On generalizing oriented matroids to a complex setting.

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Introduction

The theory of matroids arises as an axiomatization of the common combinatorial properties of linear dependency of vectors, cycle structure of graphs, intersection structured hyperplane arrangements, etc. Since matroids describe common aspects of so many different objects, they admit many alternative axiomatizations. The existence of various equivalent approaches giving rise to equivalent axiomatizations is one of the main features of this theory. For background on matroid theory we refer to the books of Welsh [15] and Oxley [14].

Oriented matroids are a more specific model for linear dependency of vectors (and, dually, for arrangements of hyperplanes), that not only takes into account dependencies of vectors, but also of the signs of coefficients in such dependencies. Oriented matroids refer especially to the case of vector configurations in real vector spaces, where one can use the standard linear order on $\mathbb{R}$ to define a sign function. As with matroids, there are alternative ways to axiomatize oriented matroids; for a complete illustration of this theory our main reference is [2]

The aim of this diploma thesis is to introduce an analogue of oriented matroids for a complex setting. The particular motivation was to search for a stratification of the complex Grassmannian $G_d(\mathbb{C}^n)$ refining the matroid stratification given by Gel'fand, Goresky, MacPherson and Serganova in [10], in analogy with the oriented matroid stratification for the real Grassmannian $G_d(\mathbb{R}^n)$.

In the literature we find existing concepts for matroids in a complex setting. Ziegler ([3], [16]) considers stratifications of $\mathbb{C}$ into finite sets of signs. He gives first an axiomatization of covectors of a 2-matroid (describing a general 2-arrangement in $\mathbb{R}^{2n}$), and then with an additional axiom mimics the case of a complex hyperplane arrangement, this way coming to the definition of complex matroids. Below, Krummeck and Richter-Gebert present in [1] the concept of a phirotope, a complex analogon of the chirotope for oriented matroids, that does not discretize $\mathbb{C}$ into a finite set of signs, but preserves the whole phase information of complex numbers.

In this thesis we suggest an axiomatization of a $\mathbb{C}$-matroid in terms of signed circuits, using signed sets over a continuous set of signs.

In chapter 1 the main ideas and some useful results of basic matroid theory are reviewed.

Chapter 2 is then entirely devoted to the definition of $\mathbb{C}$-matroids. We deal with a continuous set of signs and give an axiomatization of $\mathbb{C}$-matroids in terms of so-called signed circuits. Some considerations on orthogonality and duality for $\mathbb{C}$-matroids follow, and thereafter we define $\mathbb{C}$-chirotopes and basis signatures. These objects turn out to encode the full information; they give rise to equivalent axiomatizations for $\mathbb{C}$-matroids.

In chapter 3 we construct a commutative algebra for $\mathbb{C}$-matroids that models the algebra defined by Cordovil for oriented matroids in [6]. Both algebras have the Orlik-Terao algebra for complex hyperplane arrangements as their prototype [13]. We show that, for a $\mathbb{C}$-matroid arising from an arrangement $\mathcal{A}$, the close connection of Cordovil's algebra with the Orlik-Terao algebra of $\mathcal{A}$ is maintained in our construction. We examine the very natural question as to what extent the intersection lattice of $\mathcal{A}$ (the underlying matroid of a $\mathbb{C}$-matroid, respectively) determines the algebra. We prove that for two $\mathbb{C}$-matroids that differ only by reorientation the associated algebras are isomorphic. The case of two $\mathbb{C}$-matroids having the same underlying matroid but differing by 'more than' a reorientation turns out to be much more difficult. No final solution of this problem is given here; examples of arrangements with the same lattice of flats that give non-isomorphic algebras are not known. *Affaire à suivre...*

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Contents

Chapter 1

A review of matroid theory

Most of this chapter is an introduction to the theory of matroids, in order to give some background information and to contextualize the notions presented in the following of this work. This tractation will not be exaustive nor of particular depth: the objectives here are to present the basic concepts and state the results that we will need in the next chapters. Proofs are omitted or only sketched, in order to give a more immediate and 'panoramical' overview: one of the main features of this theory is namely the existence of different axiomatic approaches, that are shown to be equivalent (or, as sometimes said in the literature, 'cryptomorphic'), i.e. to present the same object. For the interested reader references will be given, where the theory is threated in a more exaustive way, with all proofs and details.

At the end we will briefly refer to the theory of oriented matroids, that inspired the concept of $\mathbb{C}$-matroid presented in the next chaper.

1.1 The theory of matroids

This exposition refers mainly to the book of Oxley [14].

1.1.1 Independent sets

Definition 1.1.1 *A matroid $\mathcal{M}$ is an ordered Pair $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I} \subset \mathscr{P}(E)$ satisfies following conditions:*

$$\begin{array}{ll} (\mathcal{I}0) & \varnothing \in \mathcal{I} \\ (\mathcal{I}1) & \forall I \in \mathcal{I} : \; I' \subseteq I \Rightarrow I' \in \mathcal{I} \\ (\mathcal{I}2) & \forall I_1, I_2 \in \mathcal{I} : \; |I_1| < |I_2| \Rightarrow \exists\, e \in I_2 \setminus I_1 : \; I_1 \cup e \in \mathcal{I} \end{array}$$

Example 1.1.2 (Matroid of linear dependencies) Let V be a vector space and $E := \{v_1, \ldots, v_n\} \subset V$. Define $\mathcal{I}$ to be the set of all linear independent subsets of E. $\mathcal{I}$ satisfies $(\mathcal{I}0) - (\mathcal{I}2)$ and therefore $(E, \mathcal{I})$ is a matroid, called the matroid of linear dependencies of $\{v_1, \ldots, v_n\}$.

1.1.2 Bases

We now consider in particular the family of those independent sets that are maximal with respect to the support, that is, the set of bases of $\mathcal{M}$, defined as $\mathcal{B}(\mathcal{M}) :=\mathrm{Max}(\mathcal{I}) = \{B \in \mathcal{I} \,|\, \forall I \in \mathscr{P}(E) : I \supsetneq B \Rightarrow I \notin \mathcal{I}\}$.

Example 1.1.3 (Matroid of linear dependencies, continued) In the matroid of linear dependencies of $\{v_1, \dots, v_n\}$, the bases are precisely the (linear) bases of the vector space determined by the linear hull of $\{v_1, \dots, v_n\}$.

Proposition 1.1.4 *Let $\mathcal{B}$ denote the set of bases of a matroid. $\mathcal{B}$ satisfies following conditions:*

$$(\mathcal{B}0) \quad \mathcal{B} \neq \varnothing$$
$$(\mathcal{B}1) \quad \forall B_1, B_2 \in \mathcal{B} \, \forall x \in B_1 \setminus B_2 \, \exists \, y \in B_2 \setminus B_1 : (B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}.$$

Proof: $(\mathcal{B}0)$ is immediate, and $(\mathcal{B}1)$ is proved as lemma 1.2.2 in [14].

Knowing the bases of a matroid we can determine the independent sets as the subsets of E that are contained in some basis.

Proposition 1.1.5 (1.2.3 of [14]) *Let E be a finite set, and consider $\mathcal{B} \subset \mathscr{P}(E)$ satisfying $(\mathcal{B}0), (\mathcal{B}1)$. Let $\mathcal{I} \subset \mathscr{P}(E)$ denote the collection of all $A \subseteq E$ such that $A \subseteq B$ for some $B \in \mathcal{B}$. Then $\mathcal{M} := (E, \mathcal{I})$ is a matroid and $\mathcal{B} = \mathcal{B}(\mathcal{M})$.*

From this proposition we can conclude that $\mathcal{B} \subset \mathscr{P}(E)$ is the set of bases of a matroid if and only if it satisfies $(\mathcal{B}0) - (\mathcal{B}1)$. Moreover, we can state following first alternative axiomatization of the theory of matroids:

Definition 1.1.1 (Basis-approach) *A matroid $\mathcal{M}$ on the ground set E is an ordered pair $(E, \mathcal{B})$, where E is a finite set and $\mathcal{B} \subset \mathscr{P}(E)$ satisfies $(\mathcal{B}0), (\mathcal{B}1)$.*

Axiom $(\mathcal{B}1)$ (the only one encoding some combinatorial information) is also called the *basis exchange axiom.*

Note that, as one would expect from the 'vector space interpretation', it is not hard to show that the bases of a given matroid are equicardinal.

We end this overview of the principal facts over bases in matroids by defining the notion of bases graph of a matroid.

Definition 1.1.6 *Let $\mathcal{M}$ be a matroid and $\mathcal{B}$ its set of bases. The* ***bases graph*** *of $\mathcal{M}$ is denoted by $G_{\mathcal{B}}$ and defined as follows:*

$$V[G_{\mathcal{B}}] := \mathcal{B}$$
$$E[G_{\mathcal{B}}] := \Big\{ \{B_1, B_2\} \in \mathcal{B}^2 \,\Big|\, \exists \, e \in B_1 \, \exists \, f \in B_2 : B_1 = (B_2 \setminus \{f\}) \cup \{e\}, B_2 = (B_1 \setminus \{e\}) \cup \{f\} \Big\}.$$

1.1.3 Circuits

After having considered the maximal independent sets, we turn our attention to the set $\mathcal{C}(\mathcal{M})$:=Min($\mathscr{P}(E)\setminus \mathcal{I}$) of the support minimal dependent sets. This set is called the *set of circuits* of the matroid $\mathcal{M}$ (and will be referred to only as $\mathcal{C}$ when no confusion can arise).

