100**, Definition 6]).

THEOREM 6.3.2 (Randell [**100**]). *The complement $\mathcal{M}$ of a complex arrangement is aspherical if and only if the fundamental group $\pi_1(\mathcal{M})$ has a minimal $K(\pi_1, 1)$-space and its Betti numbers satisfy $b_k(\pi_1(\mathcal{M})) = b_k(\mathcal{M})$ for all k.*

It is known that Garside groups have these propery (up to cup-product surjectivity, as far we know). The desired $K(\pi, 1)$ space is the factorization of the complex $X(P, \lambda)$ by the natural action of $G(P, \lambda)$ on it, and its minimality is proved in [**29**], where the Betti numbers are also computed: $b_k(X(P, \lambda)/G(P, \lambda))$ equals the number of elements of rank k in the poset P. This rises the following question.

QUESTION II. *For the complexification of a simplicial arrangement of hyperplanes, is it possible to compute the Betti numbers directly from the data of the posets of regions and along the lines of the proof of* [**29**]*?*

The only complexified arrangements for which the fundamental group is a Garside group with respect to the poset of regions are the Coxeter arrangements. David Bessis showed that also the fundamental groups of complex reflection arrangements (with one exeption) admit a Garside structure (which is called *dual* to the 'standard' one for Artin groups described in section 6.1.2). It is then natural to ask:

QUESTION III. *Find non-reflection arrangements whose fundamental group is a Garside group.*

6.4. On the combinatorics of Deligne's proof

One of the motivations of the above considerations was to find some new, and possibly more general, combinatorial conditions for the asphericity of arrangements that are based on the poset of regions. We will investigate this more carefully in the next chapter. In [**12**] it is shown that a real arrangement is simplicial if and only if the poset of regions is a lattice for all choices of base regions. In what follows we report on our attempt to trace the lattice property in Deligne's proof, hoping for some generalization. We find that the lattice property enters the picture in the proof of the normal form for paths in the arrangement groupoid; it is needed for any base chamber.

Our investigations show that, considering the complexes $\widehat{U}$ defined above, the topological part of the proof is almost trivial, and the focus is on the particular normal form which we call *property D* of the arrangement.

Notation: Before beginning our considerations, let us summarize the notations that we will adopt in the following. As usual we identify paths in $G(\mathcal{A})$ with morphisms in $\mathcal{G}(\mathcal{A})$. A *positive* path is a directed path in $G(\mathcal{A})$, or a morphism in the positive category $\mathcal{G}^+$. The equivalence relation defined on the morphisms of $\mathcal{G}^+$, that was denoted with R in chapter 2, will be from now on denoted by $\sim$. By $\approx$ we will denote equivalence (equality) of morphisms in $\mathcal{G}$. We refer to chapter 2 for a complete description of these constructions.

6.4.1. The positive complex. Let $\mathcal{A}$ be a real arrangement of hyperplanes, and recall that, througout this chapter, we assume the arrangement to be central. Let us fix a chamber C_0 of $\mathcal{A}$, denote by $-C_0$ its opposite (i.e., the unique chamber C such that $S(C_0, C) = \mathcal{A}$), and define a path

$$\delta := (C_0 \to -C_0)(-C_0 \to C_0),$$

which we are going to consider as a morphism of the arrangement groupoid, as usual.

The following lemma is a standard result, and we point to the given references for a proof.

LEMMA 6.4.1. *(see* [**36**]*,* [**97**]*,* [**103**, Proposition 25]*) For any path γ on the (oriented) 1-skeleton of the Salvetti complex with begin$(\gamma) = C_0$ there is an $n \in \mathbb{N}$ and a* ***positive*** *path α such that*

$$\gamma \approx \delta^n \alpha.$$

This lemma allows us to prove the following proposition, which in the meantime is also a standard fact.

PROPOSITION 6.4.2. *The universal cover complex $\Delta(Plim\widehat{\mathscr{D}})$ is contractible if the subcomplex $\Delta(Plim\widehat{\mathscr{D}})^+$ spanned by the positive paths starting at C_0 is contractible.*

Proof. (see [**97**, proof of 3.6]) Since π_1 acts by automorphisms of the universal cover, the subcomplex D_n spanned by the positive paths starting at δ^n is isomorphic to $\Delta(\mathrm{Plim}\widehat{\mathscr{D}})^+$ for any n.

By lemma 6.4.1, $\Delta(\mathrm{Plim}\widehat{\mathscr{D}}) = \bigcup_{n\in\mathbb{N}} D_n = \lim_{\mathbb{N}}\Delta(\mathrm{Plim}\widehat{\mathscr{D}})^+$. Thus, the whole complex is contractible if the positive one is. □

REMARK 6.4.3. *Salvetti* [**103**] *proves this in the slightly more general context of* ***metrical hemisphere complexes****, based on the injectivity of the functor that embeds the positive paths into the category of all paths.*

We can draw a similar consequence (with an analogous proof) for the complex $\widehat{U}$, introducing the *positive complex* $\widehat{U}^+$.

PROPOSITION 6.4.4. *The complex $\widehat{U}$ is contractible if the subcomplex $\widehat{U}^+$ spanned by the positive paths is contractible.*

Proof. See above. The 'real' reason why the same proof works is that in fact $\Delta(\mathrm{Plim}\widehat{\mathscr{D}})^+ = \mathrm{hocolim}\widehat{\mathscr{D}}^+$, where $\widehat{\mathscr{D}}^+$ is the subdiagram of $\widehat{\mathscr{D}}$ on the same poset whose spaces contain only the positive paths, and the arguments of chapter 4 only made use of positive paths. All the considerations of section 6.2 go through to show that $\widehat{U}^+ \simeq \mathrm{hocolim}\widehat{\mathscr{D}}^+$. □

6.4.2. Property D. Now the decisive fact: if the arrangement we are considering happens to be *simplicial*, i.e., all chambers are cones over simplices, then the following property holds for paths in the arrangement graph (see [**36**],[**103**],[**97**]). For terminology and definitions we refer to section 2.1.

Note that this is not the original formulation of Deligne, but it is clearly equivalent once one realizes that positive minimal paths starting from some chamber C are in bijection with elements of $\mathcal{P}_C(\mathcal{A})$.

Property D : Suppose a positive path γ is given. Then there is a chamber $K = K(\gamma)$ and a positive path γ' such that $\gamma \sim \gamma'(end(\gamma') \to end(\gamma))$ ***if, and only if,*** $end(\gamma') < end(\gamma)$ in $\mathcal{P}_K(\mathcal{A})$.

We now want to show that this can be proved directly with the assumption of the following

Strong lattice property: For any choice of a base chamber $C \in \mathcal{C}(\mathcal{A})$, the poset of regions $\mathcal{P}_C(\mathcal{A})$ is a lattice.

We take a piece of notation from Deligne's work, and denote by $D(A, B)$ the set of chambers in the interval $[A, B] \subseteq \mathcal{P}_A(\mathcal{A})$ for given chambers A and B. Then, assuming the strong lattice property we can immediately prove the statement of corollary (1.17)(i) of [**36**], which is merely the meet-semilattice part of the lattice property.

LEMMA 6.4.5 ((1.17)(i) of [**36**]). *For any three chambers A, C_1, C_2, there is a chamber C such that*

$$D(A, C_1) \cap D(A, C_2) = D(A, C).$$

Proof. Set $C := C_1 \wedge C_2$, the wedge taken in $\mathcal{P}_A(\mathcal{A})$. □

In view of this correspondence, one sees that proposition (1.15) and lemma (1.18) of [**36**] (i.e., the only two statements that are used in the proof of the key theorem (1.19), our theorem 6.4.6) correspond to easy properties of lattices: (1.15) says that a join-closed lower ideal is principal, (1.18) states that a principal ideal is join-closed. This shows the point that we would like to underline, namely that in a sense the 'real' idea underlying the theorem is that of the lattice property with respect to all chambers.

Nevertheless, we (re)prove the main theorem of Deligne from the viewpoint of order of regions and arrangement groupoids, hoping that the resulting simplification will allow us a readable exposition of an argument that by its nature is somewhat involved.

THEOREM 6.4.6 (compare [**36**], [**103**]). *Let $\mathcal{A}$ be a real arrangement. If $\mathcal{P}_C(\mathcal{A})$ is a lattice for all $C \in \mathcal{C}$, then property D holds for paths in the arrangement graph of $\mathcal{A}$.*

Proof. The idea is to split the equivalence relation $\sim$ of section 2.1 into smaller relations $\sim^i$, whose union is $\sim$. If $\sim$ relates paths that are connected by a sequence of substitutions of positive minimal paths, $\sim^i$ relates paths *of length* i that are related by an exchange of a positive minimal piece, namely at the *ending* of the path. Let us make this precise.

Consider two positive paths γ_1, γ_2 of length n, that end in the same chamber C with the last edge l_1 or l_2, respectively. For $i = 1, 2$ we let H_i denote the hyperplane crossed by l_i and denote by C_i the chamber representing the vertex before last in γ_i. Then we define $A := C_1 \vee C_2$, the join taken in $\mathcal{P}_C(\mathcal{A})$. Note that A definitely depends on γ_1 and γ_2: if specification will be needed, we will write $A = A_{\gamma_1,\gamma_2}$.
The relation $\sim^n$ is defined on positive paths of length n with same begin- and endpoint. The relation $\gamma_1 \sim^n \gamma_2$ holds if

(i) $n = 0$ or

(ii) $l_1 = l_2$ and $\gamma_1 \sim \gamma_2$ or

(iii) $l_1 \neq l_2$ and there exists a positive path α starting at $begin(\gamma_1) = begin(\gamma_2)$ and ending in A, such that $\gamma_i' \sim \alpha(A_{\gamma_1,\gamma_2} \to C_i)$ for both i's, where γ_i' denotes the subpath of γ_i ending in the last occurrence of C_i.

If $\sim^n$ is an equivalence relation for all n up to a certain $N \in \mathbb{N}$, then we can deduce:

6.4.6.1 *For all positive paths of length $n \leq N$, $\gamma_1 \sim^n \gamma_2 \Leftrightarrow \gamma_1 \sim \gamma_2$.*
The forward direction is clear. For the backward direction, note that admitting only substitutions of paths of the form $(A_{\gamma_1,\gamma_2} \to C)$ is not a restriction: in fact this means that only 'smallest possible' substitutions can be done at once - namely substitutions along the boundary of 2-cells. But clearly any substitution can be reached by a sequence of "2-cell substitutions". □

6.4.6.2 *If $\omega\gamma_1 \sim \omega\gamma_2$ for positive paths $\omega\gamma_1$, $\omega\gamma_2$ of length at most N, then $\gamma_1 \sim \gamma_2$.*
By 6.4.6.1 we know $\omega\gamma_1 \sim^n \omega\gamma_2$, and we have $end(\omega) \geq A_{\gamma_1,\gamma_2}$ in $\mathcal{P}_C(\mathcal{A})$. Thus $\omega\gamma_1 \sim^n \omega\gamma_2 \Leftrightarrow \gamma_1 \sim^n \gamma_2$ and by 6.4.6.1 we are done. □

6.4.6.3 *Given a positive path γ of length $n \leq N$, let $C := end(\gamma)$ and let $\mathscr{B}$ denote the set of all chambers B such that $\gamma \sim \beta(B \to C)$ for some positive path β. Then $\mathscr{B}$ is a principal lower ideal of the lattice $\mathcal{P}_C$.*

We first show that $\mathscr{B}$ is closed under joins.

For any two chambers B', B'' in $\mathscr{B}$ let $B := B' \wedge B''$ (here and in the following the meet is taken in $\mathcal{P}_C(\mathcal{A})$). Then $\gamma \sim \beta'(B' \to B)(B \to C))$ and $\gamma \sim \beta''(B'' \to B)(B \to C))$. Thus, by 6.4.6.2, $\beta'(B' \to B) \sim \beta''(B'' \to B)$, which implies $\beta'(B' \to B){\sim^n}\beta''(B'' \to B)$ by 6.4.6.1. By definition of meet, the open intervals (B, B') and (B, B'') in $\mathcal{P}_C$ are disjoint. Therefore any two minimal elements $M' \in (B, B')$, $M'' \in (B, B'')$ are distinct and give

$$M' \vee M'' = B' \vee B'' =: \widetilde{A}.$$

Remark that the interval $[B, \widetilde{A}]$ is the same in both $\mathcal{P}_C(\mathcal{A})$ and $\mathcal{P}_B(\mathcal{A})$ (see e.g. [**13**, 4.2.11(ii)]), then note that by item (iii) in the definition of $\sim^m$ (where m equals n minus the length of $(B \to C)$) there is a positive path $\widetilde{\alpha}$ such that

$$\widetilde{\alpha}(\widetilde{A} \to B) \sim \beta'(B' \to B) \sim \beta''(B'' \to B).$$

Then we know that γ can be written as

$$\gamma \sim \widetilde{\alpha}(\widetilde{A} \to C),$$

and therefore

$$B' \vee B'' = \widetilde{A} \in \mathscr{B}.$$

Now since $\mathscr{B}$ is clearly a lower ideal in $\mathcal{P}_C$, being closed under join means that it is a principal ideal. □

6.4.6.4 *The claim of the theorem holds for all positive paths of length at most N.*

Given a path γ of length at most N, consider the corresponding set of chambers $\mathscr{B}$. By 6.4.6.3 there is exactly one $K \in \mathcal{P}_C$ for which $\mathscr{B} = [C, K]$. In view of lemma 6.4.5, this is property D. □

The above remarks leave us to show that $\sim^n$ is indeed an equivalence relation, which we will do by recursion.

The case $\sim^0$ is trivial, and so we assume $\sim^i$ to be an equivalence relation for each $0 \leq i < n$. Symmetry and reflexivity are clear, so we have to show transitivity.

For this, assume that $\gamma_1{\sim^n}\gamma$ and $\gamma_2{\sim^n}\gamma$ for positive paths γ, γ_1, γ_2 of length n ending in C. We want to show $\gamma_1{\sim^n}\gamma_2$.

Let C', C'_1, C'_2 denote the chambers before last in γ, γ_1, γ_2, respectively. By assumption there are positive paths α_1, α_2 with

$$\gamma' \sim \alpha_1(A_{\gamma,\gamma_1} \to C') \sim \alpha_2(A_{\gamma,\gamma_2} \to C') \tag{4.1}$$

$$\gamma'_1 \sim \alpha_1(A_{\gamma,\gamma_1} \to C'_1) \tag{4.2}$$

$$\gamma'_2 \sim \alpha_2(A_{\gamma,\gamma_2} \to C'_2) \tag{4.3}$$

If $\mathscr{B}$ denotes the set of chambers associated to γ' as in 6.4.6.3, the above implies in particular $\{A_{\gamma,\gamma_1}, A_{\gamma,\gamma_2}\} \subset \mathscr{B}$, and by 6.4.6.3 also the join $M := A_{\gamma,\gamma_1} \vee A_{\gamma,\gamma_2} \in \mathscr{B}$. Therefore we can find a positive path α with

$$\gamma' \sim \alpha(M \to C'). \tag{4.4}$$

Note that

$$(M \to A_{\gamma,\gamma_1})(A_{\gamma,\gamma_1} \to C') \sim (M \to C') \sim (M \to A_{\gamma,\gamma_2})(A_{\gamma,\gamma_2} \to C'). \tag{4.5}$$

Now substitute equations 4.5 into 4.4, compare it with equations 4.2 and 4.3 and apply property 6.4.6.2 above to get

$$\alpha_1 \sim \alpha(M \to A_{\gamma,\gamma_1}),$$
$$\alpha_2 \sim \alpha(M \to A_{\gamma,\gamma_2}),$$

and thus

$$\gamma_1' \sim \alpha(M \to C_1'),$$
$$\gamma_2' \sim \alpha(M \to C_2').$$

By construction we have $M \geq C_1' \vee C_2' =: A_{\gamma_1,\gamma_2}$. Thus, $(M \to C_i')$ factorizes as $(M \to A_{\gamma_1,\gamma_2})(A_{\gamma_1,\gamma_2} \to C_i')$ for $i = 1, 2$. We may then define a positive path by $\mu := \alpha(M \to A_{\gamma_1,\gamma_2})$ and deduce

$$\gamma_1' \sim \mu(A_{\gamma_1,\gamma_2} \to C_1'),$$
$$\gamma_2' \sim \mu(A_{\gamma_1,\gamma_2} \to C_2').$$

This concludes the proof. □

Also the remaining (topological) part of the argument can be stated in a particularly nice way using the language of posets that we adopted.

Indeed, if we let $\widehat{U}_n$ denote the subcomplex of $\widehat{U}^+$ spanned by the vertices that represent paths of length smaller than n, we see the following.

Lemma 6.4.7. *If the arrangement has property D, then the link of any $\gamma \in \widehat{U}_n \setminus \widehat{U}_{n-1}$ in the complex $\widehat{U}_n$ is contained in $\widehat{U}_{n-1}$ and is isomorphic to*

$$\Delta(\mathscr{B} \setminus end(\gamma)) = \Delta(\mathcal{P}_{K(\gamma)})_{<end(\gamma)},$$

hence contractible.

This gives immediately (yet) another viewpoint on the proof of

Theorem 6.4.8. *If an arrangement has the lattice property for any base chamber, then it is aspherical.*

Proof. Since every $\widehat{U}_n$ collapses onto $\widehat{U}_{n-1}$ and $\widehat{U}_0 \simeq \{*\}$, the complex $\widehat{U}^+ = \lim \widehat{U}_n$ is contractible. Therefore by theorem 6.4.4 the complex $\widehat{U}$ (and hence the universal cover $\widehat{\mathcal{M}}$) is contractible. □

6.5. Necessity of the lattice condition

In this section we show that property D is equivalent to the strong lattice property. Theorem 6.4.6 of the previous section proves that the strong lattice property implies property D. Here we assume that the strong lattice property does not hold and prove that then property D also fails to hold. This answers a question posed by Paris in [**97**, section 4].

The first thing one should note is that the poset of regions of a real arrangement is always ranked (by the size of the sets of separating hyperplanes) and this rank is bounded (by the total number of hyperplanes). Therefore, the following lemma becomes interesting: indeed, it is the main tool used by the authors of [**12**] in the proof of the lattice property for simplicial arrangements.

LEMMA 6.5.1 (Lemma 2.1 of [**12**]). *Let $\mathcal{P}$ be a bounded poset of finite rank such that, for any x and y in $\mathcal{P}$, if x and y both cover an element z then the join $x \vee y$ exists. Then $\mathcal{P}$ is a lattice.*

So the only way for the strong lattice property to fail is that, for some base chamber R, there are chambers A, B that cover C in the poset $\mathcal{P}_R$ for which the join $A \vee B$ does not exist in $\mathcal{P}_R$. Since the interval $[C, -R] = (\mathcal{P}_R)_{\geq C}$ in $\mathcal{P}_R$ is isomorphic to the interval $[C, -R] = (\mathcal{P}_C)_{\leq -R}$ in $\mathcal{P}_C$, we may from now on consider the situation in the latter poset, that is therefore also not a lattice. In particular, the chamber C cannot be simplicial (theorem 3.1 of [**12**]) and we may then apply lemma 4.4.4 of [**13**]. Before stating the lemma, we recall that an interval $[X, Y]$ in a poset of regions is called *elementary* if the subposet $[X, Y] \setminus \{X, Y\}$ is a union of two disjoint chains.

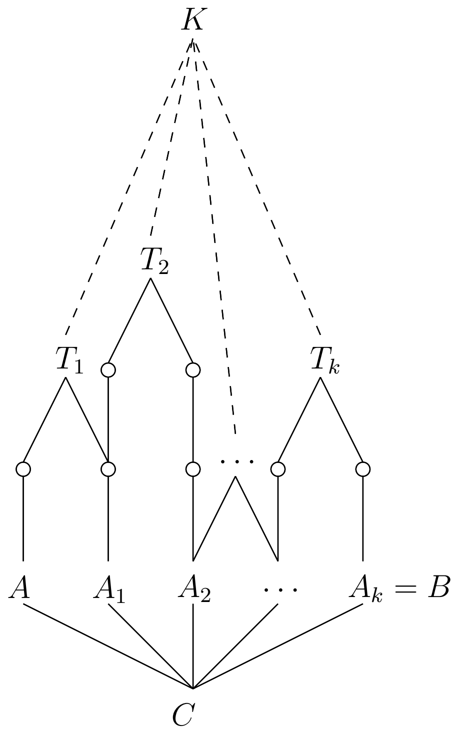


FIGURE 1. The situation of lemma 6.5.2.

LEMMA 6.5.2 (Lemma 4.4.4 of [**13**], "realizable version"). *Let A, B, C, K be chambers of $\mathcal{A}$, and suppose that A and B are atoms in the interval $[C, K]$ of $\mathcal{P}_C(\mathcal{A})$. Then there exists a sequence of atoms $A = A_0, A_1, \ldots, A_k = B$ and a sequence of other elements $T_1, T_2, \ldots, T_k$ in $[C, K]$ such that $[C, T_i]$ is elementary and contains A_i and A_{i+1}, for all $1 \leq i \leq k$. If the chamber C is simplicial, then $k = 1$. (See figure 6.5.2)*

In our setting, since the join of A and B does not exist, we have that, for any K, the minimal possible associated k is at least 2. Let us choose an element M among the minimal upper bounds of A and B. We have the following situation (figure 6.5.3): atoms $A = A_0, A_1, \ldots, A_k = B$ with the associated $T_1, T_2, \ldots T_k$, and all those elements are in the interval $[C, M]$. We will denote H_i the unique element of $S(C, A_i)$.

Consider the positive path built as follows from positive minimal ones:

$$\gamma := (M \to C)(C \to A_1).$$

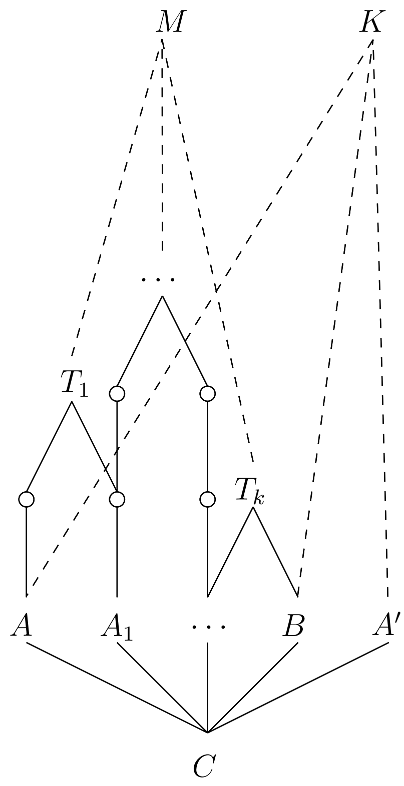


FIGURE 2. Our setup for the definition of γ.

Clearly, γ ends by both the positive paths $A \to C \to A_1$ and $B \to C \to A_1$. Let us show that *there is no chamber K such that γ ends with $K \to A_1$ and*

$$D(A_1, K) \supset D(A, A_1) \cup D(B, A_1) = \{H_0, H_1, H_k\}.$$

Indeed, for such K we would have $\{H_0, H_k\} \subset S(C, K)$ (because neither H_0 nor H_k separate A_1 from C), therefore both A and B are atoms in the interval $[C, K]$. So K is clearly incomparable with M in $\mathcal{P}_C$: $K \not\geq M$ because $H_1 \notin S(C, K)$, and $K \not< M$ by minimality of M. This means that there is an hyperplane $H' \in S(M, K) \cap S(C, K) \setminus S(C, M)$, i.e., H' separates K and C from M.

Now, since H_1 is a wall of both A_1 and C, we have $(K \to A_1) = (K \to C)(C \to A_1)$. Therefore, if such a K would exist, then γ would consist of a positive path from M to K (thus crossing H') followed by $(K \to A_1)$ (that crosses H'). But, by definition, γ does not cross H' since this hyperplane does not separate M from C (see figure 6.5.4 for a sketch of the situation). This gives a contradiction, since equivalent positive paths cross the same number of times every hyperplane (intuitively: $M \to K \to A_1$ 'turns around' H', while γ does not).

Thus, property D implies the strong lattice property, and so any arrangement satisfying property D is simplicial. We summarize.

THEOREM 6.5.3. *For a real arrangement $\mathcal{A}$, the following are equivalent:*

(i) *Property D holds in $\mathcal{A}$,*
(ii) *$\mathcal{A}$ satisfies the strong lattice property,*
(iii) *$\mathcal{A}$ is simplicial.*

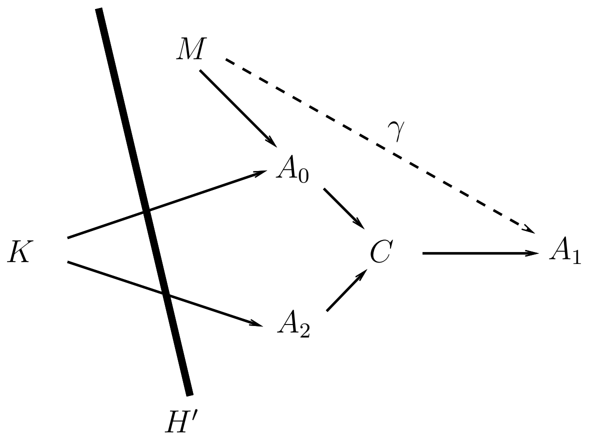


FIGURE 3. Sketch for the main argument.

REMARK 6.5.4. *As already remarked, one can construct the category of positive paths for any oriented matroid, starting from the tope graph instead of the arrangement graph. It is straightforward to translate the proof of the previous theorem to that setting, showing the equivalence of property D with the strong lattice property (for tope posets) in this more general case. Since this is not of immediate interest for us now, we omit the details.*

6.6. Some remarks on the Varchenko-Gel'fand ring

We consider the case $\rho = id$ and draw the connection with the Varchenko-Gel'fand ring. This is the ring VG of functions $f : S^{d-1} \to \mathbb{Z}$ that are constant on the regions of the arrangement. It is named after the authors of [**113**], where this object was first studied and put in relation with the cohomology ring of the complexified arrangement. The structure of the poset $\mathcal{O}$ and of the diagrams $\mathscr{G}$ gives rise to interesting similarities between VG and the cochain complex that arises from the spectral sequences derived in chapter 5.

Namely, the spaces of the diagram $\mathscr{G}$ are disjoint unions of contractible components, and therefore for any $O \in \mathcal{O}$ we have $H^t(\mathscr{G}(O)) = 0$ for all $t > 0$. In the language of chapter 5, we have that the sheaves $\mathcal{H}^t\mathscr{G}$ are all zero for $t > 0$, thus, for brevity, we set from now on $\mathcal{H}^0 := \mathcal{H}^0\mathscr{G}$. The stalk of this sheaf over the element $O \in \mathcal{O}$ can be written as

$$\mathcal{H}^0(O) := \bigoplus_{C \in O^{*+}} \mathbb{Z}_C.$$

With theorem 5.2.6, the E_1-term of the associated spectral sequence is zero everywhere else than in the O^{th} row, where we have a cochain complex that is precisely the resolution of the sheaf $\mathcal{H}^0$ on the poset $\mathcal{O}$. Therefore, with theorem 5.3.1, we may write

$$H^*(\mathcal{M}, \mathbb{Z}) = H^*(\mathcal{O}, \mathcal{H}^0).$$

One of the results of [**113**] is that the ring VG can be viewed as generated by *Heaviside functions* x_H. These are defined by choosing an orientation of each hyperplane (or equivalently, in the language of chapter 1.3, choosing one element h in each orbit of the involution $* : \check{E} \to \check{E}$). The function x_H is then defined by $x_H(C) = 1$ if C is on the positive side of H, and $x_H(C) = 0$ else. More formally, we recall from 1.4 the pseudo-hemispheres s_e that are associated to every $e \in \check{E}$ and consider a set of representatives

$(h_i)_{i=1,\dots,|E|/2}$ of the orbits of $* : \check{E} \to \check{E}$. We define

$$x_h(C) := \begin{cases} 1 & \text{if } C \subset s_h \\ 0 & \text{else.} \end{cases}$$

We remark that it is always possible to choose the h such that for a given chamber C we have $x_h(C) = 1$ for all h. Such sets $\{h_1, \dots, h_n\}$ are precisely the minimal elements of $\mathcal{O}$.

So let us fix a chamber C_0 and choose a set $\mathbf{h} := \{h_1, \dots, h_n\}$ as above, so that $\{h_1, \dots, h_n\}^+ = C_0$. Then the minimal element of $\mathcal{O}$ that corresponds to a chamber C is $O_C := \{h'_1, \dots h'_n\}$ where $h'_j = *h_j$ if and only if the pseudosphere corresponding to h_j separates C from C_0 (in particular, $O_{C_0} = \mathbf{h}$).

To any other element of $\mathcal{O}$ naturally corresponds the set $O := \bigcap_{O(C)<O} O(C)$. It is easy to see that there is a canonical element $f \in VG$ with $f(C) = 1$ if $C \subset O^+$, and $f(C) = 0$ elsewhere. This is the element

$$f_O := \prod_{h \in O \cap \mathbf{h}} x_h \prod_{h \in O \setminus \mathbf{h}} (1 - x_h).$$

The next definition gives rise to the connection between VG and $\mathcal{H}^0$.

Definition 6.6.1. *Given an element $f \in VG$, we define its support* supp f *to be the set of chambers where f is nonzero.*

With this definition we see that there is a natural reformulation of the sheaf $\mathcal{H}^0$ on $\mathcal{O}$ as

$$\mathcal{H}^0(O) = \{f \in VG \mid \mathrm{supp} f \subseteq O^{*+}\}.$$

6.6.1. The poset of ideals of VG. A point of interest in this idea is that the convex sets of the oriented matroids correspond to principal monomial ideals in VG, as the next lemma states.

Lemma 6.6.2. *For any $O \in \mathcal{O}$ we have*

$$\mathcal{H}^0(O) = \langle f_{O^*} \rangle,$$

the ideal in VG generated by f_{O^*}.
Proof. Clear. □

The notion of T-convexity (see [**13**, 4.2] or the next chapter for definitions and some background) gives to the set $\check{E}$ the structure of a convex geometry. The convex sets are precisely the elements of $\mathcal{O} \setminus \{\hat{1}\}$. Since the intersection of convex sets is convex, we see that for any ideal $J \subset VG$ there is a unique maximal $O = O_J$ such that $J \subset \langle f_{O_J} \rangle$, *"each ideal is contained in a unique minimal convex ideal"*. This leads us to take a closer look at the poset $\mathcal{J}$ of (nonzero) ideals of VG, ordered by inclusion.

Lemma 6.6.3. *The natural map $\mathcal{J} \to \mathcal{O}$, $J \mapsto O_J$ satisfies the requirement of proposition 5.3.3.*
Proof. The preimage of every $O \in O$ is the ideal $\langle f_O \rangle$. The preimage of any $\mathcal{O}_{\leq O}$ is a principal lower ideal in $\mathcal{J}$, and thus contractible. □

This correspondence of posets allows us to apply proposition 5.3.3, thereby to pull back our sheaf along $\mathcal{J} \to \mathcal{O}$, and get the following suggestive expression for the cohomology of the Salvetti complex.

Proposition 6.6.4. *For every $J \in \mathcal{J}$ define $\iota\mathcal{H}^0(J) := \langle f_{O_J} \rangle$, and let the maps $j : \iota\mathcal{H}_0(J_1) \to \iota\mathcal{H}_0(J_2)$ be induced by the maps $\mathcal{H}_0(O_{J_1}) \to \mathcal{H}_0(O_{J_2})$. We can then write*

$$H_*(\mathcal{M}, \mathbb{Z}) = H_*(\mathcal{J}, \iota\mathcal{H}_O).$$

The natural question at this point is to put the degree filtration of [**113**] into the picture of convex ideals. The point is that Varchenko and Gel'fand describe a non-natural isomorphism between the quotients of the degree filtration and the cohomology groups of the arrangement complement (see [**113**, section 6, theorem 8]). To shed some light on this question and on the differentials of the cochain complex we propose the following viewpoint.

6.6.2. Back to the face poset. We have already seen the prominent role played by the convex sets of the form $O(F)$ for some $F \in \mathcal{F}$. It is easy to see that the set $\{O(F) \mid f \in \mathcal{F}\}$ partially ordered by reverse inclusion is isomorphic to $\mathcal{F}$. We write the spectral sequence of theorem 5.2.4 for the diagram $\mathscr{D}$ of chapter 4, keeping in mind this interpretation of $\mathcal{F}$. The groups $H^*(\mathcal{M}, \mathbb{Z})$ are then the homology groups of the cochain complex

$$C^s(\mathcal{F}, \mathcal{H}^0) = \bigoplus_{rk(F)=s} \langle f_{O(F)} \rangle,$$

where the differential d^s maps a cochain $(f_F)_{rk(F)=s}$ to the cochain $(f'_F)_{rk(F')=s+1}$ that is given in component F' by the following element of VG (recall the relation R_O from equation 2.1):

$$f'_{F'}(C) := \sum_{\substack{F<F' \\ R_{O(F)}(C',C)}} [F : F'] f_F(C).$$

This chapter gave rise to more questions that answers. It is therefore not inappropriate to conclude it with a last question that may help to understand the conceptual relation of the Varchenko-Gel'fand ring with the cohomology of the arrangement.

Question IV. *Find an expression of the differentials of the above spectral sequence in terms of the "flag cochains" of* [**113**].

For this, it might be useful to pass to the "simplicial" version of the sequence (theorem 5.2.6), because then cochains correspond to the "distinguished flags of ribs" of Varchenko and Gel'fand (see [**113**]). This could also connect the cohomology of this cochain complex with the degree filtration of VG.

CHAPTER 7

Arrangements with a lattice of regions

In all what follows we will consider a real arrangement $\mathcal{A} \subset \mathbb{R}^d$ that we suppose central and essential. We will be concerned with *arrangements with a lattice of regions* (named this way after [**12**]), i.e., real arrangements for which the poset of regions is a lattice only for one basis chamber. We begin with a section that gives some point of interest of investigating this property and outlines, without technicalities, the program for the later sections. In section 7.2 we show an example of an arrangement with a lattice of regions that is not aspherical. In the attempt to understand the meaning of this weak lattice property, in section 7.5 we introduce a new, 'coarser' stratification of the universal cover complex. We show that, if the arrangement has a lattice of regions, the cells of this stratification are contractible (Corollary 7.5.8). The next step is accomplished in section 7.6, where we determine the incidence relation between the maximal cells (Theorem 7.6.3). In view of an application of the nerve lemma, we end by studying the topology of the intersections of those big cells in section 7.7.

7.1. Introduction and summary

We start by repeating the definition of the main character. Although we already gave it in chapter 6, we feel that it should stand at the beginning of this chapter.

Definition 6.1.4. *Let $\mathcal{A}$ be a real, central arrangement of hyperplanes. For any two chambers $C_1, C_2 \in \mathcal{C}(\mathcal{A})$ let*

$$S(C_1, C_2) := \{H \in \mathcal{A} |\ \textit{separates } C_1 \textit{ from } C_2\}.$$

*Fix a base chamber $C_0 \in \mathcal{C}(\mathcal{A})$ and define the **poset of regions** $\mathcal{P}_{\mathbf{C_0}}(\mathcal{A})$ **with base chamber** $\mathbf{C_0}$ on the set $\mathcal{C}(\mathcal{A})$ by setting*

$$C_1 \leq C_2 \textit{ if and only if } S(C_0, C_1) \subseteq S(C_0, C_2).$$

When no confusion on the arrangement or on the chosen base chamber will be possible, we will write $\mathcal{P}_{C_0}$ for $\mathcal{P}_{C_0}(\mathcal{A})$ and $S(C)$ for $S(C_0, C)$.

Recall that this object was first considered by Edelman in [**47**] (but see also [**50, 48**]), and further studied by Björner, Edelman and Ziegler in [**12**]. Especially the latter paper studies the situation where the above posets are lattices. The work of this chapter was motivated by the following two results:

- *The arrangement $\mathcal{A}$ is simplicial if and only if the poset $\mathcal{P}_C(\mathcal{A})$ is a lattice for any choice of base region C (we call this the **strong lattice property**)* [**12**, Theorem 3.1 and 3.4].
- *If the arrangement $\mathcal{A}$ is supersolvable, then the poset $\mathcal{P}_C(\mathcal{A})$ is a lattice whenever C is a canonical base region* [**12**, Theorem 4.6]. For the definition of canonical base regions see [**12**, p. 275]. There are 2^d such regions, if d is the rank of the arrangement.

Since all aspherical complexified arrangements that are known to us are either simplicial or supersolvable, already these facts make this property interesting in the study of the $K(\pi, 1)$-problem for complexified arrangements. But there is more than this. A class of complexified arrangements that is conjectured to be $K(\pi, 1)$ is that of *inductively factored* arrangements. These arrangements were studied by Jambu and Paris in [**84**]. Supersolvable arrangements are inductively factored, and the canonical base regions of [**12**] generalize to the *coloured regions* of [**84**, p. 277]. The following fact is proved for all *hyperfactored* arrangements, of which inductively factored arrangements are a special class:

- *If the arrangement $\mathcal{A}$ is hyperfactored, then the poset $\mathcal{P}_C(\mathcal{A})$ is a lattice whenever C is a coloured region* [**84**, Theorem 5.1]. There are 2^d colored regions, if d is the rank of the arrangement.

This motivated us to study the topological effects of a weakening of the strong lattice property. A first warning along this way is expressed in the following remark.

Remark 7.1.1. *In theorem 6.5.3 we showed that the strong lattice property and property D are indeed equivalent. It is therefore hopeless to try to show "property D" of 6.4.2 with a weaker assumption than the strong lattice property.*

In view of this, we propose to take another approach: the idea is to consider a stratification of the universal cover that is different from the ones encountered so far. Since the last sections will be rather technical, let us try here to give a global view of what will follow.

The complexes obtained in chapter 4 and 6 were stratifications of the universal cover of the arrangement by topological balls $\Omega(\tau)$ (namely, zonotopes in chapter 4 and order complexes of posets of regions in 6) indexed by classes of paths τ in the arrangement graph $G(\mathcal{A})$. We propose now to look at a coarser stratification: cells will be indexed by *closed* paths with fixed beginning point (i.e., endomorphisms of a fixed $C_0 \in \mathrm{Ob}(\mathcal{G}(\mathcal{A}))$). The cell indexed by the closed path γ will be denoted $\mathbf{\Omega}(\gamma)$ and is the union of all $\Omega(\tau)$ that are indexed by paths of the form $\tau = \gamma\alpha$, with α positive minimal.

The first point in the above definition is that it is enough to assume that the poset of regions is a lattice with respect to C_0 to show that the "big cells" $\mathbf{\Omega}(\gamma)$ are indeed contractible: this is the goal of section 7.5, which is accomplished with Corollary 7.5.8.

After this first result, we study the incidence between big cells. First of all, in section 7.6 we derive a criterion that characterizes the pairs of paths γ, γ' such that $\mathbf{\Omega}(\gamma) \cap \mathbf{\Omega}(\gamma') \neq \emptyset$ (Theorem 7.6.3). This is based on an appropriate norm that we define on paths (Definition 7.6.1).

Finally, in section 7.7 we study the topology of the intersections of big cells. Always under the assumption of the lattice property with respect only to the chamber C_0 we show that, for big cells indexed by paths whose norm difference is sufficiently small, the intersection is contractible.

If this is the case in general, then we can apply the nerve lemma and express the homotopy type of the universal cover with a simplicial complex having vertices in bijection with the elements of the fundamental group.

However, it is possible that already to prove this one has to make stronger assumptions, such as the existence of sufficiently many other chambers C_i for which the lattice property holds, and such that the sets $S(C_i)$ give rise to a partition of the arrangement in subarrangements with nice topological properties (as it is the case for inductively factored arrangements.)

Anyway, it is clear that the assumption of the existence of only one chamber that gives rise of a lattice of regions is too weak to show asphericity of the arrangement: the next section shows a counterexample to this. Nevertheless, it is instructive to see how far things can be brought with the very weak assumption of the lattice property for only one chamber.

7.2. The counterexample

If simplicial arrangements generalize Coxeter arrangements by keeping the property that all chambers are cones over simplices, *free arrangements* generalize real reflection arrangements in that they have a notion of *exponents* that reduces to the usual exponents of the Coxeter group in the reflection case. This class of arrangements was introduced by Hiroaki Terao [**109**], who conjectured asphericity for these arrangements.

This conjecture (sometimes also named after Kyoji Saito) was settled negatively by Edelman and Reiner [**51**] by the construction of a class of complexified arrangements that they show to be free but *not* aspherical. We refer the interested reader to the original work [**51**] or to section 3.1 of the survey [**56**] for a concise summary of the argument. A short account of the non-asphericity proof is given in [**55**].

These counterexamples form a family $(\mathcal{A}_t)_{t\in\mathbb{R}}$ of rank 3 complexified arrangements with defining polynomial

$$Q(\mathcal{A}_t) := xyz(x-y)(x-z)(y-z)(x-ty)(x-tz)(y-tz).$$

For our scope it is enough to consider the arrangement $\mathcal{A}_{-2}$, that is -as said- *not* aspherical. A projective picture of the real part of this arrangement is given at page 12 of [**56**]. Here we prefer to depict the actual configuration of hyperplanes in $\mathbb{R}^3$ as figure 7.2.1.

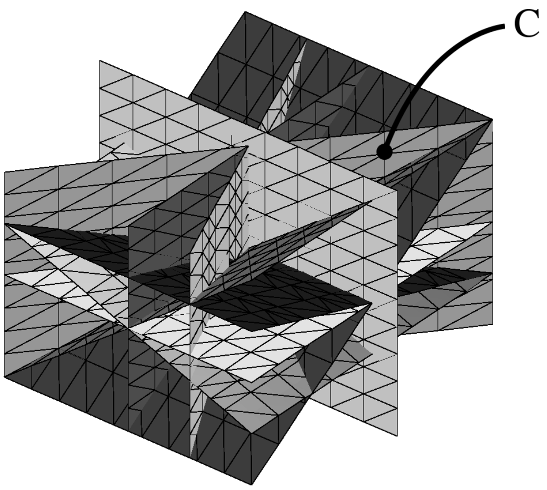


FIGURE 1. The arrangement $\mathcal{A}_{-2}$.

Looking at it for sufficiently long time one can figure out that the poset of regions of $\mathcal{A}_{-2}$ based at the chamber C indicated in the figure 7.2.2 is the lattice depicted in figure 7.2.1.

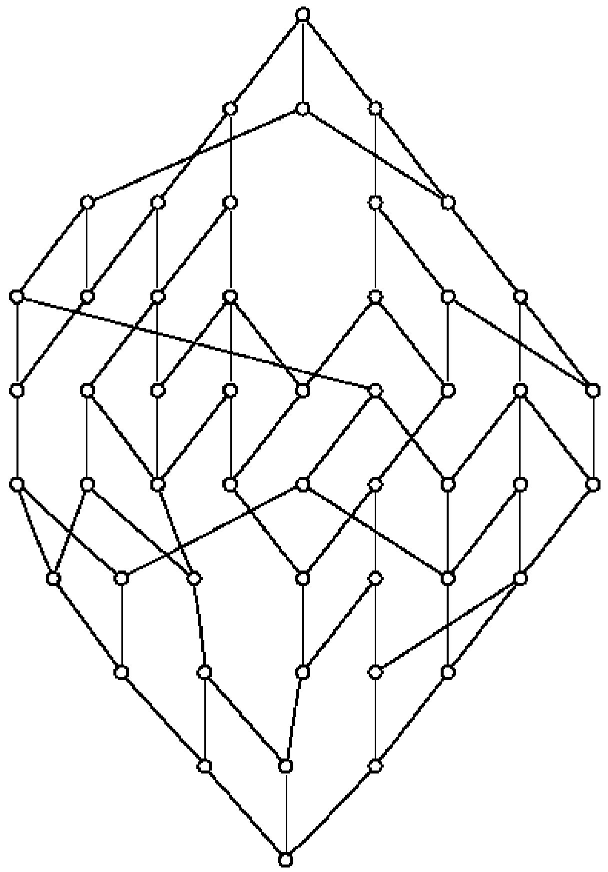


FIGURE 2. $\mathcal{P}_C(\mathcal{A}_2)$, the poset of regions of $\mathcal{A}_{-2}$, based at C (see figure 7.2.1).

Note: This can also be obtained as an application of theorem 3.2 of [**12**], because $\mathcal{A}_{-2}$ is a 3-dimensional arrangement and we start with a simplicial region. Nevertheless, we preferred to show the explicit picture.

7.3. Preliminaries

This section contains all necessary definitions and preparations to carry out the arguments of the next sections. The material is collected from [**12, 13, 102, 103**] and put into our framework. We want to point out that, especially in view of the notion of T-convexity explained below, it is convenient to return to the Salvetti-type models of chapter 4, that are tiled by the zonotope of the arrangement. This is mainly due to the immediate evidence of the contractibility of subcomplexes corresponding to T-convex sets, that are more general han the 'naturally contractible' subsets of the poset of regions, i.e., its intervals.

7.3.1. Two kinds of convexity. The next definition presents a concept that is closely related to the order of regions: the notion of *convexity* of a set of hyperplanes with respect to a given region. The point is that the convex hull of a set of hyperplanes may be the same with respect to more than one chamber.

Definition 7.3.1 (see [**12**]). *Let $\mathcal{A}$ be a real arrangement of hyperplanes with a distinguished chamber C. For any $H \in \mathcal{A}$ let H_C denote the open halfspace bounded by H and containing C. The* ***convex closure with respect to*** C *of any $X \subset \mathcal{A}$ is defined as*

$$\overline{X}^C := \left\{ H' \in \mathcal{A} \,\middle|\, \bigcap_{H \in X} H_C \subseteq H'_C \right\}.$$

A subset $X \in \mathcal{A}$ is called ***convex with respect to*** $\mathbf{C}$ *(or C-convex) if $\overline{X}^C = X$.*

There is also a (intuitive) notion of convexity on the set of chambers of an arrangement, usually called T-convexity. The original definition of this concept can be found in [**13**, 4.2.5] and is stated in terms of the tope graph of a general oriented matroid. The definition we give here is more tailored to our context and is justified by the equivalence given in [**13**, 4.1.6].

Definition 7.3.2. *Let $\mathcal{A}$ be a real arrangement of hyperplanes. For any $X \subset \mathcal{A}$ and any chamber $C \in \mathcal{C}(\mathcal{A})$ define*

$$\Delta_X^C := \left\{ C' \in \mathcal{C}(\mathcal{A}) \middle| S(C, C') \cap X = \emptyset \right\}.$$

A set of chambers $K \subset \mathcal{C}(\mathcal{A})$ is called ***T-convex*** *if $K = \Delta_X^C$ for some $X \subset \mathcal{A}$ and a $C \in K$.*

Remark 7.3.3. *To Δ_X^C naturally corresponds a subset of the $(d-1)$-sphere defined by*

$$\bigcup_{C \in \Delta_X^C} \overline{C} \subseteq S^{(d-1)}.$$

This region on the sphere is contractible whenever $X \neq \emptyset$ and never empty. We will sometimes abuse notations and denote this subset of the sphere by Δ_X^C too.

Note that T-convexity does *not* depend on any choice of base chamber, being rather a geometric notion. Indeed, a set of chambers is T-convex if and only if the union of their closures is a convex cone in $\mathbb{R}^d$.

Scholium 7.3.4. *By definition $\Delta_X^C = \Delta_{\overline{X}^C}^C$. Any two chambers in a T-convex set K can be joined by a path in the tope graph that visits only chambers in K. Any interval $[C_1, C_2]$ in the poset $\mathcal{P}_{C_0}$ (even when it is not a lattice) is T-convex because*

$$[C_1, C_2] = \Delta_{S(C_1, C_2)}^{C_1}.$$

Of course there is a connection between the two notions of convexity. To every T-convex set K is naturally associated a maximal $X \subset \mathcal{A}$ such that $K = \Delta_X^C$ for any $C \in K$, and this is convex with respect to any $C \in K$ by 7.3.4. Conversely, if a set $X \subset \mathcal{A}$ is convex with respect to some chamber C, then by definition of convexity any $H \notin X$ separates two chambers of the T-convex set Δ_X^C.

Remark 7.3.5 (On oriented matroids). *The reader familiar with the theory of oriented matroids will immediately see that all these concepts are easily defined for any oriented matroid. However, at this point we prefer not to leave the appealing intuition given by the framework of arrangements of hyperplanes.*

7.3.2. Zonotopal tiling of the universal cover complex. Recall the universal cover model of chapter 4 for a central essential complexified arrangement. We have seen (section 4.4 and corollary 4.5.3) that the poset $\mathrm{Plim}\widehat{\mathscr{D}}$ is the face poset of a cellular subdivision of this space and we wrote $[F,\gamma] = \Delta(\mathrm{Plim}_{\leq(F,\gamma)})$ for the cell that is indexed by the element $(F,\gamma) \in \mathrm{Plim}\widehat{\mathscr{D}}$. Every maximal cell can then be written $[P,\gamma]$, where P is the unique maximal element of the face poset $\mathcal{F}(\mathcal{A})$, and γ represents the homotopy class of some path in the 1-skeleton of the Salvetti complex starting at some arbitrarily chosen, fixed vertex C_0. Since $(\mathrm{Plim}\widehat{\mathscr{D}})_{\leq[P,\gamma]} \simeq \mathcal{F}$, the cell $[P,\gamma]$ corresponds to a copy the order complex $\Delta(\mathcal{F})$, which is isomorphic to the barycentric subdivision of the zonotope of the arrangement (compare remark 4.4.1).

The canonic orientation of the edges of the zonotope $[P,\gamma]$ corresponds to a *unique-source-unique-sink*-orientation of the tope graph of $\mathcal{A}$ with unique source $end(\gamma)$ and unique sink $-end(\gamma)$. Therefore we can identify the vertices of the zonotope with (homotopy classes of) positive paths starting at $end(\gamma)$. Namely, the minimal elements of $\mathrm{Plim}\widetilde{\mathscr{D}}$ that lie below $[P,\gamma]$ are of the form $[C,\gamma(end(\gamma)\to C)]$, one for every chamber of the arrangement. Conversely, it is clear that, given a path γ and a chamber C, the 0-cell $[C,\gamma(end(\gamma)\to C)]$ is a vertex of $[P,\gamma]$.

So, there is no loss of information in setting

$$\Omega(\gamma) := [P,\gamma]$$

and -when γ is understood- labeling its vertices simply by chambers.

We can then summarize some facts that are guaranteed by this setting:

REMARK 7.3.6. *Let $\gamma,\gamma_1,\gamma_2,\omega$ be paths in the oriented 1-skeleton of the Salvetti complex.*

- *The path ω represents a vertex of $\Omega(\omega)$ if $\omega \sim \gamma(end(\gamma)\to end(\omega))$.*
- *The intersection $\Omega(\gamma_1)\cap\Omega(\gamma_2)$ is the subcomplex spanned by all ω such that $\omega \sim \gamma_i(end(\gamma_i)\to end(\omega))$ for $i=1,2$, if there is any.*

In particular, from this we conclude that if there is $\omega \in \Omega(\gamma_1)\cap\Omega(\gamma_2)$, then

$$\Omega(\gamma_1)\cap\Omega(\gamma_2) \simeq \Delta^{end(\omega)}_{S(end(\gamma_1),end(\gamma_2))} \cap S^{d-1}.$$

7.3.3. On paths in $\mathcal{G}(\mathcal{A})$. We begin to see that we will have (again) to deal a lot with homotopy classes of paths in the Salvetti complex. Let us therefore state some convenient definitions.

DEFINITION 7.3.7. *Let $\mathscr{P}$ denote the set of equivalence classes of paths in the arrangement graph $G(\mathcal{A})$ that start at our fixed chamber C_0. Formally,*

$$\mathscr{P} = \bigcup_{C\in Ob(\mathcal{G}(\mathcal{A}))} Mor(C_0, C).$$

Each equivalence class will be represented by an edge path in the (CW version of the) Salvetti complex. Every such edge crosses exactly one face $F \in \mathcal{F}(F)$ of codimension 1. Therefore, for any $\gamma \in \mathscr{P}$ we may define

$$|\gamma| := \{H \in \mathcal{A} \mid H \text{ is crossed by } \gamma\} \subseteq \mathcal{A}.$$

In this chapter we denote by $\mathscr{P}^+$ the set of positive paths, i.e., the edge paths in the (directed) 1-skeleton of the Salvetti complex such that every edge is traversed along its orientation or, again, formally:

$$\mathscr{P}^+ = \bigcup_{C \in Ob(\mathcal{G}^+(\mathcal{A}))} Mor(C_0, C).$$

We represent a positive path as a word $l_1 l_2 \ldots l_k$, where each l_i stands for an oriented edge (we omit to specify the signs because we suppose anyway all edges being traversed in the positive sense). To each edge l is naturally associated a codimension 1 - face $F(l)$.

Note that from now we will abuse notation and write equality between paths whenever they correspond to the same morphism of the groupoid.

We follow [**102**] and to any positive path $\alpha := l_1 \ldots l_k$ we associate a word $W(\alpha)$ obtained by *deleting* from γ all l_i's such that

$$\sharp \left\{ l \text{ is a letter of } l_i l_{i+1} \ldots l_k \;\middle|\; |F(l)| = |F(l_i)| \right\} \equiv 1 \,(\mathrm{mod}\ 2).$$

Remark 7.3.8. Let $\alpha \in \mathscr{P}^+$, $W(\alpha) = l'_1 \ldots l'_m$, and recall the operation of faces on chambers of section 1.1, we can then write

$$\begin{aligned} \alpha \sim \big(C \to C_{F(l'_1)}\big)\big((-C)_{F(l'_1)} \to ((-C)_{F(l'_1)})_{F(l'_2)}\big) \ldots \\ \ldots \big(end(\alpha)_{F(l'_m)} \to end(\alpha)\big). \end{aligned}$$

Salvetti introduces these concepts in order to establish the following normal form for positive paths $\gamma \in \mathscr{P}$.

Definition 7.3.9. *For any positive path $\alpha = l_1 \ldots l_k \in \mathscr{P}$ let α_j denote the subpath $l_1, \ldots, l_{j-1}$. Then define*

$$^{\alpha}\beta_{F(l_j)} := \alpha_j (l_j)^2 \alpha_j^{-1}.$$

Lemma 7.3.10 (see [**102**], lemma 12). *Given a path $\alpha \in \mathscr{P}^+$ starting at C_0 and ending in C, let $W(\alpha) = l_1 \ldots l_\ell$. Then*

$$\alpha \sim {}^{\alpha}\beta_{F(l_\ell)} \ldots {}^{\alpha}\beta_{F(l_1)}(C_0 \to C).$$

We will sometimes shorten this expression writing $\beta_\alpha := {}^{\alpha}\beta_{F(l_\ell)} \ldots {}^{\alpha}\beta_{F(l_1)}$.

It has to be pointed out that for $\alpha \in \mathscr{P}^+$, the path β_α is in general *not* positive.

Since there is a clear correspondence between edges l and faces $F(l)$ we will allow ourselves to write β_F for $\beta_{F(l)}$ if there is no need to specify the edge.

We close this introductory part with a last definition and a corresponding remark.

Definition 7.3.11. *For any $\alpha \in \mathscr{P}^+$ we define*

$$\langle \alpha \rangle = \{ |F(l)| \mid l \text{ is a letter of } W(\alpha) \}.$$

Remark 7.3.12. *For a positive minimal path $\alpha = (C_1 \to C_2)$ we clearly have:*

$$|\alpha| = S(C_1, C_2),\ \langle \alpha \rangle = \emptyset\ .$$

7.4. The assumption

From now on, we will assume that the arrangement $\mathcal{A}$ is central (i.e., all hyperplanes contain the origin) and satisfies the

Weak lattice property: There is a chamber $C_0 \in \mathcal{C}(\mathcal{A})$ such that $\mathcal{P}_{C_0}(\mathcal{A})$ is a lattice.

We will call the arrangements with this property ***arrangements with a lattice of regions***. We will then choose C_0 as base point for a presentation of the fundamental group and thus as starting vertex for the paths in $\mathscr{P}$ and $\mathscr{P}^+$.