Proposition 1.1.7 (1.1.3 of [14]) *Let $\mathcal{M}$ be a matroid and $\mathcal{C}$ denote the set of circuits of $\mathcal{M}$. $\mathcal{C}$ has the following properties:*

$$
\begin{array}{ll}
(\mathcal{C}0) & \varnothing \notin \mathcal{C} \\
(\mathcal{C}1) & \forall C_1, C_2 \in \mathcal{C} : \; C_1 \subseteq C_2 \Rightarrow C_1 = C_2 \\
(\mathcal{C}2) & \forall\, C_1 \in \mathcal{C}\, \forall\, C_2 \in \mathcal{C}\setminus\{C_1\} \\
 & \qquad \forall\, e \in C_1 \cap C_2 \; \exists\, C_3 \in \mathcal{C}(\mathcal{M}) : \; C_3 \subseteq (C_1 \cup C_2) \setminus e.
\end{array}
$$

The first two statements express the 'support minimality' of the definition of $\mathcal{C}$. $(\mathcal{C}2)$ expresses the matroid structure.

In a matroid $\mathcal{M}$ we can characterize the independent sets as the sets that does not contain any member of $\mathcal{C}(\mathcal{M})$. We have following theorem:

Proposition 1.1.8 (1.1.4 of [14]) *Let E be a finite set, and consider $\mathcal{C} \subset \mathscr{P}(E)$ satisfying $(\mathcal{C}0) - (\mathcal{C}2)$. Let $\mathcal{I} \subset \mathscr{P}(E)$ be defined by the condition that no element of $\mathcal{I}$ contains any member if $\mathcal{C}$. Then $\mathcal{M} := (E, \mathcal{I})$ is a matroid and $\mathcal{C} = \mathcal{C}(\mathcal{M})$.*

Example 1.1.9 Let $G = (V, E)$ be a graph. The family of all $C \subset E$ such that the edges in C form a cycle in G satisfies $(\mathcal{C}0) - (\mathcal{C}2)$ and determines this way a matroid, called the cycle matroid of G.

In particular, we have thereby shown that $\mathcal{C} \subset \mathscr{P}(E)$ is the set of circuits of a matroid if and only if it satisfies $(\mathcal{C}0) - (\mathcal{C}2)$. We can now give another altenative axiomatization of matroid theory:

Definition 1.1.1 (Circuit-approach) *A matroid $\mathcal{M}$ is an ordered Pair $(E, \mathcal{C})$, where E is a finite set and $\mathcal{C} \subset \mathscr{P}(E)$ satisfies $(\mathcal{C}0) - (\mathcal{C}2)$.*

From these axioms a "stronger" version of $(\mathcal{C}2)$ can be derived:

Proposition 1.1.10 (1.9.2 of [15]) *Consider a matroid $\mathcal{M}$ with set of circuits $\mathcal{C}$. Then we have:*

$$
\begin{array}{ll}
(\mathcal{C}2)' & \forall\, C_1 \in \mathcal{C} \; \forall\, C_2 \in \mathcal{C}\setminus\{C_1\} \;\; \forall f \in C_1 \setminus C_2 \\
 & \qquad \forall\, e \in C_1 \cap C_2 \; \exists\, C_3 \in \mathcal{C}(\mathcal{M}) : \; f \in C_3 \subseteq (C_1 \cup C_2) \setminus e.
\end{array}
$$

This proposition can be substituted to $(\mathcal{C}2)$ in the circuits-axiomatization of matroid theory. In fact, $(\mathcal{C}2)$ is called the *weak elimination axiom* to distinguish it from $(\mathcal{C}2)'$, called the *strong elimination axiom.*

Before concluding this section we want to define the notion of basic circuit. This definition bases on the following lemma.

Lemma 1.1.11 *Let $\mathcal{M}$ be a matroid on the ground set E, $B \in \mathcal{B}(\mathcal{M})$ and consider $e \in E \setminus B$. Then $B \cup e$ contains a unique circuit that we denote by $C(B, e)$. Moreover, $e \in C(B, e)$.*

The circuit $C(B, e)$ is called *basic circuit of e with respect to B.*

1.1.4 Excursus: contraction and deletion

In this very short section we will introduce two very important matroid operations (that in section 1.1.7 will turn out to be in fact two faces of the same coin). Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and consider $F \subset E$.

The *contraction of $\mathcal{M}$ to F* is defined to be the matroid $\mathcal{M}/F$ on the ground set E, having $\mathcal{I}' := \{X \in \mathcal{I} | X \subseteq F\}$ as collection of independent sets. The verification of conditions $(\mathcal{I}0) - (\mathcal{I}2)$ is left to the reader. We have then $\mathcal{C}(\mathcal{M}/F) = \mathrm{Min}\{C \in \mathcal{C}(\mathcal{M}) | C \subseteq F\}$ (where Min denotes inclusion-minimality).

The *deletion of F from $\mathcal{M}$* is defined to be the matroid $\mathcal{M} \setminus F := (E, \mathcal{I}'')$, characterized by the collection of indipendent sets $\mathcal{I}'' := \{X \in \mathcal{I} | X \cap F = \varnothing\}$, that is easily checked to satisfy $(\mathcal{I}0) - (\mathcal{I}2)$. As an immediate consequence of this definition we have that $\mathcal{C}(\mathcal{M} \setminus F) = \{C \in \mathcal{C}(\mathcal{M}) \,|\, C \cap F = \varnothing\}$.

Related to the operation of deletion we have the concept of the *restriction of $\mathcal{M}$ to F*, called $\mathcal{M}|F$ and defined to be $\mathcal{M} \setminus (E \setminus F)$.

1.1.5 Rank

We have seen that for every matroid $\mathcal{M}$ on the ground set E and every $F \subseteq E$, the deletion $\mathcal{M} \setminus F$ is a 'good' matroid. By section 1.1.2 we know that the bases of this matroid are equicardinal. To each matroid $\mathcal{M}$ we can then associate a well-defined *rank function* ϱ by:

$$\begin{array}{rccl} \varrho : & \mathscr{P}(E) & \longrightarrow & \mathbb{N} \\ & X & \longmapsto & \varrho(X) := |B| \;\; for \; B \in \mathcal{B}(\mathcal{M} \setminus F). \end{array}$$

We will often write $\varrho(\mathcal{M})$ for $\varrho(E)$ and call this number *rank of $\mathcal{M}$*.

Proposition 1.1.12 *Let $\mathcal{M}$ be a matroid and ϱ the associated rank function. ϱ has the following properties:*

$$\begin{array}{ll} (\varrho 1) & X \subseteq E \Rightarrow 0 \leq \varrho(X) \leq |X| \\ (\varrho 2) & X \subseteq Y \subseteq E \Rightarrow \varrho(X) \leq \varrho(Y) \\ (\varrho 3) & \forall\, X, Y \subseteq E : \;\; \varrho(X \cup Y) + \varrho(X \cap Y) \leq \varrho(X) + \varrho(Y). \end{array}$$

Proof: $(\varrho 1)$ and $(\varrho 2)$ are easy, and $(\varrho 3)$ is proved as lemma 1.3.1 in [14].

As in the previous section, we are going to show that these statements characterize the matroid.

Proposition 1.1.13 *Consider a finite set E and a map $\varrho : \mathscr{P}(E) \to \mathbb{N}$ satisfying $(\varrho 1)-(\varrho 3)$. Let $\mathcal{I}$ be the collection of all $X \subset E$ such that $\varrho(X) = |X|$. Then $(E, \mathcal{I})$ is a matroid with rank function ϱ.*

In particular, $\varrho : \mathscr{P}(E) \to \mathbb{N}$ is the rank function of a matroid if and only if it satisfies $(\varrho 1) - (\varrho 3)$.

Definition 1.1.1 (Rank-function-approach) *A matroid $\mathcal{M}$ is defined to be an ordered pair (E, ϱ), where E is a finite set and $\varrho : \mathscr{P}(E) \to \mathbb{N}$ satisfies $(\varrho 1) - (\varrho 3)$.*

Carrying on the analogy of example 1.1.2, $X \subseteq E$ is called *hyperplane* iff $\varrho(X) = \varrho(E) - 1$.

1.1.6 Closure

Given a matroid $\mathcal{M}$ on the ground set E, we let ϱ be the rank function of $\mathcal{M}$ and define the *closure operator* $< \cdots >_{\mathcal{M}}$ by

$$\begin{array}{llll} < \cdots >_{\mathcal{M}}: & \mathscr{P}(E) & \longrightarrow & \mathscr{P}(E) \\ & X & \longmapsto & < X >_{\mathcal{M}} := \{x \in E \,|\, \varrho(X \cup \{x\}) = \varrho(X)\}. \end{array}$$

We follow the pattern of the former sections, to reach a fifth alternative axiomatization

Proposition 1.1.14 *Let $\mathcal{M}$ be a matroid and $< \cdots >$ the associated closure operator. $< \cdots >$ has the following properties:*

$$\begin{array}{ll} (cl\ 1) & \forall X \subseteq E : \ X \subseteq < X > \\ (cl\ 2) & X \subseteq Y \subseteq E \ \Rightarrow < X > \subseteq < Y > \\ (cl\ 3) & \forall X \subseteq E : << X >> = < X > \\ (cl\ 4) & \forall X \subseteq E \ \forall x \in E : \ y \in < X \cup \{x\} > \setminus < X > \Rightarrow x \in < X \cup \{y\} > . \end{array}$$

Proposition 1.1.15 *Consider a finite set E and a map $< \cdots >: \mathscr{P}(E) \to \mathscr{P}(E)$ satisfying $(cl\ 1)-(cl\ 4)$. Let $\mathcal{I}$ be the collection of subsets of E defined by*

$$\mathcal{I} := \{X \subseteq E \,|\, \forall x \in X : x \notin < X \setminus \{x\} > \}.$$

Then $(E, \mathcal{I})$ is a matroid with closure operator $< \cdots >$.

Definition 1.1.1 (Closure-operator-approach) *A matroid $\mathcal{M}$ is an ordered Pair $(E, < \cdots >)$, where E is a finite set and $< \cdots >: \mathscr{P}(E) \to \mathscr{P}(E)$ satisfies $(co1) - (co4)$.*

In the following we will call $X \subseteq E$ a *spanning set* if $< X > = < E >$.

1.1.7 Duality

One of the more interesting and powerful tools in matroid theory is duality: it associates to every matroid a dual matroid and therefore in practice doubles the resources in solving problems.

Definition 1.1.16 *Consider a matroid $\mathcal{M} = (E, \mathcal{B})$ given by the specification of its set of bases $\mathcal{B}$. Let $\mathcal{B}^* := \{E \setminus B \mid B \in \mathcal{B}\}$.*
*The matroid $\mathcal{M}^* := (E, \mathcal{B}^*)$ is called the **dual matroid** of $\mathcal{M}$.*

We leave to the reader to check that $\mathcal{B}^*$ satisfies $(\mathcal{B}0), (\mathcal{B}1)$. The bases of $\mathcal{M}^*$ are called *cobases* of $\mathcal{M}$, and similarly we call *cocircuits, coindependent, cospanning sets* of $\mathcal{M}$ the circuits, indipendent, resp. spanning sets of $\mathcal{M}$. We have:

$$\begin{array}{c} \mathcal{I}^*(\mathcal{M}) := \mathcal{I}(\mathcal{M}^*) := \{J \subseteq E \mid \exists B \in \mathcal{B}^* : \ B \cap J = \varnothing\} \\ \mathcal{C}^*(\mathcal{M}) = \mathcal{D}(\mathcal{M}) := \mathcal{C}(\mathcal{M}^*) := \{D \subseteq E \mid \forall C \in \mathcal{C}(\mathcal{M}) : \ |D \cap C| \neq 1\} \end{array}$$

After the work presented in the previous sections it should be clear that we could have defined $\mathcal{M}^*$ in terms of $\mathcal{C}^*$ or $\mathcal{I}^*$ as well as by $\mathcal{B}^*$. We state now some basic facts and a result that will be useful to the understanding of the rest of this paper.

Proposition 1.1.17 (2.1.6 of [14]) *Let $\mathcal{M}$ be a matroid on the ground set E. For all $X \subseteq E$ we have then:*

$$\begin{array}{ll} (i) & X \in \mathcal{I} \Leftrightarrow E \setminus X \textit{ is a cospanning set in} \mathcal{M}^* \\ (ii) & X \textit{ is a spanningset} \Leftrightarrow E \setminus X \in \mathcal{I}^* \\ (iii) & X \textit{ is a hyperplane} \Leftrightarrow E \setminus X \textit{ is a cocircuit} \\ (iv) & X \textit{ is a circuit } \Leftrightarrow E \setminus X \textit{ is a cohyperplane.} \end{array}$$

Proposition 1.1.18 (2.1.10 of [14]) *Consider a matroid $\mathcal{M}$ and disjoint sets $I \in \mathcal{I}(\mathcal{M})$, $I^* \in \mathcal{I}^*(\mathcal{M})$. Then it exists a basis $B \in \mathcal{B}$ and a cobasis $B^* \in \mathcal{B}^*$ such that $I \subseteq B$, $I^* \subseteq B^*$ and $B \cap B^* = \varnothing$.*

This last proposition in particular will play a key role in the proof of one of the most important results of the next chapter.

1.2 Oriented matroids

Matroids arise in many contexts. One of the most natural ones is the description of the combinatorics of an hyperplane arrangement in $\mathbb{K}^n$ (where $\mathbb{K}$ denotes a field). The hyperplanes can indeed be seen as kernel of linear forms: we can then identify an hyperplane with an element of $(\mathbb{K}^*)^n$ (that is well-defined up to a constant factor). Now, just as in example 1.1.2, we can consider the matroid of linear dependencies of these forms. This matroid encodes the information on the lattice of the intersections of the considered hyperplanes.

When $\mathbb{K}$ is ordered, then for each linear form α the space $\mathbb{K}^n/Ker(\alpha)$ is ordered too. Given $x \in \mathbb{K}^n$ we can therefore always say whether x is 'in front of', 'behind', or 'in' $Ker(\alpha)$ by looking at the sign $s_x(\alpha) \in \{+, -, 0\}$ of the projection of x along $Ker(\alpha)$. Given an arrangement of hyperplanes we can order them (say by indexing over an ordered set I), choose linear forms α_i as above, and then consider the family of the ordered tuples $((s_x(\alpha_i))_{i\in I})_{x\in\mathbb{K}^n}$. Since we have only finite many hyperplanes, there are only finite many different such tuples.

Forgetting the signs, we can consider the family$\{i \mid s_x(\alpha_i) \neq 0\}_{x\in\mathbb{K}^n}$. This turn out to be the family of dependent sets of the matroid given by the linear dependencies of the linear forms.

Therefore we can say that the system of ordered 'signed' tuples is basically a matroid, but there is more information: namely the signs, that refer to an orientation of $\mathbb{K}^n/Ker(\alpha_i)$. The name for this structure is then **oriented matroid**. We remark here that from the information encoded in an oriented matroid one can extract informations on the homology of the space resulting from $\mathbb{K}^n$ by deleting the hyperplanes of the arrangement

As in the case of matroids, one has different equivalent approaches to the theory. For a complete and detailed introduction to this theory we refer to [2]: our purpose in this paper is to sketch a similar theory (if possible) for the case $\mathbb{K} = \mathbb{C}$, where no linear ordering is there.

Chapter 2

$\mathbb{C}$-matroids

This chapter is devoted to the definition of $\mathbb{C}$-matroids and to establish some of their properties. We develop the theory parallel to the theory of oriented matroid as presented in [2]: in particular, we will describe different approaches that lead to the same object.

We begin by specifying the set of signs $U = S^1 \cup \{\natural\}$ that we will use, and then we state the first axiomatization of $\mathbb{C}$-matroids in terms of signed circuits. We define $\mathbb{C}$-matroids as pairs $(E, \mathfrak{C})$, where $\mathfrak{C}$ is a class of signed subsets of the ground set E. As a technical help for the proofs we introduce the notion of an oriented set: in fact all propositions will be proved using these objects, but the point is that all results can be translated in the language of signed sets. We introduce a convenient concept of duality, defined in terms of orthogonality of signed sets, and show that each $\mathbb{C}$-matroid has a well-defined dual. We then define maps $\omega, \psi : E^d \longrightarrow U$ ('$\mathbb{C}$-chirotope' and 'basis signature', respectively), and show in the remaining of the chapter that these maps determine the $\mathbb{C}$-matroid to which they are associated: in fact we will define the $\mathbb{C}$-matroid as pairs (E, ω) and (E, ψ) and show the equivalence of the definitions. In the following diagram we represent the structure of the chapter, where the numbers in brackets refer to the propositions where the corresponding implication is shown.

$$\begin{array}{ccccc} (E,\mathfrak{C}) & \longrightarrow & \text{(2.5.2)} & \longrightarrow & (E,\psi) \\ \text{(2.4.3)} \; \nwarrow & & & & \swarrow \; \text{(2.5.4)} \\ & & (E,\omega) & & \end{array}$$

Before starting with the definitions, we point out that this theory specializes the existing theory of oriented matroids, and stratification of the sets of signs used here leads to 2-matroids, resp. complex matroids as introduced by Ziegler in [16].

2.1 The definition

Definition 2.1.1 *Let E be a finite set of objects, K a set of signs. We call $A \in K^E$ a* ***signed subset*** *of E (with signs in K), and denote by $A(e)$ the component of A corresponding to $e \in E$, which we call the signum of e in A. Sometimes we will write A as the $|E|$-tuple $(A(e_1), A(e_2), \dots)$.*
For a distinguished element $\natural \in K$ we define the ***support*** *of A as:*

$$\underline{A} := \{e \in E \mid A(e) \neq \natural\},$$

and, given $A, B \in K^E$, the ***separator*** *of A and B as:*

$$S_{A,B} := \{e \in \underline{A} \cap \underline{B} | A(e) \neq B(e)\}.$$

Fix now an $A \in K^E$ and an $F \subset E$: we define $A \setminus F \in K^{(E \setminus F)}$ as:

$$(A \setminus F)(e) = A(e) \ \forall e \in E \setminus F.$$

For $Q \subset U^E$ we define the notation $\underline{Q}$ to mean the set of the supports of the elements of Q.