7.5. Local behavior

We want to investigate what happens in the (Salvetti-type) universal cover complex of a central arrangement $\mathcal{A}$ 'just around' a maximal cell indexed by a closed path starting at C_0. Our goal is to study the topology of the "big cells"

$$\mathbf{\Omega}(\gamma) := \bigcup_{C \in \mathcal{C}(\mathcal{A})} \Omega(\gamma(end(\gamma) \to C)).$$

It will be enough to consider the case $\gamma = id$. Thus let P denote the maximal element of $\mathcal{F}(\mathcal{A})$, set

$$\Omega(C_0) := [P, id],$$

and for $C \in \mathcal{P}_{C_0}(\mathcal{A})$ write

$$\Omega(C) := [P, (C_0 \to C)].$$

We want to study the intersection $\Omega(C_1) \cap \Omega(C_2)$ for $C_1, C_2 \in \mathcal{P}_{C_0}(\mathcal{A})$.

We set $\gamma_j := (C_0 \to C_j)$ for $j = 1, 2$. Every vertex $[C, \gamma] \in \Omega(C_j)$ is indexed by a path γ of the form $\gamma_j\omega_j$, where $\omega_j = (C_j \to C)$.

Because $\langle \gamma_j\omega_j \rangle = |\omega_j| \cap |\gamma_j|$, the edges in $W(\gamma_j\omega_j)$ are edges of γ_j, which is a positive minimal path. Therefore, putting $W(\gamma_j\omega_j) = l_1^j \dots l_{k_j}^j$ and $F_i^j := F(l_i^j)$, we have

$${}^{\gamma_j\omega_j}\beta_{F_i^j} = \big(C_0 \to (C_0)_{F_i^j}\big)\big(l_i^j\big)^2\big(C_0 \to (C_0)_{F_i^j}\big)^{-1}.$$

Therefore we loose no information in writing $\beta_{F_i^j} := {}^{\gamma_j\omega_j}\beta_{F_i^j}$.

By 7.3.10, we can write $\gamma_j\omega_j = \beta_j(C_0 \to C)$, with

$$\beta_j := \beta_{\gamma_j\omega_j} = \beta_{F_1^j} \dots \beta_{F_k^j}.$$

The assumption $\gamma_1\omega_1 \simeq \gamma_2\omega_2$ implies then $\beta_1 \simeq \beta_2$.

Now some technical lemma follow. The goal is to use proposition 7.5.5 and prove proposition 7.5.7, obtaining corollary 7.5.8, which says that the maximal faces 'just around' $\Omega(\gamma)$ form a contractible subcomplex, for any closed path γ.

LEMMA 7.5.1. *With the notations as above, we have*

$$\langle \gamma_1\omega_1 \rangle = \langle \gamma_2\omega_2 \rangle =: B.$$

Proof. With the presentation given in [**102**, pp. 614-615], we know that it must be possible to connect $\beta_1 = \beta_{F_1^1} \dots \beta_{F_{k_1}^1}$ with $\beta_2 = \beta_{F_1^2} \dots \beta_{F_{k_2}^2}$ using moves of the following two types:

(1) $\beta_{F_1} \dots \beta_{F_\ell} \sim \beta_{F'_{\sigma(1)}} \dots \beta_{F'_{\sigma(\ell)}}$ where $\sigma \in S_\ell$ and $|F_i| = |F'_{\sigma(i)}|$ for any i.

(2) $\beta_F = \left(\prod_{i\in I} \beta_{\tilde{F}_i}\right)^{-1} \beta_{F'} \left(\prod_{i\in I} \beta_{\tilde{F}_i}\right)$ where $|F'| = |F|$.

It is easy to see that for any hyperplane H, the *number of β_F's with $|F| = H$, counted with sign according to their exponent in the expression*, is left unchanged by both moves. More formally, the quantity

$$\sum_{\substack{\beta_F : |F|=H \text{ and} \\ \beta = \dots \beta_F^{e_1} \dots \beta_F^{e_l} \dots}} \left(\sum_{j=1..l} e_l \right)$$

remains constant under both moves for any hyperplane H.

From this the claim follows immediately: indeed, given any hyperplane H, in our β_j's there is at most one β_F with $|F| = H$. $\square$

LEMMA 7.5.2. *Keeping the notations as above,*

$$end(\gamma_1\omega_1) = end(\gamma_2\omega_2) \Rightarrow B \cap S(C_1, C_2) = \emptyset.$$

Proof. With lemma 7.5.1 we have that $H \in B \cap S(C_1, C_2)$ implies $H \in |\gamma_1| \cap |\omega_1|$ (so C is on the other side as C_1 with respect to H) and $H \in |\gamma_2| \cap |\omega_2|$ (so C is on the other side as C_2 with respect to H) this contradicts $H \in S(C_1, C_2)$. $\square$

For the rest of this section we set

$$X_1 := S(C_1, C_2) \setminus |\gamma_2|, \quad X_2 := S(C_1, C_2 \setminus |\gamma_1|).$$

The next lemma somehow justifies this definition.

LEMMA 7.5.3. *The set $|\omega_2| = S(C_2, end(\omega_2))$ contains X_2. Reciprocally, $X_1 \subset |\omega_1| = S(C_1, end(\omega_1))$.*

Proof. Draw a Venn-diagram of the four sets $|\gamma_1|$, $|\gamma_2|$, $|\omega_1|$, $|\omega_2|$ 'in general position' (see figure below), and check that:

• The parts A are empty because $|\gamma_1| \cap |\omega_1| = |\gamma_2| \cap |\omega_2|$, and what remains is then equal to B.

• The parts C are empty because $|\gamma_1| \cup |\omega_1| = |\gamma_2| \cup |\omega_2|$.

• The hyperplanes in $Y \subseteq (|\omega_1| \cap |\omega_2|) \cup (|\gamma_1| \cap |\gamma_2|)$ clearly do not separate C_1 and C_2.

We then obtain:

$$S(C_1, C_2) = \underbrace{[(\gamma_1 \cap \omega_2) \setminus (\gamma_2 \cap \omega_1)]}_{=X_1} \uplus \underbrace{[(\omega_1 \cap \gamma_2) \setminus (\omega_2 \cap \gamma_1)]}_{=X_2}.$$

$\square$

We can now make a first statement on the vertices that are candidate to lie in $\Omega(C_1) \cap \Omega(C_2)$.

LEMMA 7.5.4. *Let $(C, \gamma) \in \Omega(C_1) \cap \Omega(C_2)$, so that $C = end(\gamma)$ and $\gamma = \gamma_1\omega_1 = \gamma_2\omega_2$. Then*

$$C \subseteq \Delta_{X_2}^{(-C_0)} \cap \Delta_{X_1}^{(-C_0)} = \Delta_{X_1 \cup X_2}^{-(C_0)}.$$

Proof. The previous lemmas say that C has to satisfy:

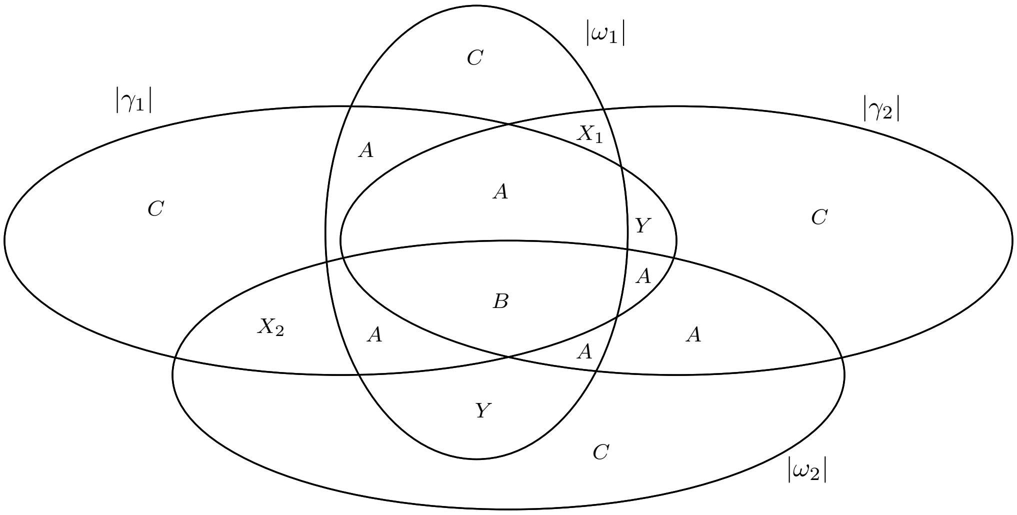


FIGURE 3. The Venn diagram for the proof of lemma 7.5.3.

(1) $\mathbf{X_2 \subseteq S(C_2, C)}$: Since every element X_2 separates $-C_0$ from C_2, this is equivalent to say that $C \subseteq \Delta_{X_2}^{(-C_0)}$.
(2) $\mathbf{X_1 \subseteq S(C_1, C)}$: Same reasoning as above, implying $C \subseteq \Delta_{X_2}^{(-C_0)}$.

□

For the next steps we need a general fact about the combinatorics of arrangements with a lattice of regions.

PROPOSITION 7.5.5. *Let $\mathcal{A}$ be a real arrangement of hyperplanes, and suppose that for some chamber C_0 the poset of regions $\mathcal{P}_{C_0}(\mathcal{A})$ is a lattice. Consider two chambers $C_1, C_2 \in \mathcal{P}_{C_0}(\mathcal{A})$ and for $X \subseteq \mathcal{A}$ let $\overline{X}^{C_0}$ denote the convex closure of X with respect to the orientation determined by letting C_0 be on the positive side of each hyperplane. Then*

$$\overline{S(C_1, C_2)}^{C_0} = S(C_1 \wedge C_2, C_1 \vee C_2).$$

Proof. Clearly $S(C_1, C_2) \subset S(C_0, C_1 \vee C_2) \cap S(-C_0, C_1 \wedge C_2)$, and both sets on the right side are convex w.r.t C_0. The set $S(C_1 \wedge C_2, C_1 \vee C_2)$ is C_0-convex and contains $S(C_1, C_2)$, so $\overline{S(C_1, C_2)}^{C_0} \subseteq S(C_1 \wedge C_2, C_1 \vee C_2)$.

Suppose now that $S(C_1 \wedge C_2, C_1 \vee C_2)$ is *not* the convex hull of $S(C_1, C_2)$. Since $S(C_1, C_2) \subset S(C_1 \wedge C_2, C_1 \vee C_2)$, we may consider the arrangement $\mathcal{A}' := S(C_1 \wedge C_2, C_1 \vee C_2)$, noting that $X \subset \mathcal{A}'$ is convex in $\mathcal{A}'$ w.r.t. $C_1 \wedge C_2$ if and only if it is convex in $\mathcal{A}$ w.r.t. C_0. Moreover, remark that $\mathcal{P}_{C_1 \wedge C_2}(\mathcal{A}') = [C_1 \wedge C_2, C_1 \vee C_2] \subseteq \mathcal{P}_{C_0}(\mathcal{A})$.

By hypothesis there is $H \in \mathcal{A}' \setminus \overline{S(C_1, C_2)}^{C_0}$. By definition, this means that there is a chamber C' of $\mathcal{A}'$ such that

$$\text{i) } H \in S(C_1 \wedge C_2, C') \text{ and}$$
$$\text{ii) } S(C_1 \wedge C_2, C') \cap \overline{S(C_1, C_2)}^{C_0} = \emptyset.$$

The hyperplane H cannot separate C_1 from C_2, and therefore, since it separates C' from $C_1 \wedge C_2$, either both C_1 and C_2 are on the same side of H as C' or on the same side of H as $C_1 \wedge C_2$.

The first possibility cannot occur, since it would contradict the definition (in particular: the maximality) of $C_1 \wedge C_2$ in $\mathcal{P}_{C_0}(\mathcal{A})$. Indeed, by ii) we have that C_1 and C_2 lie in

$\Delta^{C_1 \vee C_2}_{S(C_1 \wedge C_2, C')}$, and therefore there are positive minimal paths $\alpha_i := (C' \to C_i)$ that do not cross any hyperplane in $S(C_1 \wedge C_2, C')$. On the other side, any positive minimal path $\alpha' := (C_1 \wedge C_2 \to C')$ crosses only (and exactly) the hyperplanes in $S(C_1 \wedge C_2, C')$. So the path $\alpha\alpha_i = (C_1 \wedge C_2 \to C')(C' \to C_i)$ is positive and minimal for $i = 1, 2$. This implies that $S(C_1 \wedge C_2, C_i) = |\alpha| \cup |\alpha_i| \supseteq S(C_1 \wedge C_2, C')$, hence $C' < C_i$ for both i. This manifestly contradicts the maximality of $C_1 \wedge C_2$ because, by i), $S(C_1 \wedge C_2, C') \neq \emptyset$.

So we are left with the second possibility, where for both $i = 1, 2$ we have $H \notin S(C_1 \wedge C_2, C_i)$, and therefore $H \in \bigcap_{i=1,2} S(C_i, C_1 \vee C_2)$. We can now consider the chamber C'' opposite to C' in $\mathcal{A}'$, thus finding by duality that $H \in S(C'', C_1 \vee C_2)$ and $S(C_1 \vee C_2, C'') \cap \overline{S(C_1, C_2)} = \emptyset$. Moreover, by the previous considerations C'' is on the same side as both C_i's with respect to H. This gives then a 'dual' contradiction to the previous one. In any case, such H cannot exist, and the claim follows. □

LEMMA 7.5.6. *For any $C \subset \Delta_{X_1 \cup X_2}$ we have $S(C_2, C_2 \vee C_1) \subseteq S(C_2, C)$ and $S(C_1, C_2 \vee C_1) \subseteq S(C_1, C)$.*

Proof. Note that $X_1 \cup X_2 = S(C_1, C_2)$.

Now apply the previous proposition and note that, by general calculation rules,

$$\Delta^{(-C_0)}_{\overline{X_1 \cup X_2}(-C_0)} \subset \Delta^{(-C_0)}_{S(C_2, C_2 \vee C_1)} = \Delta^{(-C_2)}_{S(C_2, C_2 \vee C_1)}.$$

The first claim follows immediately, the second by simmetry. □

We are now prepared to prove the following characterization of the intersection of the two maximal cells $\Omega(C_1)$, $\Omega(C_2)$.

PROPOSITION 7.5.7. *Suppose that the real arrangement $\mathcal{A}$ has a chamber C_0 such that $\mathcal{P}_{C_0}(\mathcal{A})$ is a lattice. With the notations above we have*

$$\Omega(C_1) \cap \Omega(C_2) = \Delta^{(-C_0)}_{S(C_1, C_2)} \cap \partial S^{d-1}.$$

Proof. After the previous lemmas only the right-to-left inclusion needs to be proven. So let (C, γ) be a vertex of the zonotope $\Omega(C_2)$, and suppose that $C \subset \Delta^{(-C_0)}_{X_1 \cup X_2} = \Delta^{(-C_0)}_{S(C_1, C_2)}$. Let $C' := C_1 \vee C_2$, the join taken in $\mathcal{P}_{C_0}$. Then by definition $\gamma = \gamma_2 \omega$, where ω is positive minimal. By assumption we have $C \geq C'$ in $\mathcal{P}_{C_2}$, so we can write $\omega = (C_2 \to C' \to C)$. In the lattice $\mathcal{P}_{C_0}$ also $C'' := C_1 \wedge C_2$ is defined, and we can write $\gamma = \gamma_2 \omega = (C_0 \to C'')(C'' \to C_2)(C_2 \to C')(C' \to C)$.
Now $C'' \leq C'$ in $\mathcal{P}_{C_0}$ and so $(C'' \to C_1)(C_1 \to C') \sim (C'' \to C_2)(C_2 \to C')$ in the interval $[C'', C']$.
We have that $(C_1 \to C')(C' \to C)$ is positive minimal since we supposed $C \subset \Delta^+_{X_1}$. Therefore $(C, \gamma) = (C, (C_0 \to C'')(C'' \to C_2)(C_2 \to C')(C' \to C)) \in \Omega(C_1)$. □

COROLLARY 7.5.8. *For any $\gamma \in \pi_1(\mathcal{A})$ the subcomplex*

$$\mathbf{\Omega}(\gamma) := \bigcup_{C \in \mathcal{C}(\mathcal{A})} [P, \gamma(C_0 \to C)]$$

is contractible.

Proof. Since π_1 operates transitively by automorphisms on the universal cover it is enough to consider the case $\gamma = id$. In this case we have, with the previous notations,

$$\mathbf{\Omega}(id) = \bigcup_{C \in \mathcal{C}(\mathcal{A})} \Omega(C).$$

Now fix $C \in \mathcal{C}$, and note the following two facts:

• *If $C < C_1 < C_2$, then $\Omega(C) \cap \Omega(C_1) \subset \Omega(C) \cap \Omega(C_2)$.*
Proof: By proposition 7.5.7 $\Omega(C) \cap \Omega(C_j) = \Delta^{-C_0}_{S(C,C_j)} \cap \partial S^{d-1}$, and $C < C_1 < C_2$ means $S(C, C_1) \subset S(C, C_2)$, which implies $\Delta^{-C_0}_{S(C,C_2)} \subset \Delta^{-C_0}_{S(C,C_1)}$.

• *If $C' \not> C$, then $\Omega(C') \cap \Omega(C) \subset \Omega(C \wedge C') \cap \Omega(C)$.*
Proof: We have clearly $S(C', C) \supset S(C' \wedge C, C)$, which allows us to apply again proposition 7.5.7 as above.

Choose any linear extension $C_0, C_1, \ldots C_m = -C_0$ of the ordering of $\mathcal{P}_{C_0}$, and for any $j = 1, \ldots, m$ consider

$$\Omega(C_j) \cap \left[\bigcup_{i<j} \Omega(C_i) \right] = \bigcup_{i<j} \big(\Omega(C_j) \cap \Omega(C_i) \big),$$

For all $i < j$ we have $C_i \not> C_j$, and for all $C < C_j$ there is an i with $C = C_i$. Therefore we may apply the previous observations, concluding

$$\bigcup_{i<j} \big(\Omega(C_j) \cap \Omega(C_i) \big) \stackrel{ii)}{=} \bigcup_{C < C_j \text{ in } \mathcal{P}_{C_0}} \Omega(C) \cap \Omega(C_j) \stackrel{i)}{=} \bigcup_{C \prec C_j \text{ in } \mathcal{P}_{C_0}} \Omega(C) \cap \Omega(C_j).$$

The latter is a union of halfspheres, hence contractible (and always full-dimensional).

Therefore any extension if the ordering of $\mathcal{P}_{C_0}$ gives a shelling of $\mathbf{\Omega}(id)$ with no homology facets. $\square$

The latter result suggests to consider the covering $\{\mathbf{\Omega}(\gamma)\}_{\gamma \in \pi_1}$ of the universal cover complex. We will do so in the next section.

7.6. Regional behavior I: incidence

In [**103**] Salvetti gives a systematic and categorical derivation of the properties of the 'calculus of positive words' coming from the general situation of an arrangement of pseudohemispheres. We will refer to that paper for a complete exposition, and only recall some basic facts.

The main idea of this section is to consider the following norm on positive paths, that is inspired by the atomnorm of [**6**].

Definition 7.6.1. *Any positive path $\gamma \in \mathscr{P}^+$ can be written as a sequence $\alpha_1, \ldots, \alpha_k$ of positive minimal paths. The smallest k for which this is possible will be referred to as the* ***norm*** *$N(\gamma)$ of the path. To any chamber C in a central arrangement we can associate its opposite chamber, usually denoted $-C$. Accordingly, to any positive minimal path $\alpha = (C_1 \to C_2)$ we associate a path $\alpha^* := (-C_1 \to -C_2)$. For any chamber C we define $\delta_C := (C \to -C)$. If no confusion can arise about the starting chamber, we will just write δ. In particular, given a path γ we have $\gamma\delta = \gamma(end(\gamma) \to -end(\gamma))$.*

Note that with remark 7.3.8 and definition 7.3.11 we know that $N(\gamma)$ is at most the cardinality of the set $\langle\gamma\rangle$ (or the length of the word $W(\gamma)$).

The following fact is well-known.

LEMMA 7.6.2. (**[36, 103, 97]**) *Let γ_1, γ_2 be positive paths with the same beginning point, and suppose one can write $\gamma_1 = \alpha_1 \dots \alpha_k$ for positive minimal paths α. Then*

- $\gamma_1 \delta = \delta \alpha_1^* \dots \alpha_k^*$,
- *There is a positive path ϵ and $k \in \mathbb{N}$ such that $\gamma_1 \delta^k = \gamma_2 \epsilon$.*

This is already enough to investigate the incidence relations between the pieces $\mathbf{\Omega}(\gamma_i)$ in which we want to subdivide our (positive part ofthe) universal cover complex.

PROPOSITION 7.6.3. *Let γ_1, γ_2 be positive closed paths that start at a common chamber C_0. Suppose $N(\gamma_1) \leq N(\gamma_2)$. Then*

$$\mathbf{\Omega}(\gamma_1) \cap \mathbf{\Omega}(\gamma_2) \neq \emptyset \textit{ if and only if } \gamma_1 \delta^2 = \gamma_2 \epsilon \textit{ for an } \epsilon \in \mathscr{P}^+ \textit{ with } N(\epsilon) \leq 4.$$

Proof. If the intersection of the $\mathbf{\Omega}(\gamma_i)$ is not empty, then by definition there are positive minimal paths α_i, ω_i such that $\gamma_1 \alpha_1 \omega_1 = \gamma_2 \alpha_2 \omega_2$. Therefore if we consider the positive paths $\kappa' := \omega_1^{-1} \delta$, $\kappa'' := (\alpha_1^*)^{-1} \delta$ we can write $\gamma_1 \delta \alpha_1^* = \gamma_1 \alpha_1 \delta = \gamma_2 \alpha_2 \omega_2 \kappa'$, and $\gamma_1 \delta^2 = \gamma_2 \alpha_2 \omega_2 \kappa' \kappa''$, whence the claim with $\epsilon = \alpha_2 \omega_2 \kappa' \kappa''$ (in this case obviously $N(\epsilon) \leq 4$). Conversely, suppose that there is a positive path ϵ with $\gamma_1 \delta^2 = \gamma_2 \epsilon$. Write $\epsilon = \epsilon_1 \dots \epsilon_k$ as a product of positive minimal paths. Then ϵ_k is the ending of the last δ in $\gamma_1 \delta^2$, and therefore there is a positive path $\alpha := \epsilon_k^{-1} \delta$ such that $\gamma_1 \delta \alpha = \gamma_2 \epsilon_1 \dots \epsilon_{k-1}$. We note that $\gamma_1 \delta \alpha = \gamma_1 \alpha^* \delta$ and repeat the previous argument to get a positive path ω such that $\gamma_1 \alpha^* \omega = \gamma_2 \epsilon_1 \dots \epsilon_{k-2}$. Now remark that we also supposed $k \leq 4$. This concludes the proof. $\square$

Actually we can extract even more information from the preceding proposition. We formulate it for later reference.

SCHOLIUM 7.6.4. *Consider two positive closed paths γ_1 and γ_2 ordered such that $N(\gamma_1) \leq N(\gamma_2)$.*

If $\mathbf{\Omega}(\gamma_1)$ and $\mathbf{\Omega}(\gamma_2)$ have nonempty intersection, then $|N(\gamma_1) - N(\gamma_2)| \leq 2$.

Proof. We may assume that, without loss of generality, $N(\gamma_1) \leq N(\gamma_2)$. We begin by observing that, by proposition 7.6.3, if the intersection is not empty, then we may write $\gamma_1 \delta^2 = \gamma_2 \epsilon$. By definition of δ we then have $N(\gamma_1 \delta^2) = N(\gamma_1) + 2$. On the other side all we can say is $N(\gamma_2) \leq N(\gamma_2 \epsilon) \leq N(\gamma_2) + N(\epsilon)$. We can combine the two relations because $N(\gamma_1 \delta^2) = N(\gamma_2 \epsilon)$. Solving for $N(\epsilon)$ we have

$$\begin{aligned} &\text{i)} \qquad 0 \leq N(\gamma_1) - N(\gamma_2) + 2 \\ &\text{ii)}\ N(\epsilon) \geq N(\gamma_1) - N(\gamma_2) + 2 \end{aligned} \tag{6.1}$$

Already the first of these two inequalities gives $2 \geq N(\gamma_2) - N(\gamma_1) \geq 0$ (the latter inequality holds by assumption of the numbering of the γ_i's). $\square$

The above version of proposition 7.6.3 is stated in terms of positive closed paths. But it is a standard fact that *any* path starting at some vertex v of the universal cover complex has the same ending as a positive path starting at $\delta^n v$ for some n (see our lemma 6.4.1). Therefore we can generalize proposition 7.6.3 to the following corollary:

COROLLARY 7.6.5. *Let γ_1, γ_2 be two closed paths (hence elements of $\pi(\mathcal{A})$) based at the chamber C_0. Choose some vertex γ_0 that is a lift of C_0 to the 1-skeleton of the universal cover complex. Then there is $m \in \mathbb{N}$ such that both γ_1 and γ_2 are represented by*

positive closed paths γ_1^+, γ_2^+ *if* $\gamma_0\delta^m$ *is chosen as the lift of the identity path.*
The intersection $\mathbf{\Omega}(\gamma_1)\cap\mathbf{\Omega}(\gamma_2)$ *equals the intersection* $\mathbf{\Omega}(\gamma_1^+)\cap\mathbf{\Omega}(\gamma_2^+)$ *after change of the lift of the identity by* δ^m. *In particular it is not empty precisely when the conditions of proposition 7.6.3 are satisfied on* γ_1^+ *and* γ_2^+.

We will therefore carry on our considerations taking only positive paths and loops into account. Let us list some observations in the next remark

REMARK 7.6.6. Consider two positive closed paths γ_1, γ_2 satisfying the conditions of proposition 7.6.3, and assume w.l.o.g. that $N(\gamma_2)\geq N(\gamma_1)$. It is clear that

(i) If $N(\epsilon)=0$, then the $\mathbf{\Omega}(\gamma_i)$'s intersect only in a single vertex, namely $\gamma_2=\gamma_1\delta^2$.
(ii) If $N(\gamma_2)-N(\gamma_1)=2$, then $N(\epsilon)\leq 1$ and ϵ is positive minimal. But then $\omega:=\delta\epsilon^{-1}$ is positive, and we have $\gamma_1\delta\omega=\gamma_2$, which is only possible if $\omega=\delta$ because both have at least to end at the same chamber, and this is C_0. So $\epsilon=id$, and $N(\epsilon)=0$.

7.7. Regional behavior II: topology of the intersections.

We want to investigate how the intersections of the blocks behave with respect to the norm of ϵ. The idea is to point towards an application of the nerve lemma to the covering by the $\mathbf{\Omega}(\gamma)$.

REMARK 7.7.1. *We introduce the last piece of notation. Given a positive minimal path* $\epsilon=(A\to B)$, *we will write* $\overline{\epsilon}:=(-B\to A)$.

7.7.1. $\mathbf{N}(\epsilon)=\mathbf{0}$. The only positive path with zero norm is the void path. Therefore we have $\gamma_1\delta^2=\gamma_2$ and clearly the intersection is a single point:

$$\mathbf{\Omega}(\gamma_1)\cap\mathbf{\Omega}(\gamma_2)=\{\gamma_2\}.$$

7.7.2. $\mathbf{N}(\epsilon)=\mathbf{1}$. This would mean that there is a (nonzero, i.e. $|\epsilon|\neq\emptyset$) positive minimal path ϵ with $\gamma_1\delta^2=\gamma_2\epsilon$.
But we know that for a positive minimal path $|\epsilon|=S(begin(\epsilon),end(\epsilon))$, in particular if ϵ is not the zero path we must have $end(\epsilon)\neq begin(\epsilon)=$. On the other hand, $end(\epsilon)=end(\gamma_2\epsilon)=end(\gamma_1\delta^2)=end(\gamma_1)$. Both γ_i are loops, therefore have the same ending, namely C_0. It follows that $end(\epsilon)=C_0=end(\gamma_2)=begin(\epsilon)$, contradicting $N(\epsilon)=1$. This case does therefore not enter.

7.7.3. $\mathbf{N}(\epsilon)=\mathbf{2}$. Write $\epsilon=\epsilon_1\epsilon_2$ with both ϵ_i positive minimal. Then by remark 7.7.1 there are positive minimal paths $\overline{\epsilon}_i^*$ such that $\delta=\overline{\epsilon}_i^*\epsilon_i$. In particular $\delta\epsilon_i^{-1}$ is positive minimal, and we can write

$$\gamma_1\delta^2=\gamma_2\epsilon_1\epsilon_2\Rightarrow\gamma_1\bar{\epsilon}_2\delta=\gamma_1\delta\bar{\epsilon}_2^*=\gamma_2\epsilon_1\Rightarrow\gamma_1\bar{\epsilon}_2\bar{\epsilon}_1=\gamma_2.$$

We conclude $\gamma_2\in\mathbf{\Omega}(\gamma_1)$. By an analogous argumentation as in the case $N(\epsilon)=1$ one sees that if $\bar{\epsilon}_2=l_1\dots l_k$, then $\bar{\epsilon}_1=l_k\dots l_1$. Let us denote by C_ϵ the chamber where $\bar{\epsilon}_2$ ends.
If there is an element of $\mathbf{\Omega}(\gamma_1)\cap\mathbf{\Omega}(\gamma_2)$ of the form $\gamma_2\alpha$ for a positive minimal path α, then we must have $N(\bar{\epsilon}_2\bar{\epsilon}_1\alpha)\leq 2$. This is only possible if α does not cross any of the hyperplanes $|F_{l_i}|$. Therefore $end(\alpha)\in\mathcal{P}_{C_0}(\mathcal{A})_{\leq C_\epsilon}$. Conversely, if α is the beginning of some positive minimal path from C_0 to C_ϵ, then $\bar{\epsilon}_1\alpha$ is positive minimal and therefore $\gamma_2\epsilon\in\mathbf{\Omega}(\gamma_1)\cap\mathbf{\Omega}(\gamma_2)$.
Now consider an element in the intersection of the $\mathbf{\Omega}(\gamma_i)'s$ that has the form $\gamma_2\alpha\omega$ with

α, ω positive minimal. Then $N(\bar{\epsilon}_2\bar{\epsilon}_1\alpha\omega) \leq 2$, and therefore $N(\bar{\epsilon}_2\bar{\epsilon}_1\alpha) \leq 2$ By the previous argument then α is the beginning of some positive minimal path from C_0 to $-C_\epsilon$.

We want to emphasize following fact

SCHOLIUM 7.7.2. *Let ω denote a positive minimal path that is contained in $\mathbf{\Omega}(\gamma)$ for some γ. Then there exist a positive minimal path α such that ω is entirely contained in $\Omega(\gamma\alpha)$.*

Proof. If ω is contained in $\mathbf{\Omega}(\gamma)$, then there is a positive path σ such that $end(\gamma\sigma) = begin(\omega)$, and $\sigma\omega \in \mathbf{\Omega}(\gamma)$ has norm less than 2. We can therefore write $\sigma\omega = \alpha_1\alpha_2$ with positive minimal α_i's, and note that $\langle\sigma\omega\rangle = |\alpha_1| \cap |\alpha_2|$. let then α be the subpath of α_1 ending right after the last letter that is in $|\alpha_1| \cap |\alpha_2|$. Then one can write $\sigma\omega$ as $\alpha\tau$ for some positive path τ. Now note that τ is minimal, because it crosses only once every hyperplane that is crossed twice by $\sigma\omega$, and if it would cross some hyperplane twice, then the same would hold for $\sigma\omega$. Further, obviously $end(\alpha) < end(\sigma)$ in $\mathcal{P}_{C_0}$, since α is subpath of $\sigma\omega$ with length $< \sigma$ because all $l \in \langle\sigma\tau\rangle$ are already in $|\sigma|$. Therefore we know that $\sigma = \alpha\sigma'$, with σ' positive minimal. now remark that $\sigma'\omega$ is positive minimal by the same reasons as τ above, and therefore ω is contained in $\Omega(\gamma\alpha)$. $\square$

Now we go back to our situation, where we want to determine all possible paths ω that are positive minimal, and for which $\gamma_1\bar{\epsilon}_2\bar{\epsilon}_1\alpha\omega \in \mathbf{\Omega}(\gamma_1)$. After the previous considerations, there must be a positive minimal path σ with $\omega \subset \Omega(\gamma_1\sigma)$.

We already know that $\bar{\epsilon}_1\alpha$ is positive minimal, and $\bar{\epsilon}_1$ is fixed. Let E denote the set of hyperplanes crossed by $\bar{\epsilon}_1$. Then the ending of $\gamma_1\bar{\epsilon}_2\bar{\epsilon}_1\alpha$ is in $\Omega(\gamma_1\bar{\epsilon}_2) \cap \Delta_E^{-C_\epsilon}$, which we denote by $\Omega(C_\epsilon)_E$ for brevity. So our candidates for σ have to satisfy the condition that $\Omega(\gamma_1\sigma) \cap \Omega(C_\epsilon)_E \neq \emptyset$. We know $\Omega(\gamma_1\sigma) \cap \Omega(\gamma_1\bar{\epsilon}_2) = \Omega(\gamma_1\bar{\epsilon}_2) \cap \Delta^{-C_0}_{S(end(\sigma),C_\epsilon)}$. The condition reduces therefore to

$$\Omega(\gamma_1\bar{\epsilon}_2) \cap \Delta^{-C_0}_{S(end(\sigma),C_\epsilon)} \cap \Delta_E^{-C_\epsilon} \neq \emptyset,$$

which is true if and only if $S(-C_0, -C_\epsilon) \cap (S(end(\sigma), C_\epsilon) \cap E) = \emptyset$. Now remark that $S(-C_0, -C_\epsilon) = E$: the condition is then equivalent to $S(end(\sigma), C_\epsilon) \cap E = \emptyset$, meaning $end(\sigma) \geq C_\epsilon \in \mathcal{P}_{C_0}$.

After all this, we know that $\gamma_1\bar{\epsilon}_2\bar{\epsilon}_1\alpha\omega$ must be a vertex in some $\Omega(C')_E$, by which we mean $\Omega(\gamma_1(C_0 \to C')) \cap \Delta_E^{C_0}$, with $C' > C_\epsilon$ in $\mathcal{P}_{C_0}$ (strict inequality comes from the assumption that $\alpha\omega$ is not positive minimal). It is easy to see that any such vertex is contained in the intersection of the $\mathbf{\Omega}(\gamma_i)$'s. So we know

$$\mathbf{\Omega}(\gamma_1) \cap \mathbf{\Omega}(\gamma_2) = \bigcup_{C' \geq C\epsilon} \Omega(C')_E.$$

Now the same idea as in the proof of corollary 7.5.8 works to show that this intersection is contractible. Namely, any linear extension of the ordering of $\mathcal{P}_{C_0}(\mathcal{A})_{\geq C_\epsilon}$ gives a shelling of $\mathbf{\Omega}(\gamma_1) \cap \mathbf{\Omega}(\gamma_2)$ without homology facets. $\square$

7.7.4. N=3,N=4. Disappointingly enough, I could not prove the contractibility of the intersections in these last two cases. It might be that the assumption to have "enough coloured regions" could help at this point, but it is not clear to me whether it is necessary or not.

The reader that had the endurance to follow our considerations up to this point is invited to try to turn the disappointment into a mathematical challenge.

Interlude

CHAPTER 8

Nested set complexes of semilattices

The theory of building sets, nested set complexes and combinatorial blowups for general semilattices was initiated and developed by Feichtner and Kozlov, [**60**], as the combinatorial framework of the *wonderful models for subspace arrangements* by De Concini and Procesi [**32**] (see the survey [**57**] for a very readable and precise account of these models and their combinatorics).

A complex of *nested sets* is associated to any meet-semilattice and any of its *building sets* - i.e., subsets of the semilattice satisfying some conditions that are inspired by the special properties of the set of irreducible elements in a geometric lattice.
Although this theory stems from the world of hyperplane arrangements, the study of such structures was carried out in different contexts (see [**61, 63, 59, 53, 42, 62, 31**]), leading to new results or sharpening the understanding of previously studied objects. In particular, this theory has proved to be a very useful tool for the study of *complexes of trees* of various kind (see [**53, 42, 41, 1**]) and in gaining conceptual understanding and generalizations of topological facts that were regarded as interesting coincidences ([**31**]).

In this chapter we want to introduce the essential toolkit needed for the development of the second part of this thesis.

8.1. Some basics on posets

For sake of self-containedness and in order to fix notations, let us recall some basic notions on posets and their topology. For a comprehensive account of this subject see [**10**].

Let $(P, \leq)$ be a partially ordered set, for short called *poset*. We call P *bounded* if there exist elements $\hat{0}, \hat{1} \in P$ so that $\hat{0} \leq x \leq \hat{1}$ for all $x \in P$.

We will write $\overline{P}$ for the proper part of P, that is $\overline{P} = P \setminus \{\hat{0}, \hat{1}\}$. Also, let $P_{\leq x} = \{y \in P \mid y \leq x\}$. Similarly, for any $\mathcal{G} \subseteq P$ with order induced by P, let us write $\mathcal{G}_{\leq x} = \{y \in P \mid y \in \mathcal{G},\ y \leq x\}$. We say that y *covers* x if $y > x$ and there is no $z \in P$ so that $y > z > x$; in this case we will write $x \lessdot y$. For $x \leq y$, the *interval* $\{z \in P \mid x \leq z \leq y\}$ is denoted by $[x, y]$. A subset $I \subseteq P$ is an *ideal* in P if, for all $y \in I$ and $x \in P$, if $x \leq y$, then $x \in I$. An ideal I is called *proper* if $I \neq P$ and $I \neq \emptyset$.

With $\Delta(P)$ we will denote the *order complex* of the given poset P: the abstract simplicial complex whose vertices are the elements of P, and faces are all chains in P (including the empty chain). We will assume that the empty face is an element of every non-empty abstract simplicial complex.

We will almost never distinguish between an abstract simplicial complex and its geometric realization, thus speaking of 'topological properties' of an abstract simplicial complex. It is easily seen that if P possesses a maximal element (that is customarily denoted $\hat{1}$), then $\Delta(P)$ is a cone over $\Delta(P \setminus \{\hat{1}\})$. An analogous statement holds of course if P has a minimal element (usually denoted $\hat{0}$). To capture the 'essential' topological information we define the *reduced order complex* $\widetilde{\Delta}(P)$ as the order complex of the poset obtained from P by removing the maximal and the minimal elements, if P has any.

A poset $\mathcal{L}$ is called a *meet-semilattice*, or simply *semilattice*, if every pair of elements $x, y \in \mathcal{L}$ has a unique maximal lower bound, which is called *meet* of these two elements, and is denoted by $x \wedge y$. All semilattices have a unique minimal element called $\hat{0}$, and for any $A = \{a_1, \cdots, a_t\} \subseteq \mathcal{L}$, the set $\{x \in \mathcal{L} \mid x \geq a_i,\ \text{for all } i \in [t]\}$ is either empty or it has a unique minimal element, its *join*, $\bigvee A = a_1 \vee \cdots \vee a_t$. Finally, a semilattice L is a *lattice* if meet and join are defined for any pair of elements of L.

8.2. Building sets and nested set complexes

We recall here the basic definitions and refer to [**60**] for a comprehensive introduction.

Definition 8.2.1. ([**60**, Definition 2.2]) *Let $\mathcal{L}$ be a semilattice. A subset $\mathcal{G}$ of $\mathcal{L}\setminus\{\hat{0}\}$ is called a* ***building set*** *of $\mathcal{L}$ if for any $x \in \mathcal{L} \setminus \{\hat{0}\}$ and $\max \mathcal{G}_{\leq x} = \{x_1, \ldots, x_t\}$, there is an isomorphism of posets*

$$\psi_x : \prod_{i=1}^{t} [\hat{0}, x_i] \longrightarrow [\hat{0}, x],$$

so that $\psi_x(\hat{0}, \ldots, \hat{0}, x_i, \hat{0}, \ldots, \hat{0}) = x_i$, for all $i \in [t]$.

Given a building set $\mathcal{G}$, the set $F_{\mathcal{G}}(x) := \max \mathcal{G}_{\leq x}$ is called the set of ***factors of* x *in*** *$\mathcal{G}$. We may sometimes drop the suffix and write only $F(x)$ if no confusion can arise.*

The following remark is important, and in fact it was the starting point for the attempt of generalization that we report on in chapter 11.

Remark 8.2.2. Building sets can be defined as in [**60**, 2.3 (2)], independently of the existence of a join operation, for any poset with a unique minimal element. If a join operation exists in $\mathcal{L}$, the isomorphism φ_x can always be chosen such that $\varphi_x(z_1, \ldots, z_n) = z_1 \vee \cdots \vee z_n$ [**60**, Proof of 2.3].

Remark 8.2.3. In any semilattice there is a unique minimal building set, given by the set of join-irreducible elements (see [**60**]). The maximal building set is of course $\mathcal{G} = \mathcal{L}$, the whole semilattice.

With the notion of building set at hand, we can now introduce the principal topological object of the following chapters.

Definition 8.2.4. ([**60**, Definition 2.7]) *Let $\mathcal{L}$ be a semilattice and $\mathcal{G}$ a building set of $\mathcal{L}$. A (possibly empty) subset N of $\mathcal{G}$ is called* **nested** *if for any $\{x_1, \ldots, x_t\} \subseteq N$, where $t \geq 2$ and any two distinct elements x_i and x_j are incomparable, the join $x_1 \vee \cdots \vee x_t$ exists and does not belong to $\mathcal{G}$.*

The nested sets in $\mathcal{G}$ form an abstract simplicial complex, called the **nested set complex** *of $\mathcal{G}$ in $\mathcal{L}$, and denoted by $\mathcal{N}(\mathcal{L}, \mathcal{G})$. If $\mathcal{L}$ has a maximal element $\hat{1}$ and $\mathcal{G}$ contains it, then the nested set complex is a cone with apex $\{\hat{1}\}$. The base of this cone is the* ***reduced nested set complex*** *$\widetilde{\mathcal{N}}(\mathcal{L}, \mathcal{G})$.*

If it is clear which semilattice $\mathcal{L}$ is meant, we will write $\mathcal{N}(\mathcal{G})$ instead of $\mathcal{N}(\mathcal{L}, \mathcal{G})$.

Remark 8.2.5. (1) This definition implies that $F(x_1 \vee \cdots \vee x_\ell) = \{x_1, \ldots, x_\ell\}$ for any set of incomparable elements $\{x_1, \ldots, x_\ell\} \subseteq N$, see [**60**, 2.8, (2)].

(2) Consider the maximal building set $\mathcal{G} = \mathcal{L}$. In this case, a set is $\mathcal{G}$-nested if and only if it is linearly ordered. Therefore we have $\widetilde{\mathcal{N}}(\mathcal{L}, \mathcal{L}) = \widetilde{\Delta}(\mathcal{L})$, the order complex of (the proper part of) $\mathcal{L}$.

8.3. Example: the partition lattice

As pointed out at the beginning of the chapter, the definitions were inspired by the properties of geometric lattices and the choice of terminology becomes suggestive if one focuses on the special case of partitions.

Given a natural number n, a *partition of* n is a set theoretic partition of the set $[n] := \{1, 2, \ldots, n\}$, i.e., a family $\alpha := (A_j)_{j=1\ldots k}$ of disjoint sets whose union is $[n]$. The A_j are called *blocks* of the partition. A partition $\alpha = (A_j)_{j=1\ldots k}$ is called a *refinement* of (or 'finer than') the partition $\beta = (B_i)_{i=1\ldots l}$ if every block of α is entirely contained in a block of β (i.e., if for any $j = 1, \ldots, k$ there is $1 \leq i \leq l$ with $A_j \subset B_i$).

Definition 8.3.1. *The* ***partition lattice*** Π_n *is the poset of all partitions of the number n, partially ordered by refinement:* $\alpha < \beta$ *if* α *is a refinement of* β.

It is easy to see that this poset is indeed a lattice. Moreover, note that Π_n enjoys many interesting properties of posets: for instance, it is *geometric* and *modular* (see e.g. [**108**]).

Given $\pi = (X_i)_{i=1\ldots k} \in \Pi$, any element of $(\Pi_n)_{\leq \pi}$ is completely determined by the choice of a partition of every block of π. Thus, we may write

$$(\Pi_n)_{\leq \pi} = \Pi_{|X_1|} \times \Pi_{|X_2|} \times \ldots \times \Pi_{|X_k|}.$$

This shows that the only join-irreducible elements of Π_n are the partitions that have exactly one block containing more than one element (i.e., exactly one so-called *nonsingleton block*). For any given subset $X \subseteq [n]$ let π_X denote the partition with X as its only nonsingleton block. Then we can write the *minimal building set* of Π_n as

$$\mathcal{I} := \{\pi \in \Pi_n \mid \pi = \pi_X \text{ for some } X \subseteq [n], |X| \geq 2\}.$$

By the definition of the order in Π_n it is clear that two elements of $\mathcal{I}$ are comparable if and only if their nonsingleton blocks are contained into one another. Consider now two incomparable partitions $\pi_X, \pi_Y \in \mathcal{I}$. Their join is of the form $\pi_{X\cup Y}$ (and thus belongs to $\mathcal{I}$ again) precisely when the intersection $X \cap Y$ is not empty (otherwise the partition having two distinct nonsingleton blocks X, Y would be a refinement of $\pi_{X\cup Y}$ and clearly be above both π_X and π_Y).

We can now apply definition 8.2.4 and determine the nested set complex of $\mathcal{I}$ in the partition lattice:

$$\mathcal{N}(\Pi_n, \mathcal{I}) = \left\{ \{\pi_{X_1}, \ldots \pi_{X_k}\} \subseteq \mathcal{I} \;\middle|\; \begin{array}{l} \text{For all } i \neq j, \\ X_i \subset X_j, \text{ or } X_j \subset X_i, \\ \text{or } X_i \cap X_j = \emptyset. \end{array} \right\}$$

In other words, the nested sets are those for which the nonsingleton blocks are either contained into one another or disjoint, i.e., nested.

8.4. Combinatorial blowups and stellar subdivisions

We present the main topological feature of the theory of building sets and nested set complexes, as far as our work is concerned. The point is to understand the topological relation of the nested set complexes of different building sets expressed in corollary 8.4.9.

For atomic semilattices $\mathcal{L}$, Feichtner and Yuzvinsky [**63**] give an explicit geometric realization of nested set complexes of $\mathcal{L}$. If $\mathcal{H} \subseteq \mathcal{G}$ are building sets of $\mathcal{L}$, Feichtner and Müller [**59**] proved that the geometric realization of $\mathcal{N}(\mathcal{L}, \mathcal{G})$ arises from the one of $\mathcal{N}(\mathcal{L}, \mathcal{H})$ by a sequence of stellar subdivisions.

A closer look at the arguments shows that the support of the geometric realization is not strictly necessary, and indeed in [**31**] a version for arbitrary semilattices is derived, relying on the concept of stellar subdivision of an abstract simplicial complex. Here we give the main definitions and the key theorem.

Definition 8.4.1. ([**60**, Definition 3.1]) *For a semilattice $\mathcal{L}$ and an element $\alpha \in \mathcal{L}$ we define a new poset* $\mathrm{Bl}_\alpha\mathcal{L}$ *on the set of elements*

$$\{x \in \mathcal{L} \mid x \not\geq \alpha\} \cup \{\langle \alpha, x\rangle \mid x \in \mathcal{L},\ x \not\geq \alpha \text{ and } (x \vee \alpha)_{\mathcal{L}} \text{ exists}\},$$

with order relation defined as follows:

(i) $y > z$ in $\mathrm{Bl}_\alpha\mathcal{L}$ if $y > z$ in $\mathcal{L}$;
(ii) $\langle \alpha, y\rangle > \langle \alpha, z\rangle$ in $\mathrm{Bl}_\alpha\mathcal{L}$ if $y > z$ in $\mathcal{L}$;
(iii) $\langle \alpha, y\rangle > z$ in $\mathrm{Bl}_\alpha\mathcal{L}$ if $y \geq z$ in $\mathcal{L}$;

where in all three cases $y, z \not\geq \alpha$.

The poset $\mathrm{Bl}_\alpha\mathcal{L}$ *is called the* ***combinatorial blowup of $\mathcal{L}$ at*** *α.*

Note: To avoid ambiguity with the customary notation for intervals in posets, we will use the notation $\langle \alpha, y\rangle$ instead of $[\alpha, y]$ which was used in [**60**].

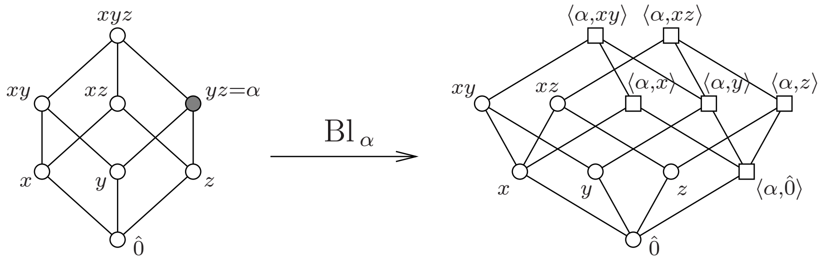


Figure 1. Example of the combinatorial blowup of the boolean lattice B_3 at the element $\alpha = \{y, z\}$.

Remark 8.4.2. *Given a semilattice $\mathcal{L}$ and an element $\alpha \in \mathcal{L}$, the poset* $\mathrm{Bl}_\alpha\mathcal{L}$ *is again a semilattice, see* [**60**, Lemma 3.2].

In the following remark we collect some facts that will be used mainly in chapter 10.

Remark 8.4.3. Let $\mathcal{F} = \mathcal{F}(K)$ be the face semilattice of some simplicial complex K, that is, elements of $\mathcal{F}(K)$ are faces of K, and they are ordered by inclusion (the minimal element of $\mathcal{F}(K)$ is the empty face of K). Let $\alpha \in \mathcal{F}$ be a face of K. By the previous remark, $\mathrm{Bl}_\alpha\mathcal{F}$ is again a semilattice. If we have two elements from $\mathrm{Bl}_\alpha\mathcal{F}$, let us see what their meet is equal to:

- If $F, G \in \mathcal{F} \cap \mathrm{Bl}_\alpha\mathcal{F}$, then $F \wedge G = F \cap G$, considered as an element of $\mathrm{Bl}_\alpha\mathcal{F}$.
- If $F, \langle \alpha, G\rangle \in \mathrm{Bl}_\alpha\mathcal{F}$, where $F, G \in \mathcal{F}$, then $F \wedge \langle \alpha, G\rangle = F \cap G \in \mathrm{Bl}_\alpha\mathcal{F}$.
- Finally, if $\langle \alpha, F\rangle, \langle \alpha, G\rangle \in \mathrm{Bl}_\alpha\mathcal{F}$, $F, G \in \mathcal{F}$, then $\langle \alpha, F\rangle \wedge \langle \alpha, G\rangle = \langle \alpha, F \cap G\rangle$.

We proceed with the definition of stellar subdivision for abstract simplicial complexes. Note that, passing to the geometric realization, this translates to the well-known corresponding geometric notion of "removing a face F and inserting a new vertex v_F in its barycenter, together with the cone on the link of F with apex v_F".

DEFINITION 8.4.4. *The* ***stellar subdivision*** *of a simplicial complex* K *with respect to a non-empty face* F *is the simplicial complex* $\mathrm{sd}_F(K)$ *whose faces are*

$$\{G \in K \mid G \not\supseteq F\} \cup \{G \cup \{v_F\} \mid G \in K,\ G \not\supseteq F, \text{ and } G \cup F \in K\}.$$

The relation with the notion of combinatorial blowup is stated in the following remark.

REMARK 8.4.5. *It was noticed in* [**70**, Section 3] *that*

$$\mathcal{F}(\mathrm{sd}_F(K)) = \mathrm{Bl}_F(\mathcal{F}(K)),$$

that is, stellar subdivisions are instances of combinatorial blowups.

To generalize the result of [**59**] we now have to show how combinatorial blowups relate to the structure of building sets and nested set complexes.

The next proposition is similar in spirit to Proposition 4.2 of [**59**], but works in the abstract case as well and does not assume atomicity of the lattice. It describes the behavior of nested set complexes under extension of the building set.

PROPOSITION 8.4.6 (Proposition 2.4 of [**31**]). *Let* $\mathcal{L}$ *be a semilattice, and let* $\mathcal{G}$ *be a building set in* $\mathcal{L}$*. If* $\alpha \in \max(\mathcal{L} \setminus \mathcal{G})$*, then* $\mathcal{G}' = \mathcal{G} \cup \{\alpha\}$ *is a building set in* $\mathcal{L}$*, and* $\mathcal{F}(\mathcal{N}(\mathcal{G}')) = \mathrm{Bl}_B(\mathcal{F}(\mathcal{N}(\mathcal{G})))$*, where* $B = \max \mathcal{G}_{\leq \alpha}$*.*

PROOF. Note first that the number of elements of B is at least 2, since otherwise Definition 8.2.1 would not be satisfied for the building set $\mathcal{G}$ and the element $\alpha \in \mathcal{L}$. If $\{\beta_1, \ldots, \beta_t\} \subseteq B$ is an antichain, with $t \geq 2$, then $\beta_1 < \vee_{i=1}^t \beta_i \leq \alpha$ and hence $\vee_{i=1}^t \beta_i \in \mathcal{L} \setminus \mathcal{G}$. Therefore $B \in \mathcal{N}(\mathcal{G})$. Let us now prove that $\mathcal{G}'$ is a building set. Since α is a maximal element of $\mathcal{L} \setminus \mathcal{G}$, it is easy to see that, for all $x \in \mathcal{L} \setminus \mathcal{G}'$, $\max \mathcal{G}'_{\leq x} = \max \mathcal{G}_{\leq x}$, and for $x \in \mathcal{G}'$, $\max \mathcal{G}'_{\leq x} = x$. Since $\mathcal{G}$ is a building set, by Definition 8.2.1, $\mathcal{G}'$ is also a building set.

Define now a map $f : \mathcal{F}(\mathcal{N}(\mathcal{G}')) \to \mathrm{Bl}_B(\mathcal{F}(\mathcal{N}(\mathcal{G})))$ in the following way:

$$f(A) = \begin{cases} A, & \text{if } \alpha \notin A; \\ \langle B, A \setminus \{\alpha\} \rangle, & \text{otherwise,} \end{cases}$$

where $A \in \mathcal{F}(\mathcal{N}(\mathcal{G}'))$. Let us prove that f is an order-preserving bijection.

• The map f is well-defined:

(1) If $\alpha \notin A$, then let $\{a_1, \ldots, a_t\}$, $t \geq 2$, be a set of incomparable elements in A, if such exists. Then $\vee_{i=1}^t a_i \in \mathcal{L} \setminus \mathcal{G}' \subset \mathcal{L} \setminus \mathcal{G}$, and hence $A \in \mathcal{N}(\mathcal{G})$. Since B is an antichain with at least two elements, $\bigvee B = \alpha$, and by assumption $A \in \mathcal{N}(\mathcal{G}')$ it follows that $A \not\supseteq B$. Therefore $f(A) = A \in \mathrm{Bl}_B(\mathcal{F}(\mathcal{N}(\mathcal{G})))$.

(2) If $\alpha \in A$, then $A \setminus \{\alpha\} \in \mathcal{N}(\mathcal{G}')$, and therefore $A \setminus \{\alpha\} \in \mathcal{N}(\mathcal{G})$ and $B \not\subseteq A \setminus \{\alpha\}$. In order to prove that $\langle B, A \setminus \{\alpha\} \rangle$ is an element of $\mathrm{Bl}_B(\mathcal{F}(\mathcal{N}(\mathcal{G})))$, we need to check that $B \vee (A \setminus \{\alpha\}) \in \mathcal{F}(\mathcal{N}(\mathcal{G}))$, i.e. that $B \cup (A \setminus \{\alpha\})$ is a nested set in $\mathcal{G}$.

Note that, for all $x \in A \setminus \{\alpha\}$, x has to be comparable with α, since otherwise $\{x, \alpha\}$ would be an antichain in A, and $x \vee \alpha$ either does not exist, or $x \vee \alpha > \alpha$, and hence $x \vee \alpha \in \mathcal{G}'$. This would contradict the fact that A is a nested set in $\mathcal{G}'$.