As set of signs we will especially consider $U := S^1 \cup \{\natural\}$, with S^1 parametrized as $\mathbb{R}/2\pi\mathbb{Z}$. Between elements of U we define a 'dotted addition':

$$\phi_1 \dot{+} \phi_2 = \begin{cases} \phi_1 + \phi_2 & \text{if } \phi_1 \neq \natural \text{ and } \phi_2 \neq \natural \\ \natural & \text{if } \phi_1 = \natural \text{ or } \phi_2 = \natural \end{cases} \quad \text{for all } \phi_1, \phi_2 \in U,$$

and a corresponding 'dotted subtraction':

$$\phi_1 \dot{-} \phi_2 = \begin{cases} \phi_1 - \phi_2 & \text{if } \phi_1 \neq \natural \text{ and } \phi_2 \neq \natural \\ \natural & \text{if } \phi_1 = \natural \text{ or } \phi_2 = \natural \end{cases} \quad \text{for all } \phi_1, \phi_2 \in U.$$

For $A \in U^E$ and $\alpha \in U$ we define a signed set $\alpha \boxplus A$ such that:

$$(\alpha \boxplus A)(e) = \alpha \dot{+} A(e) \text{ for all } e \in E.$$

Now the definition of the object we will deal with along this paper:

Definition 2.1.2 (signed-circuit approach) *Let E be a set of objects, and consider $U := S^1 \cup \{\natural\}$ as set of signs. A $\mathbb{C}$-matroid is a pair $\mathfrak{M} = (E, \mathfrak{C})$, where $\mathfrak{C} = \mathfrak{C}(\mathfrak{M}) \subseteq U^E$ is called the set of* ***signed circuits of*** *$\mathfrak{M}$ and satisfies following axioms:*

$$\begin{array}{ll} (C0) & (\natural, \natural, \dots, \natural) \notin \mathfrak{C}(\mathfrak{M}) \\ (C1) & \forall C \in \mathfrak{C}(\mathfrak{M}) \ \ \forall \alpha \in U \setminus \natural : \ \ \alpha \boxplus C \in \mathfrak{C}(\mathfrak{M}) \\ (C2) & \forall C_1, C_2 \in \mathfrak{C}(\mathfrak{M}) : \ \ \underline{C_1} \subseteq \underline{C_2} \Rightarrow \underline{C_1} = \underline{C_2} \\ (C3) & \forall C_1, C_2 \in \mathfrak{C}(\mathfrak{M}) \forall f \in S_{C_1,C_2} \exists C' \in \mathfrak{C}(\mathfrak{M}) \\ & C'(f) \neq \natural \text{ and } \forall e \in E : C_1(e) = C_2(e) \Rightarrow C'(e) = \natural \end{array}$$

Note that for such a $\mathbb{C}$-matroid, the supports of the circuits of $\mathfrak{M}$ satisfy the 'usual' circuit axioms (1.1.8), this way defining a matroid that we call *underlying matroid* of $\mathfrak{M}$ and denote by $\underline{\mathfrak{M}}$[1]. More formally, $\underline{\mathfrak{M}}$ is defined by the pair $(E, \mathcal{C})$, where $\mathcal{C} := \mathcal{C}(\underline{\mathfrak{M}}) = \underline{\mathfrak{C}}(\mathfrak{M}) = \{\underline{C} \mid C \in \mathfrak{C}(\mathfrak{M})\}$.

Remark: An oriented matroid is then a $\mathbb{C}$-matroid where in each signed circuit C and for all pairs of elements $e, f \in \underline{C}$ the difference $C(e) - C(f)$ can take only the values 0 or π. Intuitively, one can think $\natural$ taking the place of 0 in oriented matroid theory and 0, π playing the role of 1, -1.

We now define the $\mathbb{C}$-matroid operations of deletion and contraction:

Definition 2.1.3 *Given a $\mathbb{C}$-matroid $\mathfrak{M}$ and a subset F of its ground set, we define the* ***deletion*** *$\mathfrak{M} \setminus F$ as the $\mathbb{C}$-matroid with set of signed circuits:*

$$\mathfrak{C}(\mathfrak{M} \setminus F) := \{C \setminus F \,|\, C \in \mathfrak{C}(\mathfrak{M}),\ \forall x \in F :\ C(x) = \natural\},$$

where minimality is with respect to support-inclusion, and the ***contraction*** *$\mathfrak{M}/F$ as the $\mathbb{C}$-matroid with set of signed circuits:*

$$\mathfrak{C}(\mathfrak{M}/F) := Min\{C \setminus F \,|\, C \in \mathfrak{C}(\mathfrak{M})\}.$$

An immediate consequence of these definitions is:

Lemma 2.1.4 *For a $\mathbb{C}$-matroid $\mathfrak{M}$ and subsets A, B of its ground set we have:*

$$(\mathfrak{M} \setminus A) \,/\, B \;=\; (\mathfrak{M} \,/\, B) \setminus A.$$

Proof: Immediate, by checking of following statement:
$\mathfrak{C}(\mathfrak{M} \setminus A)/B = Min\{C \setminus (A \cup B) \,|\, C \in \mathfrak{C}(\mathfrak{M}), \forall x \in A : C(x) = \natural\} = \mathfrak{C}(\mathfrak{M}/B) \setminus A$. □

Remark: The definitions come in a very natural way from the corresponding concepts of classical matroid theory: in particular, the underlying matroid $\underline{\mathfrak{M}/F}$ (resp. $\underline{\mathfrak{M} \setminus F}$) is the contraction $\underline{\mathfrak{M}}/F$ (the deletion $\underline{\mathfrak{M}} \setminus F$).

A crucial concept in the theory of oriented matroids is that of ortogonality. It is the point where geometrical aspects can be brought into this theory in a very direct way: in fact the Grassmann-Plücker relations (that we have seen to encode the full geometrical information of the arrangement) are, from the point of view of oriented matroids, the specification of the orthogonality of certain signed sets, and this determines entirely the oriented matroid.

[1] To show ($\mathcal{C}3$) remark that we can choose the signatures of the circuits in such a way that the element e that we want to eliminate has the same sign in the two circuits: then ($C3$) gives C' with $C'(e) = \natural$, i.e. $e \notin \underline{C'}$. The other statements in (1.1.8) are immediate.

The idea is then to try a similar approach to the complex case: to do this we give an alternative characterisation of $\mathbb{C}$-matroids to allow multiplicative and additive structure on the signs of the circuits.

2.2 Orientation of the circuits

Here we call ***oriented set*** a signed set with signs in $\mathbb{C}$. The support of an oriented set γ is defined to be $\underline{\gamma} := \{e \in E | \gamma(e) \neq 0\}$.

Definition 2.2.1 *A legal orientation of the set $\mathfrak{C}$ of signed circuits of a $\mathbb{C}$-matroid on the ground set E is $\Gamma \subset \mathbb{C}^E$ such that*

(I) $\alpha \boxtimes \gamma \in \Gamma$ for all $\gamma \in \Gamma$ and all $\alpha \in \mathbb{C}$, where $(\alpha \boxtimes \gamma)(e) := \alpha\, \gamma(e)$ for all $e \in E$.

(II) there is a surjection $C_{(\cdot)} : \Gamma \to \mathfrak{C}$ with

$$C_{(\gamma)}(e) = \begin{cases} arg(\gamma(e)) & if \gamma(e) \neq 0 \\ \natural & if \gamma(e) = 0 \end{cases}$$

for all $e \in E$.

The fibers of $C_{(\cdot)}$ are called orientation classes.
It is clear that for every $\mathbb{C}$-matroid $\mathfrak{M}$ there is a canonical legal orientation of $\mathfrak{C}(\mathfrak{M})$: we call this $\Gamma(\mathfrak{M})$.

Remark 1 *A legal orientation Γ of the signed circuits of a $\mathbb{C}$-matroid satisfies following properties:*

$$\begin{array}{ll} (\Gamma 0) & (0,0,\ldots,0) \notin \Gamma \\ (\Gamma 1) & \forall \gamma \in \Gamma \quad \forall \alpha \in \mathbb{C}^* : \quad \alpha \boxtimes \gamma \in \Gamma \\ (\Gamma 2) & \forall \gamma_1, \gamma_2 \in \Gamma : \quad \underline{\gamma_1} \subseteq \underline{\gamma_2} \Rightarrow \underline{\gamma_1} = \underline{\gamma_2} \\ (\Gamma 3) & \forall \gamma_1, \gamma_2 \in \Gamma \, \forall f \in S_{\gamma_1,\gamma_2} \exists \gamma' \in \Gamma \\ & \gamma'(f) \neq 0 \; and \; \forall e \in E : \; \gamma_1(e) = \gamma_2(e) \Rightarrow \gamma'(e) = 0 \end{array}$$

It is clear that such a Γ gives the set of signed circuits of a $\mathbb{C}$-matroid via extracting argument, and setting $arg(0) := \natural$. This motivates the following definition:

Definition 2.2.2 *Let E be a set of objects. $\Gamma \subseteq \mathbb{C}^E$ satisfying $(\Gamma 0) - (\Gamma 3)$ is called the set of **oriented circuits of a $\mathbb{C}$-matroid** $\mathfrak{M}$.*

To point out that the relevant information is the difference between two entries of an oriented set rather than the single values, we define the concept of *geometric class*. We say that two oriented sets γ and γ' are in the same geometric class iff there is $\alpha \in \mathbb{C}$ with $\gamma = \alpha \boxtimes \gamma'$. We say that an oriented set γ is a representative of the class $[\gamma] := \{\alpha \boxtimes \gamma | \alpha \in \mathbb{C}\}$.

There is a one-to-one correspondance between the circuits of the underlying matroid and the geometric classes in Γ. Therefore, given $C \subset E$, we will sometimes denote by $[C]$ the class $\{\gamma | \underline{\gamma} = C\} = \{\alpha \boxtimes \gamma_C | \alpha \in \mathbb{C}\}$ and write γ_C to mean a (free chosed) representative of $[C]$.

It is important to notice that (essentially because of (Γ1)) no extra information is gained in the oriented case with respet to the signed case: the relevant information is the 'phase difference' between two entries of a circuit. Moreover, the whole phase-information has to be preserved if one wants to show equivalence of differents definitions: here this is carried out in a straightforward way by considering a continuous set of signs. Another way could be the one proposed by Ziegler in [16], of introducing 'cellular coefficients'. Anyway, it seems that in this case the required CW-complex should be infinite in dimension 0. In this sense the work of Wenzel and Dress [8],[9] appears inspiring: in fact $\mathbb{C}$-matroids have turned out to be very close to matroids with coefficients in $\mathbb{C}//\mathbb{R}^+$ or $\mathbb{C}$. An interesting question could be wether a theory of this type can be developed using other coefficient domains that Dress and Wenzel's "Q-bereiche"(namely 'cellular' domains as suggested by Ziegler). In the next chapter we will associate a graded complex algebra to $\mathbb{C}$-matroids: taking a fuzzy-$\mathbb{C}$-module instead of this algebra could perhaps lead to such an object (see [11] for an introduction to fuzzy algebra).

In the following we will work with the 'oriented' case, stating the main definitions and results for the 'signed' case too. For the interplay of the two approaches it will be essential to keep in mind that $\natural$ sould represent the formal expression $arg(0)$, so that $e^{i\natural} = 0$.

2.3 Cocircuits and orthogonality

Definition 2.3.1 *Let α, β be two signed sets with signs in a ring R (the 'distinguished element' of definition 2.1.1 being in this case always 0_R). Then we say that α is orthogonal to β (and write $\alpha \perp \beta$) iff*

$$\sum_{e \in \underline{\alpha} \cap \underline{\beta}} \alpha(e)\beta(e) = 0_R. \tag{2.1}$$

This definition fits for oriented sets, where the sum can be considered over whole E since $\alpha(x)\beta(x) \neq 0 \Leftrightarrow x \in \underline{\alpha} \cap \underline{\beta}$, and induces a corresponding concept in the case of signed circuits of a $\mathbb{C}$-matroid.

Definition 2.3.1 (signed-circuits-case) *Let A, B be two signed sets with signs in U. Then we say that α is orthogonal to β (and write $A \perp B$) iff* *there exists families of positive real numbers $(a_e)_{e\in E}$, $(b_e)_{e\in E}$ such that:*

$$\sum_{f\in \underline{A}\cap\underline{B}} a_f e^{iA(f)} b_f e^{iB(f)} = 0.$$

For $\mathcal{A}, \mathcal{B} \subset K^E$ we will sometimes write $\mathcal{A} \perp \mathcal{B}$ if for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$ we have $A \perp B$.

Consider now a $\mathbb{C}$-matroid $\mathfrak{M}$: we have already seen that $\mathcal{C} = \underline{\mathfrak{C}}(\mathfrak{M})$ is the circuit set of a matroid $\underline{\mathfrak{M}}$.
By $\mathcal{D} = \mathcal{D}(\underline{\mathfrak{M}}) := \{\underline{D} \mid D \in U^E,\ \forall C \in \mathfrak{C}(\mathfrak{M}) : |\underline{D} \cap \underline{C}| \neq 1\}$ we denote the set of cocircuits of $\underline{\mathfrak{M}}$.

We are now ready for the result on which bases the concept of duality for $\mathbb{C}$-matroids.

Proposition 2.3.2 *Consider a $\mathbb{C}$-matroid $\mathfrak{M}$ as in the previous definitions, given with a legal orientation Γ of $\mathfrak{C}(\mathfrak{M})$. Then there is exactly one $\Delta \subseteq \mathbb{C}^E$ with*

$$\gamma \perp \delta \quad \textit{for all } \delta \in \Delta \textit{ and } \gamma \in \Gamma$$

and closed with respect to $\boxtimes$ ('multiplication with a scalar').
Moreover, $\underline{\Delta} = \mathcal{D}(\underline{\mathfrak{M}})$.

Proof: It is clear that for δ in such a Δ there is no $\gamma \in \Gamma$ with $|\delta \cap \gamma| = 1$, since in this case the sum (2.1) reduces to an unique, nonzero term. Therefore we have $\underline{\Delta} := \{\underline{\delta} \mid \delta \in \Delta\} \subseteq \mathcal{D}(\mathfrak{M})$.

For each $D \in \mathcal{D}$ define a function

$$\begin{array}{rccc} \sigma_D : & \underline{D} \times \underline{D} & \longrightarrow & \mathbb{C} \\ & (f_1, f_2) & \longmapsto & \sigma_D(f_1, f_2) := -\dfrac{\gamma(f_1)}{\gamma(f_2)} \end{array}$$

where $\gamma \in \Gamma$ is an oriented circuit with $D \cap C = \{f_1, f_2\}$.

Claim: *The value of $\sigma_D(f_1, f_2)$ does not depend on the choice of γ.*

Proof: Consider for contradiction $\gamma' \in \Gamma$ with $\underline{\gamma'} \cap D = \{f_1, f_2\}$ but $\frac{\gamma(f_1)}{\gamma(f_2)} \neq \frac{\gamma'(f_1)}{\gamma'(f_2)}$.
Then a complex number $\alpha \neq 1$ exists with $\frac{\gamma(f_1)}{\gamma(f_2)} = \alpha \frac{\gamma'(f_1)}{\gamma'(f_2)}$. Axiom ($\Gamma$3) applied to $\frac{1}{\gamma(f_2)} \boxtimes \gamma$, $\frac{1}{\gamma'(f_2)} \boxtimes \gamma'$ and f_1 yields $\tilde{\gamma} \in \Gamma$ with $f_1 \in \underline{\tilde{\gamma}} \subseteq \underline{\gamma} \cap \underline{\gamma'}$, and therefore $|\delta \cap \tilde{\gamma}| = 1$, wich is impossible. □

Since by construction $\sigma_D(a, b)\, \sigma_D(b, c) = \sigma_D(a, c)$, the condition $\frac{\delta(e)}{\delta(f)} = \sigma_D(e, f)$ determines a geometric class $[D]$. Carrying out this construction for another cocircuit we have $D \neq D' \Rightarrow [D] \cap [D'] = \varnothing$. Therefore define:

$$\Delta = \biguplus_{D\in\mathcal{D}} [D].$$

Δ is by construction closed under $\boxtimes$ operation.

Claim: *Each* $\delta \in \Delta$ *satisfies* $\delta \perp \gamma$ *for all* $\gamma \in \Gamma$.

Proof: Following an idea from [8] we define for $\alpha, \beta \in \Gamma$, $f \in E$ the signed set $\alpha \circ_f \beta$ as $(\alpha \circ_f \beta)(e) := \alpha(e)\beta(f) - \alpha(f)\beta(e)$, and let Υ be generated by Γ and closed under $\circ_{(\cdot)}$. We show that δ is orthogonal to each $\upsilon \in \Upsilon$: this implies in particular our claim since $\Gamma \subset \Upsilon$. We proceed by induction in $n = |\upsilon \cap \delta|$, noting first of all that if $|\upsilon \cap \delta| = 0$ there is nothing to show. Axiom ($\Gamma 3$) implies further that for each $\upsilon \in \Upsilon$ and $f \in \underline{\upsilon}$ there is $\gamma \in \Gamma$ with $f \in \underline{\gamma} \subseteq \underline{\upsilon}$: this way we conlude that the case $|\upsilon \cap \delta| = 1$ cannot occur. Assume that the claim holds for all $\nu \in \Upsilon$ with $|\nu \cap \delta| < n$, and consider $\upsilon \in \Upsilon$ with $|\upsilon \cap \delta| = n \geq 2$. Let $\{e, f\} \subset \underline{\upsilon} \cap \underline{\delta}$. There is $\mu \in \Gamma$ with $\underline{\delta} \cap \underline{\mu} = \{e, f\}$ and $\mu \perp \delta$.
Consider then $\upsilon_f := \upsilon \circ_f \mu$: we have $|\underline{\upsilon_f} \cap \underline{\delta}| < |\underline{\upsilon} \cap \underline{\delta}|$ and therefore $\upsilon_f \perp \delta$. Now evaluate the sum:

$$\begin{aligned}
\sum_{x \in \underline{\upsilon_f} \cap \underline{\delta}} \upsilon_f(x)\delta(x) &= \sum_{x \in E \setminus f} \upsilon(x)\mu(f)\delta(x) - \sum_{x \in E \setminus f} \upsilon(f)\mu(x))\delta(x) \\
&= \mu(f)\Big(\sum_{x \in E \setminus f} \upsilon(x)\delta(x)\Big) - \upsilon(f)\mu(e)\delta(e) \\
&= \mu(f)\Big(\sum_{x \in E \setminus f} \upsilon(x)\delta(x)\Big) + \upsilon(f)\mu(f)\delta(f) \\
&= \mu(f) \sum_{x \in E} \upsilon(x)\delta(x).
\end{aligned}$$

This shows that $\upsilon_f \perp \delta$ implies $\upsilon \perp \delta$, and completes the proof.

We conclude that the constructed Δ fulfills all requirements. Uniqueness is given by the fact that the condition $\frac{\delta(x)}{\delta(y)} = \sigma_D(x, y)$ determines uniquely the geometric class $[D]$. □

This proposition shows that signature (orientation) of cocircuits comes in a natural (and unique!) way from the signature (orientation) of the circuits, and gives sense to the rest of this chapter: therefore it would be nice to have a directer (i.e. not requiring the construction of Υ) proof of the second claim.