Let $\{x_1, \cdots, x_t\}$, where $t \geq 2$, be a set of incomparable elements in $B \cup (A \setminus \{\alpha\})$. If there exists some $i \in [t]$ so that $x_i \geq \alpha$, then it is easy to see that $\{x_1, \cdots, x_t\} \cap B = \emptyset$, and hence $\{x_1, \cdots, x_t\} \subseteq A \setminus \{\alpha\}$. Then $\vee_{i=1}^t x_i \in \mathcal{L} \setminus \mathcal{G}$ follows from the fact that $A \setminus \{\alpha\}$ is nested in $\mathcal{G}$. Suppose now that there exists $i \in [t]$ so that $x_i \in B$. In this case we have that $x_j \leq \alpha$ for all $j \in [t]$, and $\alpha \geq \vee_{j=1}^t x_j > x_i$. Therefore $\vee_{j=1}^t x_j \in \mathcal{L} \setminus \mathcal{G}$. We conclude that $B \cup (A \setminus \{\alpha\}) \in \mathcal{N}(\mathcal{G})$.

• By definition, f is injective.

• The map f is surjective:

If $F \in \mathrm{Bl}_B(\mathcal{F}(\mathcal{N}(\mathcal{G})))$, and F is a face of $\mathcal{N}(\mathcal{G})$, then $F \not\supseteq B$. We want to prove that $F \in \mathcal{N}(\mathcal{G}')$. Let $\{y_1, \dots, y_s\}$, $s \geq 2$, be a set of incomparable elements in F. Since $\vee_{i=1}^{s} y_i \in \mathcal{L} \setminus \mathcal{G}$, and $\mathcal{G}' = \mathcal{G} \cup \{\alpha\}$, we are supposed to prove that $\vee_{i=1}^{s} y_i \neq \alpha$. Assume the contrary. Then, by [**60**, Proposition 2.8], $\{y_1, \cdots, y_s\} = \max \mathcal{G}_{\leq y_1 \vee \cdots \vee y_s} = \max \mathcal{G}_{\leq \alpha} = B$, which is a contradiction with $F \not\supseteq B$. Hence $F \in \mathcal{F}(\mathcal{N}(\mathcal{G}'))$, and $f(F) = F$.

If $\langle B, A \rangle \in \mathrm{Bl}_B(\mathcal{F}(\mathcal{N}(\mathcal{G})))$, then $A \not\supseteq B$, and $A \cup B \in \mathcal{N}(\mathcal{G})$. It is clear that $\alpha \notin A$, since $\alpha \notin \mathcal{G}$. We ought to prove that $A \cup \{\alpha\}$ is nested in $\mathcal{G}'$. Let us first prove that all elements in A are comparable with α. Suppose the contrary, i.e. that there exists $y \in A$, so that y is not comparable with α. Then $S := B \setminus \mathcal{L}_{\leq y} \neq \emptyset$, since otherwise $y \geq \bigvee B = \alpha$. Now, if $\alpha \vee y$ exists in $\mathcal{L}$, we would have $\alpha < y \vee \alpha = y \vee \bigvee B = y \vee \bigvee S$, and hence $y \vee \bigvee S \in \mathcal{G}$. This is a contradiction with $A \cup B \in \mathcal{N}(\mathcal{G})$, since $\{y\} \cup S \subseteq A \cup B$ is a set of incomparable elements of cardinality at least two.

Since any set of incomparable elements $\{a_1, \dots, a_s\}$ in $A \cup \{\alpha\}$, where $s \geq 2$, is actually a subset of A, by the same arguments as above we conclude that $\vee_{j=1}^{s} y_j \in \mathcal{L} \setminus \mathcal{G}'$. Therefore $A \cup \{\alpha\}$ is nested in $\mathcal{G}'$ and $f(A \cup \{\alpha\}) = \langle B, A \rangle$.

• It is clear that both f and f^{-1} are order-preserving.

Therefore, f is an isomorphism of posets. □

Now we only have to recall remark 8.4.5 above to derive the main result of this section.

THEOREM 8.4.7. *Consider two building sets $\mathcal{G}_1, \mathcal{G}_2$ in a meet-semilattice $\mathcal{L}$. If $\mathcal{G}_1 \subset \mathcal{G}_2$, then the simplicial complex $\mathcal{N}(\mathcal{L}, \mathcal{G}_2)$ can be obtained from $\mathcal{N}(\mathcal{L}, \mathcal{G}_1)$ by a sequence of stellar subdivisions.*

With remark 8.2.3 we have immediately the following corollary, that will be an essential tool in our considerations.

COROLLARY 8.4.8. *Let $\mathcal{L}$ be a meet-semilattice and $\mathcal{G}$ a building set in $\mathcal{L}$. Then $\widetilde{\Delta}(\mathcal{L})$ can be obtained from $\widetilde{\mathcal{N}}(\mathcal{L}, \mathcal{G})$ by a sequence of stellar subdivisions.*

Since subdivision of abstract simplicial complexes preserves homeomorphism type (see section 11.1 for the concept of homeomorphism between abstract simplicial complexes), the next result follows immediately either from the previous corollary or from remark 8.2.2 (saying that every building set of a semilattice contains the minimal one).

COROLLARY 8.4.9. *For any two building sets $\mathcal{G}_1$, $\mathcal{G}_2$ of a semilattice $\mathcal{L}$, the nested set complexes $\mathcal{N}(\mathcal{L}, \mathcal{G}_1)$ and $\mathcal{N}(\mathcal{L}, \mathcal{G}_2)$ are homeomorphic as abstract simplicial complexes.*

8.5. Combinatorial blowups and shellability

The geometric intuition of stellar subdivisions may suggest to investigate the behavior of shellability of posets with respect to combinatorial blowups. We have the following theorem, whose proof we will postpone to chapter 10, where we also give an application of it.

THEOREM 10.2.5 ([**31**]). *Let $\hat{\mathcal{F}} = \hat{\mathcal{F}}(K)$ be the face lattice of some simplicial complex K, $\mathcal{F} = \hat{\mathcal{F}} \setminus \{\hat{1}\}$, and assume that $\hat{\mathcal{F}}$ admits a recursive coatom ordering. Then $\hat{\mathcal{F}}_\alpha = \mathrm{Bl}_\alpha(\mathcal{F}) \cup \{\hat{1}\}$ also admits a recursive coatom ordering, where α is any element of $\hat{\mathcal{F}} \setminus \{\hat{0}, \hat{1}\}$.*

Part 2

Nested set complexes

CHAPTER 9

Nested set complexes of Dowling lattices and complexes of Dowling trees

Background. Dowling lattices are named after T. A. Dowling, who first studied a particular class of arrangements of hyperplanes whose intersection lattices can be obtained by enriching the partition lattice with sets of elements of the cyclic groups $\mathbb{Z}_n$ [**43**]. In a separate work ([**44**]), Dowling took the step to a general combinatorial point of view, and thoroughly studied what he called 'posets based on finite groups'. We wish to point to [**44**] as a very readable and comprehensive introduction to the subject.
Both approaches to these structures were followed since the work of Dowling. Let us mention, as two examples, the work of Ehrenborg and Readdy [**51, 52**], who introduced the notion of *Dowling transform* of an arrangement of hyperplanes and showed that this transformation preserves supersolvability, and the study of a combinatorial generalization of Dowling lattices by Hanlon [**78**].

The study of abstract simplicial complexes whose cells are indexed by combinatorial types of rooted trees on a fixed number of leaves was recently brought to a broad attention by the work of Billera, Holmes and Vogtmann [**8**]. They considered the space of all possible phylogenetic trees of a certain set of biological species, with lengths on edges representing the genetic distance of two mutations. This space is a cone with apex the unique tree with all edge lengths equal to zero. The base of this cone (the link of the apex) is obtained by considering the trees with unit edge lengths (i.e., cutting the cone by a transversal hyperplane). This space has a natural stratification, each cell corresponding to a combinatorial type of trees - this is the *complex of phylogenetic trees.*
Complexes of trees were studied for different purposes, before and after [**8**], see [**18, 77, 79, 101, 115, 111**]. One of the recent applications of the theory of nested set complexes showed that the homological relationship between the complex of trees and the partition lattice is in fact a strongly combinatorial one. In [**58**], Feichtner showed that the order complex of the partition lattice is obtained from the complex of trees by a sequence of stellar subdivisions. This property was proven to hold even for k-trees (see [**41**]).

Our purpose is to put these three themes into an organic general picture and was motivated by the work of Hultman [**81**], who defined the *complex of G-symmetric phylogenetic trees* and showed that it has the same homotopy type as a particular subposet of the Dowling lattice.

Overview. In sections 9.1 and 9.2 we give a detailed picture of Dowling lattices and G-symmetric phylogenetic trees, reviewing the basics and developing a notation that will prove to be appropriate for a direct application of the theory of nested set complexes. They will enter the picture in section 9.3, where the result of Hultman is sharpened by showing that the complex of G-symmetric phylogenetic trees actually is subdivided by the order complex of the associated subposet of the Dowling lattice (Theorem 9.3.3). In section 9.4 the notion of a *Dowling tree* is introduced as naturally associated to the nested set complex of the full Dowling lattice, thus being probably the most appropriate

Dowling generalization of the complex of phylogenetic trees (Definition 9.4.3, Theorem 9.4.5). The complex of Dowling trees contains the complex of G-symmetric phylogenetic trees as a simplicial subcomplex. In the last section we describe explicitly how the bigger complex can be obtained from the smaller one by successively coning over certain subcomplexes (Theorem 9.4.9). By keeping under control the topology of those subcomplexes, we calculate how many homology spheres arise at each step, thus explicitly relating the "numerologically" suggestive expressions of the top homology ranks of the two complexes (Corollary 9.4.11).

9.1. Posets based on finite groups

Before starting out, let us fix some general notations. In this chapter we will deal with finite partially ordered sets, briefly called *posets*. Recall the main topological structure associated to a poset P, its *order complex* $\Delta(P)$, the (abstract) simplicial complex of the totally ordered subsets of P. We will almost never distinguish between an abstract simplicial complex and its geometric realization, thus speaking of 'topological properties' of an abstract simplicial complex. Recall also that, because of its more interesting topology, we defined (see section 8.1) the *reduced order complex* $\widetilde{\Delta}(P)$ as the order complex of the poset obtained from P by removing the maximal and the minimal elements, if P has any.

In considering partitions, we will switch between the set-theoretic notation $\sigma = S_0 \amalg S_1 \amalg \cdots \amalg S_k$ and the (more customarily) 'block notation' $\sigma = S_0|\ldots|S_k$.

9.1.1. Dowling lattices. Dowling introduces in [**44**] these lattices as sets of *partial partitions* of the set $[n] := \{1, \ldots, n\}$, enriched by elements of a finite group G. It is easy to see that these enriched partitions are sets of representatives for the orbits of the simple blocks of what Hultman [**81**] calls a *G-symmetric partition*. Dowling lattices may then also be defined in these terms, as we will now do, following [**81**].

DEFINITION 9.1.1. *Let G be a finite group and n a natural number. Consider the action of G on the set $\{0\} \cup ([n] \times G)$ defined by $g((i, h)) = (i, gh)$ and $g(0) = 0$. The partitions of $\{0\} \cup ([n] \times G)$ such that this action induces an action on the blocks are called **G-symmetric partitions**. A block is called **simple** if its orbit under this action has length $|G|$. The set of G-symmetric partitions such that the only non-simple block is the block containing 0 is called $\mathcal{Q}_n(G)$. The ordering by refinement turns it into a lattice, called the **Dowling lattice**.*

We take the opportunity of sketching the connection with Dowling's definition to fix some more notation.

The 'forgetful' map $\{0\} \cup ([n] \times G) \to \{0\} \cup [n]$ defined by $(i, g) \mapsto i$ and $0 \mapsto 0$ induces a mapping of $\mathcal{Q}_n(G)$ onto the poset $\Pi_{n,0}$ of partitions of the set $\{0\} \cup [n]$. This map sends $\omega \in \mathcal{Q}_n(G)$ to its *associated partition* $\underline{\omega} \in \Pi_{n,0} \cong \Pi_{n+1}$. An element of $\Pi_{n,0}$ will be written as $\alpha := A_0|A_1|\ldots|A_k$, where we agree to choose the indexing such that 0 is always contained in the block indexed by 0. Of course, $\alpha = \underline{\omega}$ has one block A_i for every orbit of blocks in ω, and the A_i with $i > 1$ correspond to the simple blocks.

We now see that we can encode in a unique way any $\omega \in \mathcal{Q}_n(G)$ in the following data:

- A partition $\alpha \in \Pi_{n,0}$, called the *associated partition* of ω.
- For every block $A_j = \{i_1, \ldots, i_k\}$ of α with $j > 0$ an $|A_j|$-tuple $A_j^G := (id, g_{i_2}^{\alpha}, g_{i_3}^{\alpha}, \ldots, g_{i_k}^{\alpha}) \subset G^k$.

This encoding becomes biunique if we agree to always order the elements in the blocks A_j in increasing order: $i_1 < i_2 < \cdots < i_k$.

The order relation translates to the following: $\omega_1 < \omega_2$ if and only if

- the associated partitions satisfy $\underline{\omega}_1 < \underline{\omega}_2$
- for every block B of $\underline{\omega}_2$, if $C_1, \ldots, C_k$ are the blocks of $\underline{\omega}_1$ that subdivide it, there are elements $h_1, \ldots, h_k$ in G such that for every $j \in B \cap C_i$ we have $g_j^{\omega_2} = h_i g_j^{\omega_1}$.

Dowling gave a complete and very readable survey on these objects in his paper [**44**], to which we point as a general reference. The main statement on the topology of these lattices that can be deduced from [**44**] is summarized in the following proposition, of which we sketch the proof for completeness.

Proposition 9.1.2. *The order complex $\Delta(\mathcal{Q}_n(G) \setminus \{\hat{0}, \hat{1}\})$ is homotopy equivalent to a wedge of $(|G|+1)(2|G|+1)\ldots((n-1)|G|+1)$ spheres of dimension $(n-2)$.*

Proof. The proof bases on a concatenation of arguments from [**44, 107, 9**]. The fact that $\Delta(\mathcal{Q}_n(G) \setminus \{\hat{0}, \hat{1}\})$ is a wedge of spheres of dimension $(n-2)$ follows because $\mathcal{Q}_n(G)$ is a supersolvable geometric lattice ([**44**]), and therefore shellable ([**9**]).
To calculate the number of the spheres one has to determine the characteristic polynomial of $\mathcal{Q}_n(G)$. To do this, consider a maximal chain $\omega := \hat{0} < \omega_1 < \cdots < \omega_{n-1} < \omega_n = \hat{1}$ of modular elements in $\mathcal{Q}_n(G)$. By [**44**, Thm. 4] we may take

$$\omega_\ell := \underbrace{\{0\} \cup \{(j,g) \big| g \in G, j \leq \ell\}}_{S_0} \coprod_{\substack{i>\ell \\ g\in G}} \{(i,g)\}$$

$$\omega_i := \sigma_0 \amalg \coprod_{g\in G}\{1,g\} \amalg \coprod_{g\in G}\{2,g\} \amalg \cdots \amalg \coprod_{g\in G}\{i,g\}.$$

The rank of ω_ℓ equals ℓ, and with corollary 2.3 of [**44**] one finds easily that there are $\ell + \binom{\ell}{2}|G|$ atoms below ω_ℓ. Therefore, given ℓ, the number of atoms below ω_ℓ but *not* below $\omega_{\ell-1}$ is
$\ell + \binom{\ell}{2}|G| - (\ell-1) - \binom{\ell-1}{2}|G| = 1 + \left[\binom{\ell}{2} - \binom{\ell-1}{2}\right]|G|$

$$= 1 + \tfrac{(\ell-1)|G|}{2}(\ell - \ell + 2) = 1 + (\ell-1)|G|.$$

A well-known result of Stanley ([**107**]) allows us to write the characteristic polynomial as

$$\chi_{\mathcal{Q}_n(G)}(x) = \prod_{\ell=1}^{n}(x - (1 + (\ell-1)|G|)).$$

We can now calculate the Euler characteristic of $\Delta(\mathcal{Q}_n(G))$ as

$$\chi(\Delta(\mathcal{Q}_n(G))) = \chi_{\mathcal{Q}_n(G)}(0) = \prod_{\ell=1}^{n}(1 + (\ell-1)|G|).$$

This concludes the proof. $\square$

9.1.2. The subposet $\mathcal{Q}_{\mathbf{n}}^{\mathbf{G}}$. In [**81**], the author introduces a subposet of $\mathcal{Q}_n(G)$ that is of particular interest in connection with G-symmetric phylogenetic trees.

Definition 9.1.3. *We define $\mathcal{Q}_n^G$ to be the subposet of $\mathcal{Q}_n(G)$ consisting of the partitions with trivial zero block.*

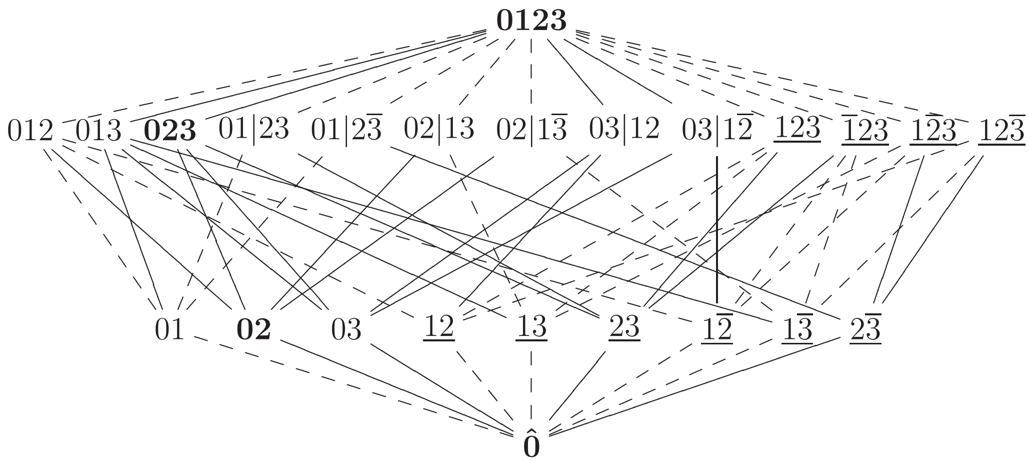


FIGURE 1. The lattice $\mathcal{Q}_3(\mathbb{Z}_2)$ and its subposet $\mathcal{Q}_3^0(\mathbb{Z}_2)$ (underlined). The bold elements give a modular chain, the dashed chains are the 15 homology chains of the corresponding shelling.

REMARK 9.1.4. *Our definition is slightly different from the one given in* [**81**]*, because we allow the minimal element of $\mathcal{Q}_n(G)$ to be in $\mathcal{Q}_n^G$.*

In particular, given $\sigma \in \mathcal{Q}_n^G$, every element of $[n] \times G$ is contained in a simple block of σ.

In general, $\mathcal{Q}_n^G$ is not a lattice. Indeed, consider the following two elements of $\mathcal{Q}_3^{\mathbb{Z}_2}$:

$$\sigma_1 := \{0\}\{(1,0),(2,0)\}\{(1,1)(2,1)\}\{(3,0)\}\{(3,1)\},$$
$$\sigma_2 := \{0\}\{(1,0),(2,1)\}\{(1,1)(2,0)\}\{(3,0)\}\{(3,1)\}.$$

Their join is the maximal element of the full Dowling lattice, but this is not contained in $\mathcal{Q}_3^{\mathbb{Z}_2}$, where there is no element that is bigger of both σ_1 and σ_2.
Nevertheless, $\mathcal{Q}_n^G$ is a meet-semilattice. We prove the following easy lemma.

LEMMA 9.1.5. *For any $\sigma \in \mathcal{Q}_n^G$, we have an isomorphism*

$$(\mathcal{Q}_n^G)_{\leq \sigma} \simeq (\Pi_n)_{\leq \underline{\sigma}}.$$

Proof. The map defined by $\sigma \mapsto \underline{\sigma}$ is clearly a poset morphism, and obviously surjective. For injectivity note that, given $\underline{\sigma}' \leq \underline{\sigma}$, any set of representatives of the orbit blocks for σ forces the choice of the ℓ-tuples associated to the blocks to $\underline{\sigma}'$. Indeed, if B is a block of $\underline{\sigma}$ that is associated to $(id, g_2^\sigma, \ldots, g_\ell^\sigma)$, then if $C = \{i_1, \ldots, i_k\} \subseteq B$ is a block of $\underline{\sigma}'$, the only possibility for a partition associated to $\underline{\sigma}'$ to be below σ is to associate to C the k-tuple $(id, g_{i_2}^\sigma (g_{i_1}^\sigma)^{-1}, \ldots, g_{i_k}^\sigma (g_{i_1}^\sigma)^{-1})$. $\square$

This makes the work of section 9.3 possible, where the theory of building sets and nested set complexes will be applied to the posets $\mathcal{Q}_n^G$.

As an example, we depict in figure 9.1.2 the Dowling lattice $\mathcal{Q}_3(\mathbb{Z}_2)$, where we write only one representant for the nonsingleton blocks. The numbers refer to the associated partitions, and an overline over an element indicates that this number is associated with the nonidentity element of $\mathbb{Z}_2$. The elements of $\mathcal{Q}_3^0(\mathbb{Z}_2)$ are underlined.

The homotopy type of $\widetilde{\Delta}(\mathcal{Q}_n(G))$ was determined in [**81**] by comparison with the complex of G-symmetric phylogenetic trees (see section 9.2), as an application of discrete Morse theory.

9.2. G-Symmetric phylogenetic trees

Recall that we fixed once and for all a natural number n. In this context, given a finite group G, a G-tree is a rooted tree whose leaves are in bijection with the set $[n] \times G$. The group G acts on the set of leaves by means of the 'standard' action described in the previous section (see Definition 9.1.1).

We now specify a class of G-trees that will be the object of our study.

Definition 9.2.1. *A* ***G-symmetric phylogenetic tree*** *is a G-tree satisfying the following additional conditions:*

(1) *Every internal vertex (except the root) has degree at least* 3.
(2) *The tree is invariant under the G-action.*
(3) *For any two different elements $g, h \in G$ and any $i \in [n]$, the (unique) shortest path connecting the leafs labelled (i, g) and (i, j) passes through the root.*

The set of G-symmetric phylogenetic trees is denoted by $\mathcal{T}_n^G$.

This is definition 3.1 of [**81**], where some properties of those trees are listed. Here we need only recall that every internal edge t of a tree $T \in \mathcal{T}_n^G$ generates an orbit $\mathfrak{o}(t)$ (called *inner orbit*) of cardinality $|G|$ under the action of G. Following [**81**], one can associate to $\mathfrak{o}(t)$ the partition $\pi(\mathfrak{o}(t))$ of $[n] \times G$ obtained by putting in the same block all labels of leaves that are in the same connected component of T after removing all the edges in $\mathfrak{o}(t)$, and adding 0 to the block corresponding to the component containing the root. We want to slightly modify this definition.

9.2.1. Some notation. Note that in all partitions of $\{0\} \cup [n]$ that are associated to some $\sigma \in \mathcal{Q}_n^G$ the element 0 is alone in its block. Therefore we may reconstruct every such partition from the corresponding partition of $[n]$ by just adding the block $\{0\}$. We then agree that, for $\sigma \in \mathcal{Q}_n^G$, in this section we will consider $\underline{\sigma} \in \Pi_n$.

Now consider a tree T satisfying definition 9.2.1. For any vertex v of T let λ_v denote the set of labels of leaves such that the path connecting them to the root traverses v. Let

$$\lambda_{Gv} := \bigcup_{g \in G} \lambda_{gv}$$

denote the set of labels of leaves that are separated from the root by a vertex of the form gv for some $g \in G$.

Then, given an inner edge t of T, let $\lambda_t := \lambda_v$ where v is the vertex of t that is further from the root. To every internal edge t we can then define a partition $\sigma(t) \in \mathcal{Q}_n^G$ as

$$\coprod_{(i,g) \notin \lambda_{Gt}} \{(i, g)\} \coprod_{g \in G} \lambda_{gt}.$$

It is clear that $\lambda_{gt} = \{(i, gh) | (i, h) \in \lambda_t\} = g\lambda_t$, and therefore we see that $\sigma(t)$ has only one nonsingleton orbit. Thus the associated partition $\underline{\sigma}(t) \in \Pi_n$ has only one nonsingleton block.

The next two subsections present some material of [**81**] in a language and from a viewpoint that are well-suited to our methods.

9.2.2. Inner orbit contraction. The contraction of all edges in an inner orbit $\mathfrak{o}(t)$ turns a G-symmetric tree T into another tree $T' \in \mathcal{T}_n^G$. We can then define following partial order on $\mathcal{T}_n^G$.

DEFINITION 9.2.2. *Given $T, T' \in \mathcal{T}_n^G$, define $T \leq T'$ if and only if T' can be obtained from T by a sequence of inner orbit contractions.*

The importance of this ordering is shown in the following proposition.

PROPOSITION 9.2.3. [**81**, Corollary 3.5] *With the partial ordering of definition 9.2.2, $\mathcal{T}_n^G$ is the face poset of a pure simplicial complex of dimension $n-2$.*

Before turning our attention to the operation of inner orbit *extension*, which is inverse to the contraction defined above, let us see what kind of relation one can draw between partitions associated to different contractions on the same tree.

So suppose again a tree $T \in \mathcal{T}_n^G$ be given, and consider two inner edges t, t' of T that are not in the same G-orbit. We have seen that the associated partition $\underline{\sigma}(t)$ (respectively $\sigma(t')$) has exactly one nonsingleton block, say B (resp. B'), associated to the unique nonsingleton orbit of $\sigma(t)$ ($\sigma(t')$), of which we consider a representative block S (S'). Suppose we first contract the edges in the orbit $\mathfrak{o}(t)$.

If, for some $h \in G$, $S' \subseteq hS$, then by definition $gS' \subseteq g(hS)$ for all $g \in G$. We conclude that in this case $\sigma(t) > \sigma(t')$, and in particular $\underline{\sigma}(t) > \underline{\sigma}(t')$. Therefore, the unique nonsingleton block B' of the partition $\underline{\sigma}(t')$ is contained in B, so $\underline{\sigma}(t') < \underline{\sigma}(t)$.

If for some $h \in G$ the reverse inclusion $S' \supset hS$ holds, then of course the conclusion above holds with t and t' switched.

The fact that T is a tree excludes the possibility that, if neither of the previous cases enters, $S' \cap hS \neq \emptyset$ for some $h \in G$.

We summarize the conclusion for later reference.

SCHOLIUM 9.2.4. *If t, t' are two inner edges of a G-symmetric tree T such that $\mathfrak{o}(t) \neq \mathfrak{o}(t')$ and if $\sigma(t)$ and $\sigma(t')$ are incomparable, then the sets λ_{Gt} and $\lambda_{Gt'}$ are disjoint. In particular the associated nonsingleton blocks B, B' of $\underline{\sigma}(t)$ and $\underline{\sigma}(t')$ are either contained into one another, or disjoint.*

REMARK 9.2.5. *Following the previous ideas, we can associate to every tree $T \in \mathcal{T}_n^G$ a subset $N(T) \subset \mathcal{Q}_n^G$, defined as*

$$N(T) := \{\underline{\sigma}(t) \mid t \text{ is (a representant of the orbit of) an inner edge of } T\}.$$

$N(T)$ has the property that the nonsingleton block - orbits of any incomparable $\sigma, \sigma' \in N(T)$ are disjoint.

9.2.3. Inner orbit extension. We now discuss the inverse of the above presented operation: *inner orbit extension*.

For this, we suppose a tree $T \in \mathcal{T}_n^G$ to be given together with a partition $\sigma \in \mathcal{Q}_n^G$ that has exactly one nonsingleton-orbit $\mathfrak{o}$ (of which we consider a representative block S). The preceding observations suggest to require following condition to be satisfied by σ:

($*$) *For any inner edge t of T, if neither $\sigma \subset \lambda_t$ nor $\sigma \supset \lambda_t$ then $\sigma \cap \lambda_t = \emptyset$.*

In the following we will show how these data give rise to a tree $T' \in \mathcal{T}_n^G$ such that T is obtained from T' by an inner orbit contraction that is represented by the partition σ.

First of all it is clear that there is a unique vertex v of T such that the component that is separated from the root by removing v is minimal with the property of containing all leafs labelled by elements of S. The family of sets $\{\lambda_{gv} \setminus gS \mid g \in G\}$ can be obtained from the representative $\lambda_v \setminus S$ by the action of G.

Because of property ($*$), we may partition the edges f incident to v and with $\lambda_f \subset \lambda_v$ into the two following classes:

$$F := \{f \mid \lambda_f \subset S\},\ E := \{f \mid \lambda_f \cap S = \emptyset\}.$$

Now everything is prepared for the extension. We first delete all edges $f \in E \cup F$, and for each of them we get a connected component T_f not containing the root.