We conclude the section with a precise definition, that formalizes what the intuition has probably already argued:

Definition 2.3.3 *Let* $\mathfrak{M}$ *be a* $\mathbb{C}$*-matroid on the ground set* E*. The set of (signed) cocircuits of* $\mathfrak{M}$ *is defined to be*

$$\mathfrak{D}(\mathfrak{M}) := \{D \in U^E | \forall C \in \mathfrak{C}(\mathfrak{M}) \; D \perp C\}.$$

An orientation of the cocircuits of a $\mathbb{C}$*-matroid is* $\Delta \in \mathbb{C}^E$ *such that*

(I) $\alpha \boxtimes \delta \in \Delta$ *for all* $\delta \in \Gamma$ *and all* $\alpha \in \mathbb{C}$.

(II) there is a surjection $D_{(\cdot)} : \Delta \to \mathfrak{D}$ *with*

$$D_{(\delta)}(e) = \begin{cases} arg(\delta(e)) & if \ \delta(e) \neq 0 \\ \natural & if \ \delta(e) = 0 \end{cases}$$

for all $e \in E$.

Note that $\mathfrak{D}(\mathfrak{M})$ satisfies $(C0)-(C3)$, and Δ as in the definition satisfies $(\Gamma 0)-(\Gamma 3)$. In fact $\mathfrak{D}(\mathfrak{M})$ is the set of cocircuits of a $\mathbb{C}$-matroid $\mathfrak{M}^*$, which is called the *dual* of $\mathfrak{M}$ since $\mathfrak{D}(\mathfrak{M}^*) = \mathfrak{C}(\mathfrak{M})$ and therefore $\mathfrak{M}^{**} = \mathfrak{M}$.

Finally remark that $\mathfrak{D}(\mathfrak{M}/F) = \mathfrak{C}(\mathfrak{M}^* \setminus F)$ and $\mathfrak{D}(\mathfrak{M} \setminus F) = \mathfrak{C}(\mathfrak{M}^*/F)$ for all $F \subset E$.

2.4 Chirotopes

We now want to show that the definition of the complex analogon of a chirotope leads to the structure of $\mathbb{C}$-matroid described above. We begin by showing how such a complex chirotope (with values in $\mathbb{C}$) determines a matroid and an orientation of his circuits. The fact that the orientation classes are uniquely determined by the argument of the complex chirotope leads to the definition of $\mathbb{C}$-chirotopes (see definiton 2.4.1) and shows that there is a one-to-one correspondence (up to scaling) between them and $\mathbb{C}$-matroids defined by circuit signature. These $\mathbb{C}$-chirotopes turn out to be in fact the same as the phirotopes introduced by Richter-Gebert et al. in [1].

2.4.1 Complex chirotopes and circuit orientation.

Definition 2.4.1 *Let* E *be a finite set and* $d \in \mathbb{N}_{>0}$. *Here and in the following let* S_n *denote the symmetric group on* n *elements. We call* ***complex chirotope*** *a map* $\vartheta : E^d \to \mathbb{C}$ *that fulfills following conditions:*

$(\vartheta 0)$ $\vartheta \not\equiv 0$

$(\vartheta 1)$ $\forall (x_1, \ldots, x_d) \in E^d \ \forall \sigma \in S_d :$
$\vartheta(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(r)}) = sign(\sigma) \vartheta(x_1, x_2, \ldots, x_d)$

$(\vartheta 2)$ $\forall (x_1, \ldots, x_d), (y_1, \ldots, y_d) \in E^d :$
$\vartheta(x_1, \ldots, x_d) \vartheta(y_1, \ldots, y_d) =$
$\sum_{i=1}^{d} \vartheta(y_i, x_2, \ldots, x_d) \vartheta(y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_d)$

The first step in costructing a $\mathbb{C}$-matroid from ϑ is the following result:

Proposition 2.4.2 $\mathcal{B}^\vartheta := \{B \subset E^d \,|\, \vartheta(B) \neq 0\}$ *is the set of bases of a matroid.*

Proof: By proposition 1.1.5 and since ($\vartheta0$) ensures that $\mathcal{B}^\vartheta \neq \varnothing$, checking the basis exchange axiom implies the claim. Let then $B, B' \in \mathcal{B}$ and $x \in B$. By definition $\vartheta(B)\vartheta(B') \neq 0$. ($\vartheta2$) implies then that there is an $y \in B'$ with

$$\vartheta((B \setminus \{x\}) \cup \{y\})\vartheta((B' \setminus \{y\}) = \{x\}) \neq 0,$$

and in particular $\vartheta((B \setminus \{x\}) \cup \{y\}) \neq 0$, which means $(B \setminus \{x\}) \cup \{y\} \in \mathcal{B}^\vartheta$. □

We denote by $\mathcal{M}^\vartheta$ the matroid defined by the pair $(E, \mathcal{B}^\vartheta)$ and call $\mathcal{C}^\vartheta, \mathcal{D}^\vartheta$ its set of circuits, resp. cocircuits. Now we state the main proposition of this section: before proving it, we will need to state some lemma.

Proposition 2.4.3 *Let ϑ be a complex chirotope. ϑ induces $\Gamma^\vartheta \subset \mathbb{C}^E$ with $\underline{\Gamma^\vartheta} = \mathcal{C}^\vartheta$ and $\Delta^\vartheta \subset \mathbb{C}^E$ with $\underline{\Delta^\vartheta} = \mathcal{D}^\vartheta$ such that:*

$$\delta \perp \gamma \text{ for all } \delta \in \Delta^\vartheta \text{ and } \gamma \in \Gamma^\vartheta.$$

Γ^ϑ and Δ^ϑ satisfy $(\Gamma0) - (\Gamma3)$.

Lemma 2.4.4 *Let $C = \{e, f, x_2, \ldots, x_l\} \in \mathcal{C}^\vartheta$ and choose $x_{l+1}, \ldots, x_d \in E$ such that $\{f, x_1, \ldots, x_d\} \in \mathcal{B}^\vartheta$. Then*

$$\sigma_f{}^e := \frac{\vartheta(e, x_2, \ldots, x_d)}{\vartheta(f, x_2, \ldots, x_d)}$$

does not depend on the choice of $x_{l+1}, \ldots, x_d \in E$.

Proof: Consider $\{f, y_2, \ldots, y_d\} \in \mathcal{B}^\vartheta$ with $y_i = x_i$ for all $i \leq l$. By definition of $\mathcal{B}^\vartheta$ we have $\vartheta(e, x_2, \ldots, x_d)\vartheta(f, y_2, \ldots, y_d) \neq 0$, and by ($\vartheta3$):

$$\begin{aligned}&\vartheta(e, x_2, \ldots, x_d)\vartheta(f, y_2, \ldots, y_d) = \\ &\qquad \textstyle\sum_{j=1}^{d} \vartheta(y_j, x_2, \ldots, x_d)\vartheta(f, y_2, \ldots, y_{j-1}, e, y_{j+1}, \ldots, y_d),\end{aligned} \tag{2.2}$$

where $y_1 := f$. Now remark the following:

- If $j > l$, then $y_j \notin C$, i.e. $C \subseteq \{f, y_2, \ldots, y_{j-1}, e, y_{j+1}, \ldots, y_d\}$. This means $\{f, y_2, \ldots, y_{j-1}, e, y_{j+1}, \ldots, y_d\} \notin \mathcal{B}^\vartheta$ and therefore $\vartheta(f, y_2, \ldots, y_{j-1}, e, y_{j+1}, \ldots, y_d) = 0$.
- If $l \geq j > 1$, then $y_j \in C \setminus \{f\}$. It follows $y_j \in \{x_2, \ldots, x_l\}$ and ($\vartheta1$) implies then $\vartheta(y_j, x_2, \ldots, x_d) = 0$.

The only nonzero summand in (2.2) is then $\vartheta(f, x_2, \ldots, x_d)\vartheta(e, y_2, \ldots, y_d)$, and (2.2) becomes

$$\vartheta(e, x_2, \ldots, x_d)\vartheta(f, y_2, \ldots, y_d) = \vartheta(f, x_2, \ldots, x_d)\vartheta(e, y_2, \ldots, y_d).$$

Rewriting this we have

$$\frac{\vartheta(e, x_2, \ldots, x_d)}{\vartheta(f, x_2, \ldots, x_d)} = \frac{\vartheta(e, y_2, \ldots, y_d)}{\vartheta(f, y_2, \ldots, y_d)} \quad .\square$$

Lemma 2.4.5 *For every $C \in \mathcal{C}^\vartheta$ choose $e \in C$ and define*

$$\gamma_C(f) := \begin{cases} 1 & \text{if } f = e, \\ -{\sigma_e}^f & \text{else.} \end{cases}$$

Then for all $C \in \mathcal{C}^\vartheta$ we have $\underline{\gamma_C} = C$, and the family $(\gamma_C)_{C\in\mathcal{C}^\vartheta}$ is a representing system of the geometric classes in a set $\Gamma^\vartheta \in \mathbb{C}^E$ satisfying $(\Gamma 0) - (\Gamma 2)$. The class $[\gamma_C] := \{\alpha \boxtimes \gamma | \alpha \in \mathbb{C}\}$ is well-defined (does not depend on the choice of e).

Proof: $\underline{\gamma_C} = C$ is clear. The geometric class $[\gamma_C]$ is well-defined, because by definition ${\sigma_b}^a {\sigma_c}^b = {\sigma_c}^a$, and choosing $e' \neq e$ gives $\gamma'_C = {\sigma'_e}^e \boxtimes \gamma_C$. The fact that $\underline{\gamma_C} = C$ and the circuit axioms for matroids suffice then to show that the sum

$$\Gamma^\vartheta = \biguplus_{C\in\mathcal{C}^\vartheta} [\gamma_C]$$

is disjoint and satisfies $(\Gamma 0) - (\Gamma 2)$.□

Lemma 2.4.6 *For every $D \in \mathcal{D}^\vartheta$ choose $e \in D$ and $y_2, \dots, y_d$ that spans the hyperplane $E \setminus D$. Recall the construction of lemma 2.4.5 and define:*

$$\delta_D(f) := \begin{cases} 1 & \text{if } f = e, \\ {\sigma_f}^e & \text{else.} \end{cases}$$

Then for all $D \in \mathcal{D}^\vartheta$ we have $\underline{\delta_D} = D$, and the family $(\delta_D)_{D\in\mathcal{D}^\vartheta}$ is a representing system of the geometric classes of a uniquely determined $\Delta^\vartheta \in \mathbb{C}^E$ with $\Delta^\vartheta \perp \Gamma^\vartheta$, satisfying $(\Gamma 0) - (\Gamma 2)$. The class $[\delta_D] := \{\alpha \boxtimes \delta | \alpha \in \mathbb{C}\}$ is well-defined (does not depend on the choice of e).

Proof: The same arguments as in lemma (2.4.4) show that for all $D \in \mathcal{D}^\vartheta$ we have $\underline{\delta_D} = D$, that

$$\Delta^\vartheta := \biguplus_{D\in\mathcal{D}^\vartheta} [\delta_D]$$

is unique, and the geometric class $[\delta_D]$ does not depend on the choice of e. Now we have to show orthogonality. Let $D \in \mathcal{D}^\vartheta$, $C := \{x_0, x_1, \dots, x_l\} \in \mathcal{C}^\vartheta$. Choose elements $x_{l+1}, \dots, x_d$ with $\{x_1, \dots, x_d\} \in \mathcal{B}^\vartheta$, and a maximal indipendent set $\{y_2, \dots, y_d\} \subseteq E \setminus D$. W.l.o.g. suppose $x_0 \in C \cap D$. For all j we have then:

$$\begin{aligned} &\vartheta(x_1, \dots, x_{j-1}, x_0, x_{j+1}, \dots, x_d)\vartheta(x_j, y_2, \dots, y_d) \\ &\qquad = -\frac{\gamma_C(x_j)}{\gamma_C(x_0)} \frac{\delta_D(x_j)}{\delta_D(x_0)} \vartheta(x_1, \dots, x_d)\vartheta(x_0, y_2, \dots, y_d) \end{aligned}$$

(recall that this expression is nonzero iff $x_j \in C \cap D$). Now ($\theta 2$) says:

$$\begin{aligned}
&\vartheta(x_1, \ldots, x_d)\vartheta(x_0, y_2, \ldots, y_d) \\
&\quad = \textstyle\sum_{j=1}^{d} \vartheta(x_1, \ldots, x_{j-1}, x_0, x_{j+1}, \ldots, x_d)\vartheta(x_j, y_2, \ldots, y_d) \\
&\qquad = \textstyle\sum_{j=1}^{d} -\frac{\gamma_C(x_j)}{\gamma_C(x_0)}\frac{\delta_D(x_j)}{\delta_D(x_0)}\vartheta(x_1, \ldots, x_d)\vartheta(x_0, y_2, \ldots, y_d).
\end{aligned}$$

We have $\vartheta(x_1, \ldots, x_d)\vartheta(x_0, y_2, \ldots, y_d) \neq 0$ because $\{x_1, \ldots, x_d\} \in \mathcal{B}^\vartheta$ and $x_0 \in D$, and followly

$$\sum_{\substack{x_j \in C \cap D \\ j \neq 0}} \frac{\gamma_C(x_j)}{\gamma_C(x_0)}\frac{\delta_D(x_j)}{\delta_D(x_0)} = -1.$$

So we can finally write:

$$\sum_{x_j \in C \cap D} \gamma_C(x_j)\delta_D(x_j) = 0.\square$$

By construction, $\{\underline{\gamma} \,|\, \gamma \in \Gamma^\vartheta\} = \mathcal{C}^\vartheta$ satisfies the circuit axioms for matroids (and so does $\{\underline{\delta} \,|\, \delta \in \Delta^\vartheta\} = \mathcal{D}^\vartheta$).

But ($\Gamma 3$) requires more than that. We need following lemma:

Lemma 2.4.7 *Γ^ϑ satisfies ($\Gamma 3$) and is then really a circuit orientation of a $\mathbb{C}$-matroid.*

Proof: We have to show:

$$(\Gamma 3) \quad \begin{array}{l} \forall \gamma_1, \gamma_2 \in \Gamma^\vartheta \, \forall f \in S_{\gamma_1,\gamma_2} \exists \, \gamma' \in \Gamma^\vartheta \\ \gamma'(f) \neq 0 \textit{ and } \forall e \in E: \; \gamma_1(e) = \gamma_2(e) \Rightarrow \gamma'(e) = 0 \end{array}.$$

In the following we let $x_i = e_i$ for all $i = 1, \ldots, k$.

We begin by considering the eventuality of a cocircuit $D \in \mathcal{D}^\vartheta$ such that $D \cap C_1 = J_f$. Each orientation $\delta_D \in [D]$ satisfies then $\gamma_1(f)\delta_D(f) = -\sum_{i=1}^{k} \delta_D(e_i)\gamma_1(e_i)$. For C_2 we have a different situation: $\gamma_1(f) \neq \gamma_2(f)$ implies

$$-\sum_{i=1}^{k} \delta_D(e_i)\gamma_2(e_i) = -\sum_{i=1}^{k} \delta_D(e_i)\gamma_1(e_i) = \delta_D(f)\gamma_1(f) \neq \delta_D(f)\gamma_2(f).$$

Since orthogonality must hold, we conclude that the second sum on the right side of

$$-\delta_D(f)\gamma_2(f) = \sum_{i=1}^{k} \delta_D(e_i)\gamma_2(e_i) + \sum_{i=k+1}^{m} \delta_D(y_i)\gamma_2(y_i)$$

has at least one nonzero summand.

This shows that, for each cocircuit $D \in \mathcal{D}^\vartheta$ with $f \in D \subset J_f$, we have $A := D \cap (C_2 \setminus C_1) \neq \varnothing$.

In the sequel it will often be useful to consider $\mathcal{M}^\vartheta|(C_1 \cup C_2)$, that throughout the remainder of the proof we will denote by $\widetilde{\mathcal{M}}$. We agree to call $\widetilde{\mathcal{C}}$, $\widetilde{\mathcal{D}}$, $\widetilde{\mathcal{B}}$, $\widetilde{\mathcal{B}}^*$ the set of circuits, cocircuits, bases, cobases of $\widetilde{\mathcal{M}}$.

Recall from chapter 1 that $\widetilde{\mathcal{M}}$ is a matroid, and that

$$\begin{aligned} \widetilde{\mathcal{C}} &= \{C \in \mathcal{C}(\mathcal{M}^\vartheta) \,|\, C \subset (C_1 \cup C_2)\}, \\ \widetilde{\mathcal{D}} &= \mathrm{Min}\{D \cap (C_1 \cup C_2) \,|\, D \in \mathcal{D}^\vartheta\}, \end{aligned}$$

where Min denotes inclusion minimality.

The previous considerations prove:

Claim 1: *There is no cocircuit $D \in \widetilde{\mathcal{D}}$ with $f \in D \subset J_f$.*

Now, the set J could be coindependent: this is the better situation that we can expect to have, since then following claim, applied to $\widetilde{\mathcal{M}} = \mathcal{M}^\vartheta|(C_1 \cup C_2)$, suffices to complete the proof:

Claim 2: *Let $\mathcal{M}$ be a matroid obtained from $\mathcal{M}^\vartheta$ by restriction on a set $F \subseteq (C_1 \cup C_2)$, and $G \subseteq F$ be a coindependent set of $\mathcal{M}$, containing f and including $J \cap F$. Then there is $\gamma' \in \Gamma^\vartheta$ satisfying* ($\Gamma3$).

Proof: We can expand G to a cobasis B^* of $\mathcal{M}$, and consider the basis B complementar to B^*. Clearly, $f \notin B$. Now set $C' := C(\widetilde{B}, f)$: this is the basic circuit of f, and therefore contains f. Moreover, C' contains no e_i (recall: $C' \subset F$, $C' \cap (J \cap F) = \varnothing$ since $(J \cap F) \subset G$) and, viewed as a circuit of $\mathcal{M}^\vartheta$, is contained in $C_1 \cup C_2$. Therefore $\gamma' \in [C'] \subset \Gamma^\vartheta$ satisfies

$$\begin{gathered} \gamma'(f) \neq 0 \\ \gamma_1(e) = \gamma_2(e) \Rightarrow \gamma'(e) = 0. \end{gathered}$$

Summarizing the situation: Now we are able to construct a γ' satisfying the requirements of the lemma, in the case where J_f is a coindependent set of $\widetilde{\mathcal{M}}$. We will now show that the situation can always be reconducted to this case.

Claim 3: *Let $\mathcal{M}$ be a matroid on the ground set F, let $J \subset F$, $f \in F \setminus J$. Suppose that there is a circuit $\hat{C}$ with $\hat{C} \cap J \neq \varnothing$, but that no cocircuit containing f and contained in $J \cup \{f\}$ exists. We want to prove: J is coindependent, or there is a circuit C' of $\mathcal{M}$ with $f \in C' \subseteq F \setminus J$.*

Proof: We proceed by induction on $k := |J|$, since the case $k = 1$ is trivial: J is then coindependent, because else J *is* a cocircuit, and then $|\hat{C} \cap J| = 1$, a contradiction.