We then grow an edge t below v appending all edges in E (and their whole components) to v, and the edges of F (with their connected components) to the vertex of t that is further from the root.

In the tree that we have now constructed we clearly have $\lambda_t = S$. Of course we may repeat the whole process inserting edges gt below gv for any $g \in G$, eventually reaching a tree T' satisfying the requirements.

The tree T could have been reached by a sequence of inner orbit extensions. If we consider the set N of all partitions that correspond to an edge orbit of T we indeed may reconstruct T starting from the unique tree without inner edges by recursively performing the above process with all elements of N. We have seen that the unique condition to be able to perform such an extension is given at each step by $(*)$. We summarize our considerations with the following two statements.

Condition N: *Given a subset $X \subset \mathcal{Q}_n^G$ of partitions that have only one orbit consisting of non-singleton blocks, we say that X* satisfies condition N *if for any two incomparable $\sigma, \sigma' \in X$ the only non-singleton blocks of $\underline{\sigma}, \underline{\sigma}'$ are disjoint.*

Scholium 9.2.6. *To any set N of one-nonsingleton-orbit partitions from $\mathcal{Q}_n^G$ satisfying condition N we can naturally associate a tree T, and this is such that, with the notation of remark 9.2.5, $N = N(T)$.*

9.3. Homeomorphism through subdivisions

The reader familiar with the subject will have already remarked that in the previous section all material has been prepared for a direct application of the theory of building sets and nested set complexes. We refer to chapter 8 for the basic definitions and notations of this theory.

In analogy with the partition lattice (see section 8.3) let us define a subset $\mathcal{I}^G \subset \mathcal{Q}_n^G$ as follows:

$$\mathcal{I}^G := \{\sigma \in \mathcal{Q}_n^G | \underline{\sigma} \in \mathcal{I}\}.$$

The following proposition shows that this is indeed "the right definition".

Proposition 9.3.1. *$\mathcal{I}^G$ is the minimal building set of $\mathcal{Q}_n^G$.*

Proof. With lemma 9.1.5 it is straightforward to show by comparison with Π_n that $\mathcal{I}^G$ is the set of join-irreducible elements of $\mathcal{Q}_n^G$. □

Proposition 9.3.2. *The complexes $\mathcal{T}_n^G$ and $\mathcal{N}(\mathcal{I}^G, \mathcal{Q}_n^G)$ are isomorphic.*

Proof. The condition of being nested in $\mathcal{I}^G$ is equivalent to condition N of the previous section. □

We are ready to state the main result of this section, which is now an easy application of corollary 8.4.8.

Theorem 9.3.3. *The order complex $\widetilde{\Delta}(\mathcal{Q}_n^G)$ is obtained from the complex of G-symmetric trees $\mathcal{T}_n^G$ by a sequence of stellar subdivisions.*

Hultman calculated the homotopy type of $\mathcal{T}_n^G$ in [**81**]. We include this result in the following corollary, that is intended to summarize our topological knowledge about G-symmetric partitions and G-symmetric phylogenetic trees.

Corollary 9.3.4. *The simplicial complexes $\widetilde{\Delta}(\mathcal{Q}_n(G))$ and $\mathcal{T}_n^G$ are PL-homeomorphic. They are homotopy equivalent to a wedge of*

$$(|G|-1)(2|G|-1)\dots((n-1)|G|-1)$$

spheres of dimension $(n-2)$.

9.4. Dowling Trees

The natural task at this point is to study of the nested set complexes of the full Dowling lattice $\mathcal{Q}_n(G)$.

9.4.1. Nested set complexes in $\mathcal{Q}_n(G)$. We want to determine the minimal building set $\mathcal{J}^G$ of $\mathcal{Q}_n(G)$. By an easy check (or by comparing Theorem 2 of [**44**]) one sees that, given any $\omega \in \mathcal{Q}_n(G)$ with zero block S_0 and fixed chosen orbit representatives S_i, $i = 1, \dots, k$, there is a natural isomorphism

$$(\mathcal{Q}_n(G))_{\leq\omega} \xrightarrow{\sim} \mathcal{Q}_{m(\omega)}(G) \times \Pi_{|S_1|} \times \dots \times \Pi_{|S_k|}$$

where $m(\omega) := \frac{(|S_0|-1)}{|G|}$.

With this decomposition, we see that any irreducible element having a nonzero block that is not a singleton must have $S_0 = \{0\}$, thus be an element of $\mathcal{I}^G$. Moreover, $\mathcal{J}^G \setminus \mathcal{I}^G$ consists of the partitions where all simple blocks are singletons.

We will distinguish these two types of elements in $\mathcal{J}^G$ by calling any $x \in \mathcal{J}^G \cap \mathcal{I}^G$ *of type* 1, while we will refer to the elements in $\mathcal{J}^G \setminus \mathcal{I}^G$ as to those *of type* 0.

Remark 9.4.1. *A subset X of $\mathcal{J}^G$ is nested if and only if, for any $\omega, \omega' \in X$, the only nonsingleton blocks of the associated partitions $\underline{\omega}, \underline{\omega}'$ of $\{0\} \cup [n]$ are either disjoint or contained into one another.*

The following facts are now at hand, and we collect them for later reference.

Scholium 9.4.2. *Let $\mathcal{J}^G$ denote the minimal building set of $\mathcal{Q}_n(G)$.*

(1) $\mathcal{J}^G \cap \mathcal{Q}_n^G = \mathcal{I}^G$.
(2) $\mathcal{N}(\mathcal{I}^G, \mathcal{Q}_n^G) \subseteq \mathcal{N}(\mathcal{J}^G, \mathcal{Q}_n(G))$
(3) *For any $X \in \mathcal{N}(\mathcal{J}^G, \mathcal{Q}_n(G))$, $X \cap \mathcal{I}^G \in \mathcal{N}(\mathcal{I}^G, \mathcal{Q}_n^G)$.*
(4) *If $X \in \mathcal{N}(\mathcal{J}^G, \mathcal{Q}_n(G))$, the subset $X \setminus \mathcal{I}^G$ of the elements of type 0 is linearly ordered.*

9.4.2. Dowling trees. The last question we want to address is whether the nested set complex of the full Dowling lattice has an interpretation in terms of trees. The answer is positive, and leads to the definition of what we would like to call ***the complex of Dowling trees***.

Definition 9.4.3. *Given a natural number n and a finite group G, a Dowling tree is a G-tree T with some distinguished vertices, called **zero vertices** , satisfying following conditions:*

(0) *The root is a zero vertex.*
(1) *Every internal vertex (except the root) has degree at least 3.*

(2) *The tree is invariant under the G-action, and the zero vertices are fixed by this action.*
(3) *For any two different elements $g, h \in G$ and any $i \in [n]$, the (unique) shortest path connecting the leafs labelled (i, g) and (i, j) passes through exactly one zero vertex.*
(4) *The zero vertices form a path of strictly decreasing high, beginning at the root.*

On Dowling trees the operation of inner orbit contraction and extension are defined analogously as in $\mathcal{T}_n^G$ with the only difference that for every edge t that connects two zero vertices we have $\mathfrak{o}(t) = \{t\}$. Therefore the Dowling trees form an abstract simplicial complex that we will denote by $\mathcal{T}_n(G)$.

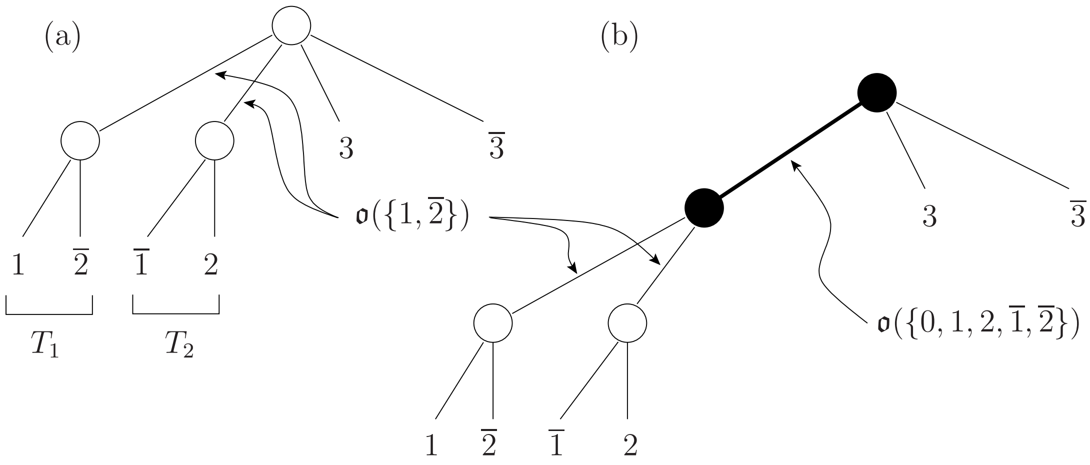


FIGURE 2. (a): the tree corresponding to $\{1, \overline{2}\}$. (b): the Dowling tree constructed from the nested set $\{1\overline{2}, 012\}$. The zero vertices are black. *As above, only a representant of every nonsingleton orbit is indicated. See figure 9.1.2.*

The goal of this section will be to show that in fact $\mathcal{T}_n(G)$ is isomorphic to $\widetilde{\mathcal{N}}(\mathcal{J}^G, \mathcal{Q}_n(G))$ as an abstract simplicial complex.

To this end, consider a simplex $X \in \widetilde{\mathcal{N}}(\mathcal{J}^G, \mathcal{Q}_n(G))$. Since $\mathcal{Q}_n^G$ is an order ideal in $\mathcal{Q}_n(G)$, we may choose a linear extension of the ordering in X such that all elements of type 1 come before all those of type 0. We will perform our inner orbit extensions according to the chosen linear order of X. After having exhausted all elements of type 1 we are clearly left with a tree $T \in \mathcal{T}_n^G$, that can be turned into a good Dowling tree just by declaring the root as the only zero vertex. On this tree we have now to perform the 'type 0' - orbit extensions.

So let ω be a type-0 partition, with zero block S_0. As above, there is a vertex v of T such that the union of the leaves in the connected components $T_1, \ldots, T_s$ of T not containing the root that arise by deleting v is minimal with the property of containing the set $S_0 \setminus \{0\}$. In particular, this v is fixed by the action of G and therefore is a zero vertex - but by the chosen linear order v can be only the root.

Let $\tau_1, \ldots, \tau_k$ denote the elements of X that are maximal among those below ω. By construction, to every τ_i correspond an inner orbit of edges that are incident to the root. Again by construction, all elements in $S_0 \setminus \{0\}$ that are not contained in a nonsingleton block of some τ_i are directly appended to the root. We may then renumber the T_i's in such a way that the union of the labels of the leafs of the first s' trees is exactly $S_0 \setminus \{0\}$.

Note that $s' < s$ because $\hat{1} \notin X$ (in figure 9.4.2 we have $s' = 2$ and the corresponding trees T_1, T_2 are indicated).

Then we build a tree starting with an edge t with ending in the root and in a new vertex w which we declare to be a zero vertex. Below w we grow s' edges $e_1, \dots e_{s'}$, and append to those the trees $T_1, \dots T_{s'}$. The trees $T_{s'+1}, \dots T_s$ will be appended to the root via edges $e_{s'+1}, \dots e_s$.

Now check that this is again a Dowling tree: we have only to worry about the zero vertices. Both the sets $\bigcup_{i \le s'} \lambda_{e_i}$ and $\bigcup_{i > s'} \lambda_{e_i}$ contain the full orbit of each of their elements, and therefore property (1), (2) and (3) follow immediately. For property (4) recall scholium 9.4.2 to see that all zero vertices (except the root) are in some T_i with $i \le s'$.

We have thus constructed a unique Dowling tree $T(X)$ from a nested set $X \in \mathcal{N}(\mathcal{J}^G, \mathcal{Q}_n(G))$. The inverse operation is now easy: given a Dowling tree T identify the orbits of all inner edges under the action of G, and note that the set of corresponding one-block-orbit elements of $\mathcal{Q}_n(G)$ is nested.

Scholium 9.4.4. *The bijection $T : \widetilde{\mathcal{N}}(\mathcal{J}^G, Dl) \to \mathcal{T}_n(G)$ extends to an isomorphism of simplicial complexes, if we take the operation of orbit contraction as boundary operator in $\mathcal{T}_n(G)$.*

Summarizing, we can formulate the following theorem, with a corollary that is a suggestive counterpart of corollary 9.3.4.

Theorem 9.4.5. *The complex of Dowling trees $\mathcal{T}_n(G)$ is a pure simplicial complex of dimension $(n-2)$. It is subdivided by the reduced order complex $\widetilde{\Delta}(\mathcal{Q}_n(G))$ of the Dowling lattice. Their realizations are therefore PL-homeomorphic.*

Corollary 9.4.6. *The reduced order complex $\widetilde{\Delta}(\mathcal{Q}_n(G))$ subdivides the complex of Dowling trees $\mathcal{T}_n(G)$, thus the two simplicial complexes are PL-homeomorphic. They are homotopy equivalent to a wedge of*

$$(|G|+1)(2|G|+1)\dots((n-1)|G|+1)$$

spheres of dimension $(n-2)$.

Example. For the examples considered above, where $n = 3$ and $G = \mathbb{Z}_2$, we have that the complexes $\widetilde{\Delta}(\mathcal{Q}_3^0(\mathbb{Z}_2))$ and $\mathcal{T}_3^{\mathbb{Z}_2}$ are homotopy equivalent of a wedge of $(2-1)(4-1) = 3$ circles, while $\widetilde{\Delta}(\mathcal{Q}_3(\mathbb{Z}_2))$ and $\mathcal{T}_3(\mathbb{Z}_2)$ have the homotopy type of a wedge of $(2+1)(2\cdot 2+1) =$ 15 circles.

9.4.3. From $\mathcal{T}_n^G$ to $\mathcal{T}_n(G)$. The description in terms of nested set complexes allows us to explicitly reconstruct $\mathcal{T}_n(G)$ from $\mathcal{T}_n^G$ by successively coning over subcomplexes having the homotopy type of wedges of $(n-3)$-spheres. This gives another proof of the fact that $\mathcal{T}_n^G$ is Cohen-Macaulay and allows us to explicitly calculate the difference of the number of spheres in the homotopy type of $\mathcal{T}_n^G$ and $\mathcal{T}_n(G)$.

First of all we want to distinguish three types of simplices in $\mathcal{T}_n(G)$. We call simplices of *type 0*, respectively of *type 1*, those simplices consisting only of elements of type 0, respectively of type 1. The nested sets containing elements of both type will be called simplices *of mixed type*. Remark that the subcomplex given by the simplices of type 1 is exactly $\mathcal{T}_n^G$, and that any simplex X of mixed type is contained in the star of a unique maximal simplex X_0 of type 0, namely $X_0 = X \setminus \mathcal{I}^G$.

The idea is therefore to start with $\mathcal{T}_n^G$ and glue successively the star of all simplices of type 0. Topologically, this means coning over the link of those simplices: to keep track of the change of topology, we need some definitions and a lemma.

DEFINITION 9.4.7. *Let $\mathcal{T}_J$ denote the subcomplex of $\mathcal{T}_n(G)$ consisting of all simplices of type* 0*, i.e.*

$$\mathcal{T}_J := \{X \in \mathcal{T}_n(G) | X \subset \mathcal{J}^G\}.$$

From the above considerations we know that any $X \in \mathcal{T}_J$ is a chain $\omega = \omega_1 < \omega_2 < \cdots < \omega_\ell$ of elements of type 0*. The length of the chain is the number of its elements and will be denoted $\ell(\omega) = \ell(X)$. The associated partitions $\underline{\omega}_i \in \Pi_{n,0}$ have only one nonsingleton block, namely the one containing* 0*, which we call w_i. Setting $w_0 := \{0\}$ and $w_{\ell+1} := \{0, 1, 2, \ldots, n\}$, we define numbers $p_0(\omega), \ldots p_\ell(\omega) \in \mathbb{N}$ as*

$$p_i(\omega) := |w_{i+1} \setminus w_i|.$$

If the chain is understood, we will simply write p_i. For $m = 1, \ldots n-1$ we define the subcomplex of $\mathcal{T}_n(G)$ consisting of $\mathcal{T}_n^G$ and the stars of all simplices $X \in \mathcal{T}_J$ with $\ell(X) \leq m$:

$$\mathcal{K}_m := \mathcal{T}_n^G \cup \{X \in \mathcal{T}_n(G) | \, |X \cap \mathcal{J}^G| \leq m\}.$$

LEMMA 9.4.8. *The link of any $X \in \mathcal{T}_J$ with $\ell(X) = m$ in $\mathcal{K}_m$ is*

$$lk_{\mathcal{K}_m}(X) \simeq \widetilde{\Delta}(B_m) * \widetilde{\Delta}(\mathcal{Q}_{p_0}^G) * \cdots * \widetilde{\Delta}(\mathcal{Q}_{p_m}^G),$$

where B_m denotes the boolean lattice on m elements.

Proof. Any simplex Y in the link can be written as

$$Y = Y' \amalg Y_0 \amalg \ldots Y_m,$$

where Y' is a (proper!) subset of X, and Y_i is a nested subset of $\mathcal{I}^G$ such that the only nonsingleton block of the associated partitions in $\Pi_{n,0}$ contains only elements from $w_{i+1} \setminus w_i$. The subcomplex of such Y_i can of course be identified with $\widetilde{\mathcal{N}}(\mathcal{I}^G, \mathcal{Q}_{p_i}^G)$, whereas the possible choices of Y' give a subcomplex with a face lattice that can be identified with the proper part of B_m, the boolean lattice on m elements. Remark that any choice of $Y' \in B_m \setminus \{\hat{1}\}$ and $Y_i \in \widetilde{\mathcal{N}}(\mathcal{I}^G, \mathcal{Q}_{p_i}^G)$ gives a simplex in the link.

Since all complexes $\widetilde{\Delta}(B_m)$ and $\widetilde{\mathcal{N}}(\mathcal{I}^G, \mathcal{Q}_{p_i}^G)$ are flag complexes, we have that the link of X is a simplicial complex that is isomorphic to the join

$$\widetilde{\Delta}(B_m) * \widetilde{\mathcal{N}}(\mathcal{I}^G, \mathcal{Q}_{p_0}^G) * \cdots * \widetilde{\mathcal{N}}(\mathcal{I}^G, \mathcal{Q}_{p_m}^G).$$

With corollary 8.4.8 the claim follows. $\square$

In order to simplify notation, let us define numbers q_i^ω associated to any chain ω that gives rise to a simplex in $\mathcal{T}_J$. Recall definition 9.4.7 and let

$$q_i^\omega := \prod_{j=1}^{p_i(\omega)-1} (j|G| - 1).$$

The numbers $Q(\omega)$ are then defined for any chain ω as

$$Q(\omega) := q_0^\omega q_1^\omega \ldots q_{\ell(\omega)}^\omega.$$

The convenience of these definitions can be seen already in the following corollary.

COROLLARY 9.4.9. *The link of any $X \in \mathcal{T}_J$ with $\ell(X) = m$ in $\mathcal{K}_m$ is homotopy equivalent to a wedge of $Q(\omega)$ spheres of dimension $(n-3)$, where ω is the chain obtained by ordering the elements of X. Each of those spheres is the boundary of an $(n-2)$-ball in $\mathcal{K}_m$.*

Proof. After Hultman [**81**] we know that, for any $p_i(\omega)$, $\widetilde{\Delta}(\mathcal{Q}^G_{p_i(\omega)})$ is homotopy equivalent to a wedge of q_i^ω spheres of dimension $(p_i(\omega)-2)$. It is a standard fact that $\widetilde{\Delta}(B_m) \simeq S^{(m-1)}$. We have then to compute the homotopy type of

$$S^{(m-1)} * \bigvee_{q_0^\omega} S^{(p_0-2)} * \cdots * \bigvee_{q_m^\omega} S^{(p_m-2)},$$

where the index under the wedges indicates how many copies of the corresponding sphere come into the game.

At this point we recall two basic topological facts, referring to any book on topology, e.g. [**22**], for a proof. One has namely $S^i * S^j \simeq S^{i+j+1}$, and $A*(B \vee C) \simeq (A \vee B) * (A \vee C)$ for topological spaces A, B, C and integers $i, j \geq 0$.

With these two rules, we may rewrite the above expression as

$$S^{(m-2)} * \bigvee_{q_0^\omega \cdots q_m^\omega} S^{\sum_{i=0}^m (p_i-2)+m} = \bigvee_{Q(\omega)} S^{m-2+n-2(m+1)+m+1} = \bigvee_{Q(\omega)} S^{(m-3)},$$

where in the second equality we used that $p_0 + p_1 + \ldots p_m = n$. This proves the first part of the corollary.

The last assertion is proved by induction on m, after remarking that actually the link of X in $\mathcal{K}_m$ is contained in $\mathcal{K}_{m-1}$ (where we define $\mathcal{K}_0 = \mathcal{T}_n^G$). For $m = 1$ the assertion holds because $\mathcal{T}_n^G$ is CM of dimension $(n-2)$, thus each $(n-3)$-sphere bounds a ball. Let the claim hold for $m \geq 1$. Then in particular $\mathcal{K}_m$ was obtained by repeatedly coning over spheres that were already boundaries - therefore $\mathcal{K}_m$ is also CM of dimension $(n-2)$, and any of its $(n-3)$-cycles bounds. □

Note: Since $p_i(\omega) < n$, we need the result of [**81**] only in dimension strictly smaller than the one in which the conclusion of the corollary hold. Therefore we may in principle omit the use of [**81**], thus reproving fully independently the result, by an induction on n - corollary 9.4.9 is clear for $n = 2$ - and a separate proof of corollary 9.4.11. We preferred not to do this in order to keep the argument simpler and more readable.

SCHOLIUM 9.4.10. *We have proved that any chain $\omega \in \widetilde{\Delta}(B_n)$ indexes a simplex of $\mathcal{T}_J$ that contributes $Q(\omega)$ times to the difference of the number of spheres between the homotopy types of $\mathcal{T}_n^G$ and $\mathcal{T}_n(G)$.*

Example: For our favourite example $\mathcal{Q}_3(\mathbb{Z}_2)$, we have 12 chains in $\widetilde{\Delta}(B_3)$, each with $Q(\omega) = 1$, therefore $\sum_{\omega \in \widetilde{\Delta}(B_3)} Q(\omega) = 12$, which in fact gives $12 + 3 = 15$.

We may even combine the results of Dowling about $\widetilde{\Delta}(\mathcal{Q}_n(G))$, of Hultman about $\mathcal{T}_n^G$ and our scholium 9.4.10 to state following arithmetic equality:

COROLLARY 9.4.11. *Let integers $k \geq 1$ and $n \geq 2$ be given, and for $\pi \in \Pi_n$ let $h(\pi, j)$ denote the height of the j-th column of the Young tableau of π. Then*

$$\prod_{j=1}^{n}(jk+1) - \prod_{j=1}^{n}(jk-1) = \sum_{\sigma \in \Pi_n} \prod_{j=1}^{n}(jk-1)^{h(\sigma,j)}.$$

CHAPTER 10

Simplicial shellable spheres via combinatorial blowups

Background. Let K denote an abstract simplicial complex on the vertex set $[n] := \{1, \ldots, n\}$, and write $\mathcal{F}(K)$ for the poset of its faces. The (combinatorial) *Alexander dual* for K is the simplicial complex $A(K)$ whose faces are the complements (in $[n]$) of the *non*-faces of K. Thus, $A(K) = \{[n] \setminus \sigma \mid \sigma \in 2^{[n]} \setminus K\}$. The topological motivation for this suggestive name is that, in fact, K and $A(K)$ can be 'put together' to a sphere. A very nice construction of this sphere is due to Thomas Bier [**7**] and can be found in [**89**, p. 111-116]. Bier showed that the deleted join $\mathrm{Bier}_n(K) := (K * A(K))_\Delta$ is an $(n-2)$-sphere with at most $2n$ vertices [**89**, Theorem 5.6.2]. The idea behind this proof is to embed $\mathcal{F}(K)$ in the boolean lattice B_n and see that $\mathrm{Bier}_n(K)$ is in fact a subdivision of the boundary of $\Delta(B_n)$.

Björner et al. generalize this construction in [**14**] to obtain a *Bier poset* $\mathrm{Bier}(I, P)$ associated to any proper lower ideal I in any bounded poset P. They show that, for any such P, the order complex of $\overline{\mathrm{Bier}(P, I)}$ is PL-homeomorphic to that of $\overline{P}$. In the boolean case $P = B_n$ we have $\mathrm{Bier}(B_n, I) = \mathrm{Bier}_n(I)$ (at the right hand side of the equality I is seen as an abstract simplicial complex). In the same paper shellability of $\mathrm{Bier}(B_n, I)$ is proven, together with a characterization of its g-vector. In particular, it is pointed out that, for large n, this construction leads to 'many simplicial shellable $(n-2)$-spheres', most of them lacking convex realization (see [**68**]).

Overview. We put this construction in the context of the theory of nested set complexes. Specifically, we find a conceptual way of proving that P and $\mathrm{Bier}(P, I)$ are PL-homeomorphic if P is a lattice.

Section 10.1 relates the poset P to a building set in the poset $\mathrm{Bier}(P, I)$ and then, restricting to the case when $\mathrm{Bier}(P, I)$ is a semilattice, shows how combinatorial blowups relate order complexes of $\overline{P}$ and $\overline{\mathrm{Bier}(P, I)}$.

Moreover, we obtain shellability of $\mathrm{Bier}(P, I)$ for any shellable lattice P and 'even more' simplicial shellable spheres. Indeed, the core of Section 10.2 is Theorem 10.2.5, where we show that combinatorial blowups preserve shellability. This applies in particular to the Bier construction and implies that $\mathrm{Bier}(P, I)$ is shellable and homotopy equivalent to a wedge of spheres whenever P is a shellable poset. Note that this allows us even to iterate the Bier construction to get shellable triangulations of spheres with any number of vertices, whose isomorphism type depends on P, I, and the number of iterations (Remark 10.2.9).

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10.1. Bier lattices and their building sets

DEFINITION 10.1.1. ([**14**, Definition 1.1]) *Let P be a bounded poset of finite length and $I \subset P$ a proper ideal. Then the poset* $\mathrm{Bier}(P, I)$ *is defined as follows:*

- *elements are all intervals* $[x, y] \subseteq P$ *such that* $x \in I$ *and* $y \notin I$*, together with an additional top element* $\hat{1}$*,*
- *intervals are ordered by reverse inclusion, i.e.* $[x, y] \leq [v, w]$ *if and only if* $x \leq v < w \leq y$*.*

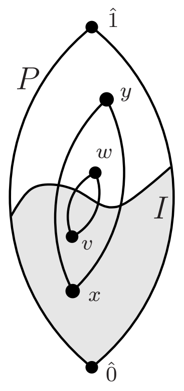

FIGURE 1. Two elements $[x, y]$ and $[v, w]$ of $\mathrm{Bier}(P, I)$ with $[x, y] \leq [v, w]$.

Clearly, $\mathrm{Bier}(P, I)$ is a bounded poset. Furthermore, by [**14**, Lemma 1.2], if P is a lattice, then $\mathrm{Bier}(P, I)$ is also a lattice.

From now on, unless stated otherwise, we will assume that L is a lattice. Then $\mathrm{Bier}(L, I)$ is also a lattice, and we can therefore apply the theory of nested set complexes. We begin by describing a building set in $\mathrm{Bier}(L, I)_{<\hat{1}}$ that is naturally associated to L.

PROPOSITION 10.1.2. *For any Bier lattice* $\mathrm{Bier}(L, I)$*, where* $I \subseteq L$ *is a proper ideal,*

$$\mathcal{G} := \{[\hat{0}, y] \mid y \in \overline{L} \setminus I\} \cup \{[x, \hat{1}] \mid x \in I \setminus \{\hat{0}\}\}$$

is a building set in $\mathrm{Bier}(L, I)_{<\hat{1}}$*.*

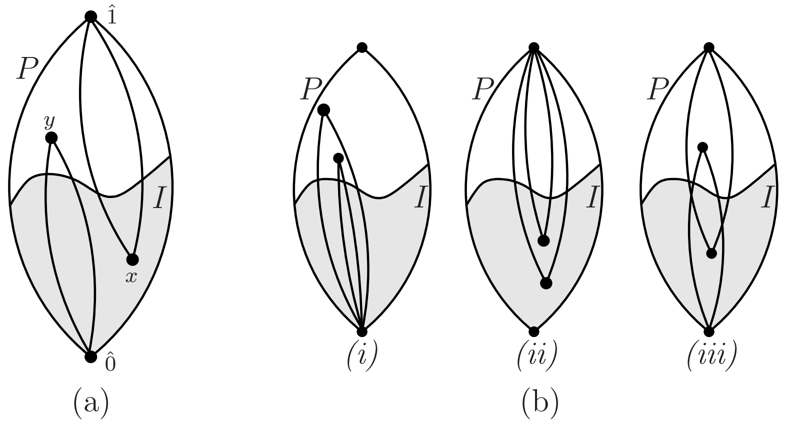

FIGURE 2. (a) Elements of the building set $\mathcal{G}$. (b) The three cases considered in Lemma 10.1.3.

PROOF. Let $[x, y] \in \overline{\mathrm{Bier}(L, I)}$. If $[x, y] \in \mathcal{G}$, then Definition 8.2.1 is clearly satisfied. Suppose then that $[x, y] \notin \mathcal{G}$, i.e. $x \neq \hat{0}$ and $y \neq \hat{1}$. It is clear that

$$\max \mathcal{G}_{\leq [x,y]} = \{[x, \hat{1}], [\hat{0}, y]\}.$$

Since in $\mathrm{Bier}(L,I)$ we have that $\big[[\hat{0},\hat{1}],[x,\hat{1}]\big] = \{[x',\hat{1}] \mid x' \leq x\}$, $\big[[\hat{0},\hat{1}],[\hat{0},y]\big] = \{[\hat{0},y'] \mid y' \geq y\}$, and $\big[[\hat{0},\hat{1}],[x,y]\big] = \{[x',y'] \mid x' \leq x,\ y' \geq y\}$, it is easy to see that the function

$$\begin{array}{rcl}\psi_{[x,y]} : \big[[\hat{0},\hat{1}],[x,\hat{1}]\big] \times \big[[\hat{0},\hat{1}],[\hat{0},y]\big] & \longrightarrow & \big[[\hat{0},\hat{1}],[x,y]\big]\,, \text{ where} \\ ([x',\hat{1}],[\hat{0},y']) & \longmapsto & [x',y'],\end{array}$$

is an order-preserving bijection of these posets.

It is also easy to see that $\psi_{[x,y]}([\hat{0},\hat{1}],[\hat{0},y']) = [\hat{0},y']$ and $\psi_{[x,y]}([x',\hat{1}],[\hat{0},\hat{1}]) = [x',\hat{1}]$. Therefore the function $\psi_{[x,y]}$ satisfies the conditions of Definition 8.2.1, and $\mathcal{G}$ is a building set in $\mathrm{Bier}(L,I)_{<\hat{1}}$. □

It is also true that $\mathcal{G}$ is a building set in $\mathrm{Bier}(L,I)_{<\hat{1}}$ if L is any bounded poset, since the definition we gave is independent of the existence of meet and join operations. However, in that case it is not clear how to characterize the concept of nested sets, even for the special case of Bier posets.

In the lattice case the standard definition works, and therefore we move towards our next goal, the characterization of the nested set complex of $\mathcal{G}$, which will be reached in Proposition 10.1.4. We need a preparatory lemma.

LEMMA 10.1.3. *Let $\mathcal{G}$ be the building set of* $\mathrm{Bier}(L,I)_{<\hat{1}}$ *defined in Proposition 10.1.2. Then a (possibly empty) subset A of $\mathcal{G}$ is nested if and only if the following three conditions are satisfied:*

(i) If $[\hat{0},y_1],[\hat{0},y_2] \in A$, then y_1 and y_2 are comparable in L.
(ii) If $[x_1,\hat{1}],[x_2,\hat{1}] \in A$, then x_1 and x_2 are comparable in L.
(iii) If $[x,\hat{1}],[\hat{0},y] \in A$, then $x < y$ in L.

PROOF. ($\Longrightarrow$) Suppose that A is nested, and suppose that (i) is not true, i.e. that there exists a pair of elements $[\hat{0},y_1],[\hat{0},y_2] \in A$ so that y_1 and y_2 are not comparable. Then $\big\{[\hat{0},y_1],[\hat{0},y_2]\big\}$ is a set of incomparable elements in A. But

$$[\hat{0},y_1] \vee [\hat{0},y_2] = \begin{cases} [\hat{0},y_1 \wedge y_2] \in \mathcal{G}, & \text{if } y_1 \wedge y_2 \notin I; \\ \text{doesn't exist}, & \text{otherwise.}\end{cases}$$

This is a contradiction with A being nested. Condition (ii) is proved analogously.

It is left to prove that (iii) is true. Since $[x,\hat{1}]$ and $[\hat{0},y]$ are incomparable, $[x,\hat{1}]\vee[\hat{0},y] = [p,q] \in \mathrm{Bier}(L,I)_{<\hat{1}} \setminus \mathcal{G}$, where $x \leq p < q < \hat{1}$ and $\hat{0} < p < q \leq y$. Therefore $x < y$.

($\Longleftarrow$) Suppose that $A = \{[x_1,\hat{1}],\dots,[x_l,\hat{1}],[\hat{0},y_1],\dots,[\hat{0},y_k]\}$, and $x_1 < x_2 < \cdots < x_l < y_1 < \cdots < y_k$. If $l = 0$ or $k = 0$, then A is a chain, and hence nested. Otherwise, it is clear that the cardinality of any set of incomparable elements is at most two. Take any two incomparable elements $[x_p,\hat{1}]$ and $[\hat{0},y_q]$ from A. Then $[x_p,\hat{1}] \vee [\hat{0},y_q] = [x_p,y_q] \notin \mathcal{G}$. Therefore A is a nested set. □

The lemma immediately implies the following proposition.

PROPOSITION 10.1.4. *Let $\mathcal{G}$ be the building set of* $\mathrm{Bier}(L,I)_{<\hat{1}}$ *defined in Proposition 10.1.2. Then $\mathcal{N}(\mathcal{G}) = \Delta(\overline{L})$.*

From the previous two propositions, we can directly deduce the lattice case of [**14**, Theorem 2.2]:

COROLLARY 10.1.5. *Let L be a lattice with finite length $l(L) < \infty$, and let $I \subset L$ be a proper ideal. Then $\Delta(\overline{\mathrm{Bier}(L,I)})$ is obtained from $\Delta(\overline{L})$ by a sequence of stellar* *subdivisions on all the edges from the set* $\mathcal{S} = \big\{\{x,y\} \mid x \in I \setminus \{\hat{0}\},\ y \in \overline{L} \setminus I\big\}$, *where these subdivisions are performed in an order of increasing length* $l(x,y)$.

Proof. If $\mathcal{G}$ is the building set defined in Proposition 10.1.2, then it is not hard to see that $\overline{\mathrm{Bier}(L,I)} = \mathcal{G} \cup \mathcal{S}'$, where $\mathcal{S}' = \{[x,y] \mid x \in I \setminus \{\hat{0}\} \text{ and } y \in \overline{L} \setminus I\}$. Let $\mathcal{S} = \{e_1, \dots, e_k\}$, where if $e_i = \{x_i, y_i\}$, $e_j = \{x_j, y_j\}$, and $i < j \Rightarrow l(x_i, y_i) \le l(x_j, y_j)$. Set $\mathcal{G}_0 = \mathcal{G}$, and for $i \in [k]$, $\mathcal{G}_i = \mathcal{G}_{i-1} \cup \{[x_i, y_i]\}$, where $e_i = \{x_i, y_i\}$. It is clear that $[x_i, y_i] \in \max(\mathrm{Bier}(L,I)_{<\hat{1}} \setminus \mathcal{G}_{i-1})$, and therefore, by Proposition 8.4.6, $\mathcal{F}(\mathcal{N}(\mathcal{G}_i)) = \mathrm{Bl}_{\{[x_i,\hat{1}],[\hat{0},y_i]\}}(\mathcal{F}(\mathcal{N}(\mathcal{G}_{i-1})))$. In other words, since $\Delta(\overline{L}) = \mathcal{N}(\mathcal{G}_0)$, and blowup in this case corresponds to stellar subdivision of the edge $\{x_i, y_i\}$, we have that

$$\mathcal{N}(\mathcal{G}_i) = \mathrm{sd}_{e_i}(\mathrm{sd}_{e_{i-1}}(\dots(\mathrm{sd}_{e_1}\Delta(\overline{L})))).$$

We finish the proof remarking that

$$\mathcal{N}(\mathrm{Bier}(L,I)_{<\hat{1}}, \overline{\mathrm{Bier}(L,I)}) = \Delta(\overline{\mathrm{Bier}(L,I)}),$$

since $\overline{\mathrm{Bier}(L,I)}$ is the maximal building set in $\mathrm{Bier}(L,I)_{<\hat{1}}$. □

It is important to emphasize the following:

Corollary 10.1.6. *For any lattice* L *and any proper ideal* $I \subset L$, *the simplicial complexes* $\|\Delta(\overline{\mathrm{Bier}(L,I)})\|$ *and* $\|\Delta(\overline{L})\|$ *are PL homeomorphic. Furthermore, if* L *is a face lattice of a strongly regular PL CW-sphere, then so is* $\mathrm{Bier}(L,I)$.

10.2. Recursive coatom orderings and shellability of Bier lattices

We now proceed to study the case of a shellable lattice or poset. After recalling the definition of shellability of a simplicial complex we will prove a proposition asserting that combinatorial blowups, and thus stellar subdivisions, preserve shellability.

Definition 10.2.1. *Let* P *be a bounded poset. We will say that* P *admits a* **recursive coatom ordering** *if* $P = \{\hat{0}, \hat{1}\}$, *or if there exists a coatom ordering* $c_1, \dots, c_r$ *so that the following two conditions are satisfied:*

(R) *For all* $j \in [r]$, *the poset* $[\hat{0}, c_j]$ *admits a recursive coatom ordering in which coatoms of* $[\hat{0}, c_j]$ *which are contained in* $[\hat{0}, c_i]$, *for some* $i < j$, *come before all other coatoms.*

(S) *For all* $1 \le i < k \le r$ *and all* $x \in P$, *if* $x \le c_i$ *and* $x \le c_k$, *then there exists some* $j < k$ *and some coatom* ω *of* $[\hat{0}, c_k]$ *so that* $x \le \omega \le c_j$.

Remark 10.2.2. *It was noticed in* [**105**, Proposition 2.13] *that, in the case when* L *is a finite lattice and* $c_1, \dots, c_r$ *is some coatom ordering of* L, *then this ordering satisfies condition (S) of Definition 10.2.1 if and only if it satisfies the following condition:*

(T) For all $1 \le i < j \le r$, there exists some $k < j$ so that

$$c_i \wedge c_j \le c_k \wedge c_j \lessdot c_j.$$

Since we will work with face lattices of simplicial complexes, we will verify conditions (R) and (T). The face lattice of a simplicial complex K *we will denote by* $\hat{\mathcal{F}}(K)$, *that is* $\hat{\mathcal{F}}(K) = \mathcal{F}(K) \cup \{\hat{1}\}$.

We will now state two propositions which will be used further in the paper. For their proofs we refer the reader to the corresponding papers.

Proposition 10.2.3. [**16**, Theorem 5.13] *A simplicial complex* K *is shellable if and only if* $\hat{\mathcal{F}}(K)$ *admits a recursive coatom ordering.*

PROPOSITION 10.2.4. [**15**, Theorem 5.1] *A graded poset P is totally semimodular if and only if for every interval $[x, y]$ of P, every atom ordering in $[x, y]$ is a recursive atom ordering.*

The next theorem is the main result of this section.

THEOREM 10.2.5. *Let $\hat{\mathcal{F}} = \hat{\mathcal{F}}(K)$ be a face lattice of some simplicial complex K, $\mathcal{F} = \hat{\mathcal{F}} \setminus \{\hat{1}\}$, and assume that $\hat{\mathcal{F}}$ admits a recursive coatom ordering. Then $\hat{\mathcal{F}}_\alpha = \mathrm{Bl}_\alpha(\mathcal{F}) \cup \{\hat{1}\}$ also admits a recursive coatom ordering, where α is any element of $\hat{\mathcal{F}} \setminus \{\hat{0}, \hat{1}\}$.*

PROOF. Since $\hat{\mathcal{F}}_\alpha$ is the face lattice of the simplicial complex $\mathrm{sd}_\alpha(K)$, intervals below maximal faces in $\hat{\mathcal{F}}_\alpha$ are boolean. Having in mind that every boolean lattice is self-dual, graded, and totally semimodular (and that a recursive atom ordering of a poset is a recursive coatom ordering of its dual), by Proposition 10.2.4 every coatom ordering of these intervals is recursive. Therefore, to check that some ordering of coatoms of $\hat{\mathcal{F}}_\alpha$ is recursive, it suffices to prove that it satisfies condition (T) from the remark above.

Suppose now that $\mathcal{O} = \{F_1, \dots, F_n\}$ is a recursive coatom ordering of $\hat{\mathcal{F}}$. Define $\mathcal{C} \subset \mathcal{F}$ in the following way:

$$\mathcal{C} = \max\{G \in \mathcal{F} \mid G \not\supseteq \alpha \text{ and } G \cup \alpha \text{ is an element of } \mathcal{F}\}.$$

Set $I = \{i \in [n] \mid F_i \supseteq \alpha\}$. Then it is not hard to see that all coatoms of $\hat{\mathcal{F}}_\alpha$ are

$$\mathrm{coat}(\hat{\mathcal{F}}_\alpha) = \{F_i \mid i \in [n] \setminus I\} \cup \{\langle \alpha, G \rangle \mid G \in \mathcal{C}\}.$$

Denote the elements of I with $i_1, \dots, i_t$, where $t = |I|$, and $i_1 < i_2 < \dots < i_t$. Having in mind that the F_i's are the coatoms of $\hat{\mathcal{F}}$, and that $F_i \supseteq \alpha$, for all $i \in I$, it is not hard to see that

$$\mathcal{C} = \{G \in \mathcal{F} \mid G \not\supseteq \alpha \text{ and } G \text{ is a codimension 1 face of } F_i, \text{ for some } i \in I\}.$$

Let now, for all $l \in [n]$, $\mathcal{A}_l = \{G \in \mathcal{C} \mid G \subseteq F_l \setminus (\bigcup_{j=1}^{l-1} F_j)\}$. It is easy to see that $\mathcal{C} = \bigsqcup_{1 \le l \le i_t} \mathcal{A}_l$, where the symbol $\bigsqcup$ denotes the disjoint union. For any $G \in \mathcal{C}$, let $A(G) = \min\{l \in [n] \mid F_l \supseteq G\}$ and let $\mathrm{mf}(G) = \min\{s \in I \mid G \lessdot F_s\}$. It is clear that $A(G) \le \mathrm{mf}(G)$.

Define a relation $\prec$ between coatoms of $\hat{\mathcal{F}}_\alpha$ in the following way:

- For any $i, j \in [n] \setminus I$, $F_i \prec F_j$ if and only if $i < j$.
- For any $i \in [n] \setminus I$ and $G \in \mathcal{C}$, $F_i \prec \langle \alpha, G \rangle$ if and only if $i < \mathrm{mf}(G)$, and $F_i \succ \langle \alpha, G \rangle$ if and only if $i > \mathrm{mf}(G)$.
- For any $E, G \in \mathcal{C}$, $\langle \alpha, E \rangle \prec \langle \alpha, G \rangle$ if and only if either $\mathrm{mf}(E) < \mathrm{mf}(G)$, or $\mathrm{mf}(E) = \mathrm{mf}(G)$ and $A(E) < A(G)$.

It is not hard to check that $\preceq$ is indeed a partial ordering. We will choose a linear extension thereof and denote it by $\mathcal{O}_\alpha$. In order to prove that condition (T) holds for $\mathcal{O}_\alpha$, we need to prove the following four cases:

(1) Assume $F_i, F_j \in \mathrm{coat}(\hat{\mathcal{F}}_\alpha)$, and let F_i come before F_j in $\mathcal{O}_\alpha$. Then, by definition, $i < j$, and $\alpha \not\subseteq F_i, F_j$. Since $\mathcal{O}$ is a recursive coatom ordering, there exists $k < j$ so that $F_i \cap F_j \subseteq F_k \cap F_j = F_j \setminus \{v\}$, where v is some vertex of F_j.

(1.1) If $\alpha \not\subseteq F_k$, then $F_k \in \mathrm{coat}(\hat{\mathcal{F}}_\alpha)$, clearly F_k comes before F_j in $\mathcal{O}_\alpha$, and $F_i \wedge F_j \le F_k \wedge F_j \lessdot F_j$.

(1.2) If $\alpha \subseteq F_k$, pick any vertex $w \in \alpha \setminus F_j \subset F_k$. Then $F_k \setminus \{w\}$ is a codimension 1 face of F_k which does not contain α, and therefore $\langle \alpha, F_k \setminus \{w\} \rangle \in \mathrm{coat}(\hat{\mathcal{F}}_\alpha)$. Noticing that $\mathrm{mf}(F_k \setminus \{w\}) \le k < j$, it is clear that $\langle \alpha, F_k \setminus \{w\} \rangle$ comes before

F_j in $\mathcal{O}_\alpha$. Since $F_j \wedge \langle \alpha, F_k \setminus \{w\}\rangle = F_j \cap (F_k \setminus \{w\}) = F_j \setminus \{v\}$, we have that $F_i \wedge F_j \leq \langle \alpha, F_k \setminus \{w\}\rangle \wedge F_j \lessdot F_j$.

(2) Let $F_i, \langle \alpha, G\rangle \in \mathrm{coat}(\hat{\mathcal{F}}_\alpha)$, and suppose that F_i comes before $\langle \alpha, G\rangle$ in $\mathcal{O}_\alpha$. Then $\alpha \nsubseteq F_i$ and $i < \mathrm{mf}(G)$. Suppose first that $A(G) < \mathrm{mf}(G)$. Then $G \subseteq F_{A(G)}$, where $\alpha \nsubseteq F_{A(G)}$, and $F_{A(G)}$ comes before $\langle \alpha, G\rangle$ in $\mathcal{O}_\alpha$. Then we have that $F_i \wedge \langle \alpha, G\rangle = F_i \cap G \leq G = F_{A(G)} \cap G = F_{A(G)} \wedge \langle \alpha, G\rangle \lessdot \langle \alpha, G\rangle$, and condition (T) is satisfied in this case.

Let now $A(G) = \mathrm{mf}(G) = j$. By the assumption, there exists $k < j$ so that $F_i \cap F_j \subseteq F_k \cap F_j = F_j \setminus \{v\}$, for some $v \in F_j$. Note that $v \in G$, since otherwise $G \subseteq F_k$, and therefore $A(G) \leq k < j$, which would give a contradiction.

Since $G \lessdot F_j$, denote by w the vertex such that $F_j \setminus \{w\} = G$, where clearly $w \in \alpha$, and let

$$H = \begin{cases} F_j \setminus \{v\} \lessdot F_j, & \text{if } \alpha \nsubseteq F_k; \\ F_k \setminus \{w\} \lessdot F_k, & \text{if } \alpha \subseteq F_k. \end{cases} \tag{2.1}$$

By simple checking, one can see that $\alpha \nsubseteq H$, and $H \in \mathcal{C}$, since H is a coatom in some face that contains α. Seeing that $H \subseteq F_k$, we conclude that either $\mathrm{mf}(H) < \mathrm{mf}(G)$, or $\mathrm{mf}(H) = \mathrm{mf}(G)$ and $A(H) < A(G)$, and therefore $\langle \alpha, H\rangle$ comes before $\langle \alpha, G\rangle$ in $\mathcal{O}_\alpha$. Now it is easy to see that

$$F_i \wedge \langle \alpha, G\rangle = F_i \cap G \leq G \setminus \{v\} < \langle \alpha, G \setminus \{v\}\rangle = \langle \alpha, G\rangle \wedge \langle \alpha, H\rangle \lessdot \langle \alpha, G\rangle.$$

(3) If $\langle \alpha, G\rangle, F_j \in \mathrm{coat}(\hat{\mathcal{F}}_\alpha)$ and $\langle \alpha, G\rangle$ comes before F_j in $\mathcal{O}_\alpha$, then it must be $\mathrm{mf}(G) < j$. The proof for this case is the same as for case (1), having in mind that $\langle \alpha, G\rangle \wedge F_j = G \cap F_j \leq F_{\mathrm{mf}(G)} \cap F_j$, and since in case (1) we didn't use the fact that $\alpha \nsubseteq F_i$.

(4) Finally, the case when $\langle \alpha, E\rangle, \langle \alpha, G\rangle \in \mathrm{coat}(\hat{\mathcal{F}}_\alpha)$, and $\langle \alpha, E\rangle$ comes before $\langle \alpha, G\rangle$ in $\mathcal{O}_\alpha$. Then there are two possibilities, either $\mathrm{mf}(E) = \mathrm{mf}(G)$ or $\mathrm{mf}(E) < \mathrm{mf}(G)$.

The case when $j := \mathrm{mf}(E) = \mathrm{mf}(G)$ is easy, namely both E and G are codimension 1 faces in F_j, and hence $E \cap G$ is a codimension 1 face of G. Therefore, we have that $\langle \alpha, E\rangle \wedge \langle \alpha, G\rangle = \langle \alpha, E \cap G\rangle \lessdot \langle \alpha, G\rangle$.

If $\mathrm{mf}(E) < \mathrm{mf}(G)$, let $i = \mathrm{mf}(E)$, $j = \mathrm{mf}(G)$, and let $k < j$ be an integer such that $F_i \cap F_j \subseteq F_k \cap F_j = F_j \setminus \{v\}$, for some $v \in F_j$. Define w to be the vertex so that $F_j \setminus \{w\} = G$, and H as in equation (2.1). Then, since $E \cap G \subseteq F_i \cap G \subseteq G \setminus \{v\}$, we have that

$$\langle \alpha, E\rangle \wedge \langle \alpha, G\rangle = \langle \alpha, E \cap G\rangle \leq \langle \alpha, G \setminus \{v\}\rangle = \langle \alpha, G\rangle \wedge \langle \alpha, H\rangle \lessdot \langle \alpha, G\rangle.$$

Therefore we have proved that $\mathcal{O}_\alpha$ satisfies condition (T) and hence is a recursive coatom ordering of $\hat{\mathcal{F}}_\alpha$. □

Since stellar subdivisions are described by combinatorial blowups in the face poset, we can formulate the following corollary.

COROLLARY 10.2.6. *If K is a shellable simplicial complex, then so is $\mathrm{sd}_F(K)$, where F is any nonempty face of K.*

Now we return to the general Bier poset construction, and conclude that:

COROLLARY 10.2.7. *Let L be a lattice so that $\Delta(\overline{L})$ is shellable. If $I \subset L$ is a proper ideal in L, then $\Delta(\overline{\mathrm{Bier}(L, I)})$ is also shellable.*

REMARK 10.2.8 (The general case). If we consider any bounded poset P, the first part of Corollary 10.1.6 remains true (see [**14**, Corollaries 2.3 and 2.4] for a proof of this fact). Since stellar subdivisions are described by combinatorial blowups in the face poset, Corollary 10.2.7 also remains true in the general case.

Let us also mention here that, by a remark at the end of Section 2 and [**14**, Theorem 2.2], for every bounded poset P and for any proper ideal $I \subset P$, the simplicial complexes $\Delta(\overline{P})$ and $\Delta(\overline{\mathrm{Bier}(P, I)})$ have the same simple homotopy type.

Remark 10.2.9 (Shellable spheres). It is clear that using the Bier poset construction together with Corollaries 10.1.6 and 10.2.7, one can obtain numerous simplicial shellable n-spheres with more than $2(n+2)$ vertices, therefore answering one of the questions asked in [**14**].

Choosing an appropriate poset L we also obtain numerous shellable simplicial complexes with the homotopy type of wedges of spheres, with any number of vertices.

CHAPTER 11

Subdivision of complexes of k-Trees

Background. The complex of k-trees $\mathcal{T}_n^k$ is the abstract simplicial complex whose faces correspond to combinatorial types of rooted trees with $m = (n-1)k+1$ labelled leaves and all outdegrees congruent to 1 modulo k (and larger than k). Face relations among these trees are such that T_2 is a face of T_1 if and only if T_2 is obtained from T_1 by contraction of internal edges. Complexes of k-trees were defined by Hanlon in [**77**] as a generalization of complexes of trees [**18**]. The complexes $\mathcal{T}_n^k$ are pure simplicial complexes of dimension $n-3$. They were shown to be Cohen-Macaulay by Hanlon in [**77**], and shellability was proven independently by Trappmann and Ziegler [**111**] and by Wachs [**116**].

The symmetric group S_m acts naturally on $\mathcal{T}_n^k$ by permuting the leaves. In fact, the complex $\mathcal{T}_n^k$ was introduced by Hanlon because of the relation of its top homology group to the 1^m homogeneous component of the free Lie-k-algebra with m odd generators, both as S_m-modules. Lie-k-algebras are a generalization of Lie algebras introduced by Hanlon and Wachs in [**79**]. Here the poset $\Pi_m^{(k)}$ of partitions of $\{1,\ldots,m\}$ with block sizes congruent to 1 modulo k enters our picture. This poset was shown to be Cohen-Macaulay by Björner [**9**], and was studied by Calderbank, Hanlon and Robinson in [**27**] from the point of view of representation theory. Combining the results of [**77**] and [**79**] one finds that there is a S_m-module isomorphism between $\widetilde{H}_{n-3}(\mathcal{T}_n^k)$ and $\widetilde{H}_{n-3}(\Pi_m^{(k)})$, since both are isomorphic to the 1^m homogeneous piece of the free Lie-k-algebra with m odd generators.

This situation leads to the very natural question of the geometric relationship between the two complexes.

The case $k=1$ is interesting on its own right. The complex $\mathcal{T}_n = \mathcal{T}_n^1$ (called *complex of phylogenetic trees* or just 'complex of trees') first appeared in [**18**], in the context of homotopy theory. Its S_m-representation theory was studied by Robinson and Whitehouse [**101**], it figured in geometric group theory in work of Vogtmann [**115**], and recently attracted interest again in the study of phylogenetic trees [**8**]. In fact, the order complex of the partition lattice Π_m is PL-homeomorphic to $\mathcal{T}_n$ [**1**]. More specifically, $\Delta(\Pi_m)$ is obtained from $\mathcal{T}_n$ by a sequence of stellar subdivisions. This was proved in [**58**] as an application of the theory of combinatorial nested set complexes. In [**58**, Question 5.2], Feichtner asks whether for general k the complexes $\Pi_m^{(k)}$ and $\mathcal{T}_n^k$ are related by a subdivision.

Overview. We answer the above question positively. The direct approach via nested set complexes does not work: other than Π_m, $\Pi_m^{(k)}$ is not a lattice, which is an essential condition for the applicability of the theory of nested sets. However, in our context we can introduce analogous concepts that allow us to reduce the problem to a situation that can be handled by the existing theory.

An essential stepping stone is the precise understanding of the topological framework for subdivisions of simplicial complexes. We study it in section 11.1, ending with lemma 11.1.4: the key tool that will enable us to prove our result. In section 11.2 the analogies with the classical framework of nested sets will be put to work, giving explicite subdivisions of each simplex of $\mathcal{T}_n^k$ (Proposition 11.2.11). By lemma 11.1.4 we will then be able to construct a subdivision of the whole complex (Theorem 11.2.12). Although our method is very much tailored to the specific situation of the partition posets, we have kept the notations and the arguments as close as possible to the original theory of nested sets, hoping that this may be a first step towards a generalization of the theory to non-semilattices.

11.1. Subdivisions of abstract simplicial complexes

Let K be an abstract simplicial complex on the vertex set $V(K)$. We denote its face poset by $\mathcal{F}(K)$. Given $s \in \mathcal{F}(K)$, we will write K_s for the simplicial complex of all nonempty subsets of the vertex set $V(K_s) = s$.

Recall that the *realization* of a simplicial complex K is the topological space $|K|$ whose points are the formal convex combinations $\Sigma t_v v$ of vertices (i.e., the coefficients t_v are all nonnegative and $\sum_{v \in V(K)} t_v = 1$) such that the set $\{v \in V(K) | \, t_v \neq 0\}$ belongs to the family K. For any simplex $s \in K$ define $|s| := |K_s| = \{\Sigma t_v v \in |K| | \, t_v \neq 0 \Rightarrow v \in s\}$. If $s \in K$ contains n vertices there is a natural correspondence between $|s|$ and the standard Euclidean n-simplex Δ^n. The topology on $|K|$ is such that $A \subseteq |K|$ is closed if and only if $A \cap |s|$ is closed (in the Euclidean topology induced by Δ^n) for each simplex s of K.

The interior of a simplex $s \in K$ is then $\langle s \rangle := |s| \setminus \partial |s|$, and for each point $x \in |K|$ we can now define the *carrier* of x as $\tau(x) := \min\{s \in K | x \in \langle s \rangle\}$.

Definition 11.1.1. (see [**106**, p.121]). *Let P, Q be simplicial complexes. P is a* **subdivision** *of Q if there exists a poset map $\phi : \mathcal{F}(P) \to \mathcal{F}(Q)$ and an (injective) mapping $f^0 : V(P) \to |Q|$ such that $f^0(V(P_p)) \subseteq |Q_{\phi(p)}|$ for all simplices $p \in P$, and the linear extension $f : |P| \to |Q|$ of f^0 to $|P|$ is a homeomorphism.*

The map ϕ will be called the *carrier map*, and f^0 the *vertex map* of the subdivision.

Our point of view will be to take a given map between the face posets, and look for conditions under which this map is the carrier map of a subdivision (or, as we will say in the following: *is* a carrier map).

Suppose that we have a good candidate for f^0. Its linear extension f is continuous because it is linear -hence continuous- on every closed simplex. Realizations of simplicial complexes are compact, Hausdorff spaces. Therefore it suffices to check bijectivity of f to conclude that f is a homeomorphism. We summarize this in the following lemma for later reference.

Lemma 11.1.2. (see [**106**, p.122]) *Let P, Q be simplicial complexes, and let a vertex map $f^0 : V(P) \to |Q|$ be given. The linear extension*

$$\begin{array}{rccc} f: & |P| & \to & |Q| \\ & \Sigma t_v v & \mapsto & \Sigma t_v f^0(v) \end{array}$$

of f^0 to $|P|$ is a homeomorphism if and only if for every simplex q of Q the set $\{f(\langle p \rangle) | f^0(p) \subseteq |q|\}$ is a finite partition of $\langle q \rangle$.

We now define the concept of *compatibility* of carrier maps. This will lead to a lemma that will be one of the key tools in our argument.

Definition 11.1.3. *Let P, Q be simplicial complexes, P_1, P_2 and Q_1, Q_2 subcomplexes of P, resp. of Q. Two poset maps $\phi_1 : \mathcal{F}(P_1) \to \mathcal{F}(Q_1)$, $\phi_2 : \mathcal{F}(P_2) \to \mathcal{F}(Q_2)$ are called*

compatible carrier maps *if for* $i = 1, 2$ *there are vertex maps* $f_i^0 : V(P_i) \to V(Q_i)$ *so that* P_i *is a subdivision of* Q_i *with carrier map* ϕ_i, *and moreover* $f_1^0 = f_2^0$ *on* $V(P_1) \cap V(P_2)$.

LEMMA 11.1.4. *A poset map* $\phi : \mathcal{F}(P) \to \mathcal{F}(Q)$ *is the carrier map of a subdivision if the maps* $\phi_q : \phi^{-1}(\mathcal{F}(Q)_{\leq q}) \to \mathcal{F}(Q)_{\leq q}$ *are compatible carrier maps for all* $q \in \mathcal{F}(Q)$.

Proof. Compatibility of the ϕ_q ensures that we can choose the f_q^0 such that if $q_1 \leq q_2$, $f_{q_1}^0(v) = f_{q_2}^0(v)$ for all v where it is defined. Then, for all $v \in V(P)$, we define

$$f^0(v) := f_{\phi(v)}^0(v).$$

Clearly $f^0(p) = f_{\phi(p)}^0(p) \subseteq |\phi(p)|$ for all simplices $p \in P$. Now take any $q \in Q$ and note that in particular $f = f_q$ on $\{v \in V(P) | \{v\} \in \phi^{-1}(q)\}$. Hence $\{f(\langle p \rangle) | f(p) \subseteq |q|\} = \{f_q(\langle p \rangle) | p \leq \phi_q(q)\}$, and this is by assumption a finite partition of $|q|$. $\square$

The condition of compatibility may be hard to check, and therefore Lemma 11.1.4 does not give a handy tool in general. However, it becomes useful in our situation, where compatibility will emerge from an explicit description of 'partial' subdivision processes.

11.2. $\Delta(\Pi_m^{(k)})$ is a subdivision of $\mathcal{T}_n^k$

11.2.1. Complexes of k-trees. We recall the definition of complexes of k-trees from [**77**] and view them in the context of a nested-set-like construction.

DEFINITION 11.2.1. *For positive integers* k, n *let* $\overline{\mathcal{T}_n^k}$ *denote the collection of all trees with* $(n-1)k+2$ *labelled leaves and such that every internal vertex has degree* $rk+2$ *for some* $r > 0$. *We can partially order this set by saying that* $T_1 > T_2$ *if* T_2 *can be obtained from* T_1 *by contraction of some set of the internal edges.*

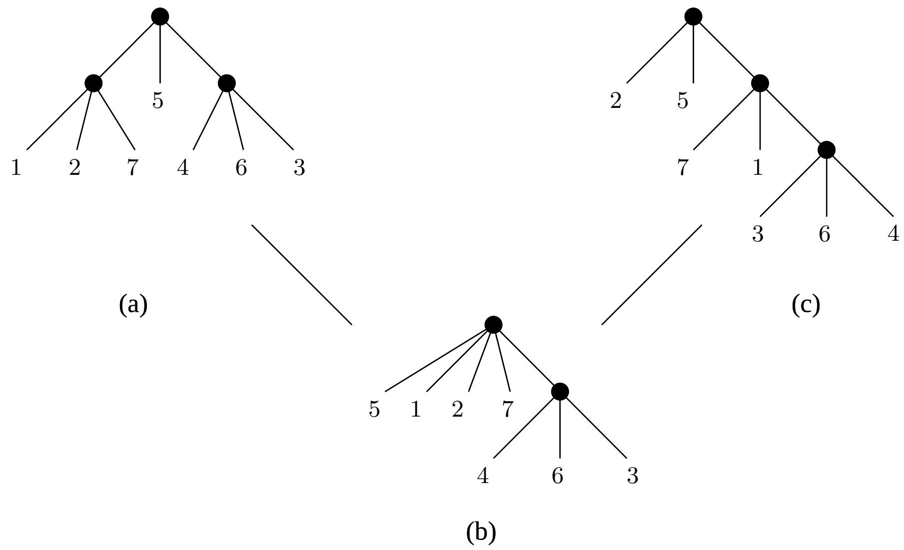

FIGURE 1. Two maximal cells of $\mathcal{T}_4^2$ ((a),(c)) and their common vertex ((b)).

For us it will be convenient to work with rooted trees. We rewrite the previous definition, and let $\mathcal{T}_n^k$ denote the abstract simplicial complex of rooted trees on $m = (n-1)k+1$ leaves, with all outdegrees of internal vertices larger than 1 and congruent to 1 modulo k. The face relations are such that T_2 is a face of T_1 if T_2 is obtained from T_1 by a contraction of a set of internal edges.

At this point it is worth to remind that $\Pi_m^{(k)}$ is not a lattice. For example, the join of the partitions (123)4567 and 1(234)567, considered as elements of Π_7, is clearly (1234)567. But as elements of $\Pi_7^{(2)}$ they have three different minimal upper bounds: (12345)67, (12346)57 and (12347)56.

This is the reason why the theory of combinatorial nested sets cannot be directly applied here. Nevertheless, in the following we will reduce our problem, at least 'locally', to a situation where the existing theory can be applied.

Let $\mathcal{I}$ denote the minimal building set of Π_m. It consists of all partitions with only one block of size bigger than 1.

We define a subset of $\Pi_m^{(k)}$,

$$\mathcal{G} := \{x \in \mathcal{I} |\ rk(x) \equiv 0 \pmod{k}\} \subseteq \Pi_m^{(k)}.$$

It will serve as an analogue of the building set $\mathcal{I}$ in our non-lattice context.

Proposition 11.2.2. *There is a bijective correspondence between the simplicial complex $\mathcal{T}_n^k$ and the family of sets $\{N \in \mathcal{N}(\Pi_m, \mathcal{I}) | N \subseteq \mathcal{G}\}$. In particular, the latter is a simplicial complex.*

Proof. We follow [**58**, proof of 3.1], where an explicit correspondence is given in the case of $k = 1$.

"$\subseteq$": Let $T \in \mathcal{T}_n^k$ be a (rooted) tree with inner vertices $t_1, \ldots, t_k$. The set of leaves below the vertex t will be denoted by $\ell(t)$. For $N(T) := \{\ell(t_i) | i = 1, \ldots, k\}$ we have $N(T) \in \mathcal{N}(\Pi_m, \mathcal{I})$ [**58**], and, by the condition on outdegrees, clearly $N(T) \in \mathcal{G}$.

"$\supseteq$": To every $N = \{S_1, \ldots, S_k\} \in \mathcal{N}(\Pi_m, \mathcal{I})$ we can construct a tree $T(N)$ as in [**58**]. The set of internal vertices is $\tilde{N} = N \cup \{[m]\}$ (where $[m]$ corresponds to the root), and the covering relation is defined by $U_1 < U_2$ if and only if $U_2 \in \max N_{<U_1}$ for all $U_1, U_2 \in \tilde{N}$. The outdegree $out(U)$ of an internal vertex $U \in \tilde{N}$ is:

$out(U) = |U \setminus \bigcup_{S \in N, S<U} S| + |\{S \in N | S < U\}| = |U| - \sum_{S \in N, S<U} |S| + |\{S \in N | S < U\}|$.

Moreover, if $N \in \mathcal{G}$, then $|S| \equiv 1 \pmod{k}$, and with $s := |\{S \in N | S < U\}|$ we have $out(U) \equiv 1 - s + s \equiv 1 \pmod{k}$. ☐

In the following we will consider $\Pi_m^{(k)}$ as a subset of Π_m. Given two elements $a, b \in \Pi_m^{(k)}$ we shall denote by $a \vee b$ the join of them as elements of Π_m, and by $a \vee^k b$ we will mean the set of all minimal upper bounds of a and b in $\Pi_m^{(k)}$. In the example above we would then have $(123)4567 \vee 1(234)567 = (1234)567$, $(123)4567 \vee^2 1(234)567 = \{(12345)67, (12346)57, (12347)56\}$. If it happens that a and b have a unique least upper bound we will write $\exists!\ a \vee^k b$, and then denote by $a \vee^k b$ the (unique) element $x \in a \vee^k b$.

We write $F(x)$ for $F_{\mathcal{I}}(x)$, the set of factors of x with respect to the building set $\mathcal{I}$ of Π_m (see Definition 10.1).

Lemma 11.2.3. ***(Properties of** $\mathcal{G} \subseteq \Pi_m^{(k)}$**)***

a) *For all $x \in \Pi_m^{(k)}$, we have $F(x) = max\mathcal{G}_{\leq x}$.*
b) *Let A, B denote the non-singleton blocks of two partitions $a, b \in \mathcal{G}$. Then we have:*

$$A \cap B = \emptyset \quad \Leftrightarrow \quad \exists!\ a \vee^k b \text{ and } a \vee^k b \notin \mathcal{G}.$$

Proof.

a) The only non-singleton block of any partition in $F(x)$ is always a block of x, therefore has size equal to 1 modulo k.

b) If $A \cap B = \emptyset$, the only non-singleton blocks of $a \vee b$ are A and B, and therefore $a \vee b \in \Pi_m^{(k)}$. So $\exists!\ a \vee^k b$, and since $a \vee b \notin \mathcal{I}$ we have $a \vee^k b \notin \mathcal{G}$.
Suppose $A \cap B \neq \emptyset$. Then $a \vee b$ has only one non-singleton block, namely $A \cup B$. If $|A \cup B| \equiv 1 \pmod{k}$ then $\exists!\ a \vee^k b$, but $a \vee^k b \in \mathcal{G}$. If $|A \cup B| \not\equiv 1 \pmod{k}$, then $|a \vee^k b| > 1$. In both cases the claim follows. □

This Lemma suggests the following definition, which mimics Definition 8.2.4.

Definition 11.2.4. *Let $\mathcal{N}$ denote the family of all subsets $N \subseteq \mathcal{G}$ such that for any set of more than two incomparable elements $\{x_1, \ldots, x_\ell\} \subseteq N$ we have that $\exists!\ x_1 \vee^k \ldots \vee^k x_\ell$ and $x_1 \vee^k \ldots \vee^k x_\ell \notin \mathcal{G}$.*

The following statement can be viewed as the analogue of [**58**, Theorem 3.1] in our context. After the above preparations, the proof is an easy check of the definitions.

Lemma 11.2.5. *With the identification of the elements of $\mathcal{G}$ with the vertices of $\mathcal{T}_n^k$ given above, $\mathcal{N} = \mathcal{T}_n^k$ as simplicial complexes.* □

We carry on the analogy. For $x \in \Pi_m^{(k)}$ let us now write $F^k(x) := \max \mathcal{G}_{\leq x}$, and given a chain $\omega \in \Delta(\Pi_m^{(k)})$, let $F^k(\omega) := \bigcup_{x\in\omega} F^k(x)$. Then we have following suggestive corollary:

Corollary 11.2.6. *For any $\omega \in \Delta(\Pi_m^{(k)})$, $F^k(\omega) \in \mathcal{N}$.*

Proof. First note that $F^k(\omega) \in \mathcal{T}_n^k$, because $F^k(\omega) \subseteq \mathcal{G}$ by Lemma 11.2.3, and $F^k(\omega) \in \mathcal{N}(\Pi_m, \mathcal{I})$ by [**60**][2.8 (3)]. Then apply Lemma 11.2.5. □

11.2.2. The poset $\Sigma(N)$. For $N \in \mathcal{N}$ consider

$$\Sigma(N) := \left\{ \bigvee_{x\in X}^k x |\ X \subseteq N \right\} \subseteq \Pi_m^{(k)},$$

the set of all joins of subsets of N, with the induced order (the join over the empty set gives the minimal element). Note that $N \in \mathcal{N}$ implies in particular that for every $A \subseteq N$ the join $\bigvee_{x\in A}^k x$ exists and is unique. The join in $\Sigma(N)$ will be denoted by $\vee$.

Remark 11.2.7. *For all $a \in \Sigma(N)$ we have $a = \bigvee^k N_{\leq a} = \bigvee max N_{\leq a}$.*

Proposition 11.2.8. *The poset $\Sigma(N)$ is a lattice, and N is a building set for $\Sigma(N)$.*

Proof. We divide the proof in two parts.

Lattice: $\Sigma(N)$ has a maximal and a minimal element (the join over all N and the join over the empty set). For the meet operation take $a, b \in \Sigma(N)$ and define

$$A := \max(N_{\leq a} \cap N_{\leq b}).$$

This is a set of incomparable elements in N, so $\exists!\ \bigvee^k A$. We claim that this is the minimal lower bound for a and b.
Indeed, for every $c \in \Sigma(N)$ with $c \leq a$ and $c \leq b$, clearly $N_{\leq c} \subseteq N_{\leq a}$ and $N_{\leq c} \subseteq N_{\leq b}$. Therefore $c = \bigvee^k \max N_{\leq c} \leq \bigvee^k A$.

N is building set: Take any $x \in \Sigma(N)$ and let $\{x_1, \ldots, x_\ell\} = \max N_{\leq x}$. The map

$$\begin{array}{rccc} \phi: & \prod_{j=1}^{\ell}[\hat{0}, x_j] & \to & [\hat{0}, x] \\ & (a_1, \ldots, a_\ell) & \mapsto & a_1 \vee \cdots \vee a_\ell \end{array}$$

is surjective by definition of $\Sigma(N)$.

For injectivity, note that $\{x_1, \ldots, x_l\} \in \mathcal{N}$ as a set of incomparable elements, and by 11.2.3 we then know the block decomposition of the partition $x = \bigvee^k x_j = \bigvee x_j$. From this we conclude $F(x) = \{x_1, \ldots, x_l\}$, and the defining property of building sets in lattices gives an isomorphism $\psi : \prod[\hat{0}, x_j] \to [\hat{0}, x]$ *in the full partition lattice*, defined by $\psi(x_1, \ldots, x_\ell) = x_1 \vee \cdots \vee x_\ell$ (see Remark 8.2.2 (2)).

Now suppose $\phi(a) = \phi(b)$ for $a \neq b$. By construction then $\psi(a) = \phi(a) = \phi(b) = \psi(b)$ - which contradicts bijectivity of ψ. $\square$

Recall that, given a face s of a simplicial complex K, we denote by K_s the subcomplex of all faces of s.

Corollary 11.2.9. *$\Delta(\Sigma(N))$ is obtained from $\mathcal{N}_N$ by a sequence of stellar subdivisions.*

Proof. This is obtained using the main result of [**59**], because $\mathcal{N}_N = \mathcal{N}(\Sigma(N), N)$. The atomicity condition - needed in [**59**] to construct an explicit realization of the complex - can here be dropped: $\mathcal{N}_N$ is a simplex and we have a standard geometric realization for it. The subdivisions can be still described as combinatorial blowups in the face poset. $\square$

We have to take a closer look at the carrier maps of that subdivision.

Proposition 11.2.10. *Let $\mathcal{L}$ be an atomic semi-lattice, $\mathcal{H} \subseteq \mathcal{G}$ two building sets of $\mathcal{L}$. The carrier $\tau(N)$ of a simplex N of the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$ in the subdivision of $\mathcal{N}(\mathcal{L}, \mathcal{H})$ by $\mathcal{N}(\mathcal{L}, \mathcal{G})$ is given by*

$$\tau(N) = \bigvee_{x \in N} F_{\mathcal{H}}(x) \ \in \mathcal{F}(\mathcal{N}(\mathcal{L}, \mathcal{H})).$$

Proof. Every linear extension $h_1, h_2, h_3, \ldots, h_\ell$ of the order of $\mathcal{L}$ on $\mathcal{G} \setminus \mathcal{H}$ gives a sequence of stellar subdivisions from $\mathcal{N}(\mathcal{L}, \mathcal{H})$ to $\mathcal{N}(\mathcal{L}, \mathcal{G})$ (see [**59**]). We proceed by induction on ℓ.

If $\ell = 1$, then $\mathcal{N}(\mathcal{L}, \mathcal{G})$ is obtained from $\mathcal{N}(\mathcal{L}, \mathcal{H})$ by stellar subdivision of the simplex $F_{\mathcal{H}}(h_1)$, so that the vertex map $f^0 : V(\mathcal{N}(\mathcal{L}, \mathcal{G})) \to |\mathcal{N}(\mathcal{L}, \mathcal{H})|$ maps h_1 to the barycenter of $F_{\mathcal{H}}(h_1)$, whereas $f^0(v) = |v|$ for all $v \neq h_1$. The carrier of $N \in \mathcal{N}(\mathcal{L}, \mathcal{G})$ is the smallest simplex $\tau(N)$ in $\mathcal{N}(\mathcal{L}, \mathcal{H})$ such that $f^0(N) \subseteq |\tau(N)|$. This is the same to say that $\tau(N)$ is the smallest simplex containing the carrier of every vertex of N, hence $\tau(N) = \bigvee_{x \in N} F_{\mathcal{H}}(x)$.

Now let $\ell > 1$, define $\mathcal{H}_\ell := \mathcal{G} \setminus \{h_\ell\}$ and let $\tau_\ell(N)$ be the carrier of $N \in \mathcal{N}(\mathcal{L}, \mathcal{G})$ in $\mathcal{N}(\mathcal{L}, \mathcal{H}_\ell)$. By the argument given in the case $\ell = 1$, we have $\tau_\ell(N) = \bigvee_{x \in N} F_{\mathcal{H}_\ell}(x)$.

Note that, with respect to $\mathcal{N}(\mathcal{L}, \mathcal{H})$, the carrier of $N \in \mathcal{N}(\mathcal{L}, \mathcal{G})$ is the same as the carrier of $\tau_\ell(N) \in \mathcal{N}(\mathcal{L}, \mathcal{H}_\ell)$. Using the induction hypothesis we can now write:

$$\tau(N) = \bigvee_{y \in \tau_\ell(N)} F_{\mathcal{H}}(y) = \begin{cases} \bigvee_{x \in N} F_{\mathcal{H}}(x) & \text{if } h_\ell \notin N, \\ \bigvee_{x \in N \setminus \{h_\ell\}} F_{\mathcal{H}}(x) \vee F_{\mathcal{H}}(F_{\mathcal{H}_\ell}(h_\ell)) & \text{if } h_\ell \in N. \end{cases}$$

Because $\mathcal{H} \subset \mathcal{H}_\ell$, we have $F_{\mathcal{H}}(F_{\mathcal{H}_\ell}(h_\ell)) = F_{\mathcal{H}}(h_\ell)$. Hence in any case the claim follows.$\square$

As in the proof of Corollary 11.2.9, the atomicity condition is needed in the general statement to construct an explicit geometric realization of the nested set complex. We again note that, in our case, the lattice $\Sigma(N)$ is not atomic but the nested set complexes to which we apply the result of [**59**] are always simplices. For a general discussion about stellar subdivisions and combinatorial blowups on abstract simplicial complexes see [**31**]. We then have the following application of Proposition 11.2.10:

PROPOSITION 11.2.11. *For $N \in \mathcal{N}$, the map*

$$\begin{array}{rccc} \phi_N: & \mathcal{F}(\Delta(\Sigma(N))) & \to & \mathcal{F}(\mathcal{N}_N) \\ & \omega & \mapsto & F^{(k)}(\omega) \end{array}$$

is the carrier map of the subdivision of $\mathcal{N}_N$ by $\Delta(\Sigma(N))$ described in Corollary 11.2.9.

Proof. Proposition 11.2.10 gives $\phi_N(\omega) = \bigvee_{x\in\omega} F^{(k)}(x)$. So we only have to show that the right hand side equals $F^{(k)}(\omega)$. For this, it is enough to show

Claim: *Let $\mathcal{L}$ be a semi-lattice and $\mathcal{G}$ a building set in $\mathcal{L}$. Then $x > y \in \mathcal{L}$ implies $F(x) \vee F(y) = F(x) \cup F(y)$ in $\mathcal{F}(\mathcal{N}(\mathcal{L}, \mathcal{G}))$.*

Proof: We show that $F(x) \cup F(y)$ is nested. Since the set of factors of an element is nested, we only need to show that for two incomparable elements $z_1 \in F(x)$ and $z_2 \in F(y)$, the join $z_1 \vee z_2$ is not in $\mathcal{G}$. From $x > y$ we know that $z_1 \vee z_2 \leq x$ and there is exactly one $z \in F(x)$ such that $z_2 \leq z$ (see [**60**, 2.5]). Recall the isomorphism φ_x in definition 10.1: w.l.o.g. we can write $z_1 \vee z_2 = \varphi_x(z_1, z_2, 0, \ldots, 0)$, and this cannot be in $\mathcal{G}$ because the preimage of any element of $\mathcal{G}_{\leq x} = \bigcup_{z\in F(x)}[\hat{0}, z]$ has only one nonzero entry. □

11.2.3. The subdivision. After explicit study of the geometric structure of the subdivisions, we are now able to check the conditions of Lemma 11.1.4, and use it to prove our theorem.

THEOREM 11.2.12. *$\Delta(\Pi_m^{(k)})$ is a subdivision of $\mathcal{T}_n^k$*

Proof. Consider the map

$$\begin{array}{rccc} \phi: & \mathcal{F}(\Delta(\Pi_m^{(k)})) & \to & \mathcal{F}(\mathcal{N}) \\ & \omega & \mapsto & F^{(k)}(\omega). \end{array}$$

It is easy to see that $\phi^{-1}(\mathcal{F}(\mathcal{N})_{\leq N}) = \mathcal{F}(\Delta(\Sigma(N)))$, hence by Proposition 11.2.11 the maps $\phi_N : \phi^{-1}(\mathcal{F}(\mathcal{N})_{\leq N}) \to \mathcal{F}(\mathcal{N})_{\leq N}$ are carrier maps.

For compatibility note that $\Delta(\Sigma(N_1)) \cap \Delta(\Sigma(N_2)) = \Delta(\Sigma(N_1 \cap N_2))$ and that, for any N, $f_N^0(v)$ is the barycenter of $|F^{(k)}(v)|$. It is then clear that $f_{N_1}^0 \restriction_{V(\Delta(\Sigma(N_1)))\cap V(\Delta(\Sigma(N_2)))} = f_{N_1\cap N_2}^0 = f_{N_2}^0 \restriction_{V(\Delta(\Sigma(N_1)))\cap V(\Delta(\Sigma(N_2)))}$.

With Lemma 11.1.4 we conclude that ϕ is a carrier map. □

REMARK 11.2.13. As in [**59**], we see that every linear extension of $\Sigma(N) \setminus N$ gives a valid sequence of stellar subdivisions leading from $\mathcal{N}_N$ to $\Delta(\Sigma(N))$. By the compatibility checked in 11.2.11 and 11.2.12 it is easy to see that a valid sequence of stellar subdivisions leading from $\mathcal{T}_n^k$ to $\Delta(\Pi_m^{(k)})$ can be obtained from any linear extension of $\Pi_m^{(k)} \setminus \mathcal{G}$.

To conclude this note let us consider the natural action of the symmetric group S_m on the complex $\mathcal{T}_n^k$ (by permutation of leaves) and on the poset $\Pi_m^{(k)}$ (by permutation of elements). The correspondence of Lemma 11.2.5 is obviously S_m-equivariant. One can check that the carrier map of the (full) subdivision are also S_m-equivariant, even if at individual subdivision steps this may not be true. We then recover the result of [**77**], [**79**]:

COROLLARY 11.2.14. *The top degree homology modules of* $\mathcal{T}_n^k$ *and* $\Pi_m^{(k)}$ *are isomorphic as* S_m*-modules:*

$$\widetilde{H}_{n-3}(\mathcal{T}_n^k) \cong_{S_m} \widetilde{H}_{n-3}(\Pi_m^{(k)}).$$

REMARK 11.2.15 (On shellability). In view of the results of [**111**] and [**31**], the relation of the complexes $\mathcal{T}_n^k$ and $\Delta(\Pi_m^{(k)})$ by stellar subdivisions gives a new proof of the shellability of $\Pi_m^{(k)}$, which was shown first in [**9**] using the fact that $\Pi_m^{(k)}$ is locally semimodular. The shelling obtained as in [**31**] can be described in a fairly explicit way. Nevertheless, it is still unclear to us whether these two methods actually give the same shelling of $\Pi_m^{(k)}$.

REMARK 11.2.16 (On operads). As was pointed out to us by Bruno Vallette, k-Trees arise in the theory of the operad `k-Com` of k-commutative algebras over a field $\mathbb{K}$. For this operad the *normalized bar construction* in arity n is isomorphic to $\Delta(\Pi_m^{(k)})$ as a complex (see [**112**]), and the *differential bar construction* in arity n is given by $B(\texttt{k-Com})(n)^* = \bigoplus_{T\in\mathcal{T}_n^k}\mathbb{K}$, where the grading of the right hand side is given by the number of internal edges of T (see [**67**, 3.1.9 and 3.4.5]).

A general theorem (see [**67**, 4.7.5], [**87**, II.3]) states the quasi-isomorphism between the two complexes. In the case of `k-Com` this follows from our result.

Index

This index starts with a list of the most used and less standard notations. As standard we assume the basic notations of combinatorics (for which we refer, e.g., to [**108**]) and topology (see [**106**]). **Note:** the subscript ρ always refers to a cover of the arrangement groupoid. With $\hat{\rho}$ we always mean the universal covering morphism. The subscript $\hat{\rho}$ is often replaced by a " $\widehat{\ }$ " over the notation. Thus, $\mathcal{G}_{\hat{\rho}}$, $\mathscr{D}_{\hat{\rho}}$, $\mathscr{G}_{\hat{\rho}}$, $U_{\hat{\rho}}$ etc. are written $\widehat{\mathcal{G}}$, $\widehat{\mathscr{D}}$, $\widehat{\mathscr{G}}$, $\widehat{U}$ etc.

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Curriculum Vitae

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1990-1994:	Secondary school in Lugano.
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Awards

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