Let then $|J| \geq 2$, and suppose J not coindependent. Consider then a cocircuit $D \subset J$, and let $e \in D$. The set $(D \setminus \{e\}) \cup \{f\}$ is coindependent, because a cocircuit in this set would either contain f (impossible by assumption) or be strictly contained in D, contradicting the inclusion minimality of cocircuits. $\{e\}$ is clearly an independent set. Proposition (1.1.18) gives then $B^* \in \mathcal{B}^*(\mathcal{M})$ and $B \in \mathcal{B}(\mathcal{M})$ with $(D \setminus \{e\}) \cup \{f\} \subset B^*$ and $e \in B$.

Now consider $C' := C(B, f)$, the basic circuit of f with respect to B. We know that $f \in C'$. Moreover, we have $e \notin C'$, since the contrary would imply $D \cap C' = \{e\}$, which is impossible.

I) If $C' \cap J = \varnothing$, C' satisfies the requirements of claim 3.

II) If $C' \cap J \neq \varnothing$, then in particular $J \setminus D \neq \varnothing$. We consider the matroid $\mathcal{M}' := \mathcal{M} \setminus D$ with ground set $F' := F \setminus D$. We can then say that C' is a circuit of $\mathcal{M}'$, there is $J' := J \setminus D \subset F'$ with $J' \cap C' \neq \varnothing$ and, although $f \in F'$, there is no cocircuit containing f and contained in $J' \cup \{f\}$ (this would imply the existence of a cocircuit of $\mathcal{M}$ containing f and contained in $J' \cup \{f\} \cup D = J \cup \{f\}$). Since $|J'| < |J|$, we can apply the induction hypotesis. This shows that if there is no circuit C'' satisfying $f \in C'' \subset F' \setminus J' = F \setminus J$, then J' is coindependent. In the second case we conclude that $J' \cup \{f\}$ is coindependent: expanding it to a cobasis B'^* we can consider the basis $B' := F' \setminus B'^*$ and get $C'' := C(B', f)$ with $f \in C'' \subset F' \setminus (B'^* \setminus \{f\}) \subset F \setminus J$.
In both cases C'', that is also a circuit of $\mathcal{M}$, satisfies the requirements of claim 3.

If claim 3 applied to $\widetilde{\mathcal{M}} := \mathcal{M}^{\vartheta}|(C_1 \cup C_2)$ does not give C' such that $\gamma' \in [C']$, then it ensures that J is coindependent in $\widetilde{\mathcal{M}}$, which is enough in order to conclude (with claim 1) that J_f is coindependent. An application of claim 2 gives then γ' as desired and completes the proof. □

Proof of proposition (2.4.3): Immediate by summarizing the results of lemma (2.4.5), (2.4.6), and (2.4.7).□

2.4.2 $\mathbb{C}$-chirotopes and circuit signature.

We now state the results of this section considering the signed-circuit description of a $\mathbb{C}$-matroid (recall the convention $e^{i\natural} = 0$). The proofs are an immediate 'traduction' of the corresponding proofs for the 'oriented' case. Nevertheless, we preferred to work with oriented sets since the computations have there a more suggestive and intuitive form.

Definition 2.4.1 (signed-circuit case) *Let E be a finite set, $U = S^1 \cup \{\natural\}$ the set of signs introduced in section 2.1 and $d \in \mathbb{N}_{>0}$. We call $\mathbb{C}$-**chirotope** a map $\omega : E^d \to U$ that fulfills following conditions:*

- $(\omega 0)$ $\omega \not\equiv \natural$
- $(\omega 1)$ $|\{x_1, \ldots, x_d\}| < r \Rightarrow \omega(x_1, \ldots, x_d) = \natural$
- $(\omega 2)$ $\forall (x_1, \ldots, x_d) \in \omega^{-1}(S^1)\ \forall \sigma \in S_d$:
  $\omega(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(r)}) = \frac{(1-sign(\sigma))}{2}\pi \dot{+} \omega(x_1, x_2, \ldots, x_d)$
- $(\omega 3)$ $\forall (x_1, \ldots, x_d), (y_1, \ldots, y_d) \in E^d \exists\, r, s \in {\mathbb{R}_+}^d$:
  $e^{\omega(x_1,\ldots,x_d)} e^{\omega(y_1,\ldots,y_d)} = \sum_{i=1}^d r_i e^{\omega(y_i,x_2,\ldots,x_d)} s_i e^{\omega(y_1,\ldots,y_{i-1},x_1,y_{i+1},\ldots,y_d)}$

Proposition 2.4.2 (signed-circuit case) $\mathcal{B}^\omega := \{B \subset E^d \,|\, \omega(B) \neq \natural\}$ *is the set of bases of a matroid.*

We denote by $\mathcal{M}^\omega$ the matroid defined by the pair $(E, \mathcal{B}^\omega)$ and call $\mathcal{C}^\omega, \mathcal{D}^\omega$ its set of circuits, resp. cocircuits.

Proposition 2.4.3 (signed-circuit case) *Let ω be a $\mathbb{C}$-chirotope. ω induces a circuit signature $\mathfrak{C}^\omega$ of a $\mathbb{C}$-matroid $\mathfrak{M}^\omega$ with $\underline{\mathfrak{M}^\omega} = \mathcal{M}^\omega$, and $\mathfrak{D}^\omega \subset U^E$ with $\underline{\mathfrak{D}^\omega} = \mathcal{D}^\omega$ such that*

$$D \perp C \text{ for all } D \in \mathfrak{D}^\omega \text{ and } C \in \mathfrak{C}^\omega.$$

It is clear that in the case where $\omega(x_1, \ldots, x_d) = arg(\vartheta(x_1, \ldots, x_d))$ for all $\{x_1, \ldots, x_d\} \in E^d$, Γ^ϑ is a legal orientation of $\mathfrak{C}^\omega$, and then $\mathfrak{M}^\vartheta$ and $\mathfrak{M}^\omega$ are in fact the same $\mathbb{C}$-matroid.

We can now state:

Definition 2.4.1 ($\mathbb{C}$-chirotope approach) *A $\mathbb{C}$-matroid is a pair $\mathfrak{M} = (E, \omega)$, where E is a finite set, and ω is a $\mathbb{C}$-chirotope on E.*

2.5 Signature of bases

In this section we want to give another definition of $\mathbb{C}$-matroid via signing (orienting) the bases of a matroid. We show that a basis signature (orientation) is naturally associated with every $\mathbb{C}$-matroid.

Basis orientation

Definition 2.5.1 *Let $\mathfrak{M}$ be a rank d $\mathbb{C}$-matroid on the ground set E, given by the orientation Γ of its circuits. An orientation of the set $\mathcal{B}_o$ of ordered bases of the underlying matroid $\mathcal{M}$ is a map $\xi : \mathcal{B}_o \to \mathbb{C}$ satisfying following conditions:*

$$\begin{array}{ll} (\xi 0) & \xi \not\equiv 0 \\ (\xi 1) & \forall (x_1, \dots, x_d) \in E^d \ \forall \sigma \in S_d : \\ & \xi(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(r)}) = sign(\sigma)\xi(x_1, x_2, \dots, x_d) \\ (\xi 2) & \forall (e, x_2, \dots, x_d), (f, x_2, \dots, x_d) \in \mathcal{B}_o : \\ & \xi(e, x_2, \dots, x_d) = -\frac{\gamma_C(f)}{\gamma_C(e)}\xi(f, x_2, \dots, x_d) \\ & for\ the\ unique\ circuit\ C \in \{e, f, x_2, \dots, x_d\}. \end{array}$$

Remark: The condition $(\xi 2)$ can be replaced by the following 'dual' statement:

$$\begin{array}{ll} (\xi 2)^* & \forall (e, x_2, \dots, x_d), (f, x_2, \dots, x_d) \in \mathcal{B}_o : \\ & \xi(e, x_2, \dots, x_d) = -\frac{\delta_D(f)}{\delta_D(e)}\xi(f, x_2, \dots, x_d), \\ & for\ the\ unique\ cocircuit\ D\ complementar\ to\ < x_2, \dots, x_d > . \end{array}$$

Proof: Let $(e, x_2, \dots, x_d), (f, x_2, \dots, x_d) \in \mathcal{B}_o$ and C, D as above. Then we have $D \cap C = \{e, f\}$, and the orthogonality of circuits and cocircuits implies $\gamma_C(e)\delta_D(e) = \gamma_C(f)\delta_D(f)$. So we have $\frac{\gamma_C(e)}{\gamma_C(f)} = -\frac{\delta_D(f)}{\delta_D(e)}$.□

Proposition 2.5.2 *Consider a $\mathbb{C}$-matroid $\mathfrak{M}$ given with a circuit orientation Γ. Let $\mathcal{B}$ denote the set of bases of $\underline{\mathfrak{M}}$, and $(b_1, \dots, b_d)$ a distinguished ordered basis of $\underline{\mathfrak{M}}$.*

There is a unique alternating map $\xi : \mathcal{B} \to \mathbb{C}$ satisfying $(\xi 2)^*$ and such that $\xi(b_1, \dots, b_d) = 1$. This map satisfies $(\xi 2)$ too and this way determines $\mathfrak{M}$.

Proof: We apply induction on $|E|$. If $|E| = 0$ the claim is trivial. Let $|E| > d$, $a \in E \setminus \{b_1, \dots, b_d\}$ and suppose for induction that there is a unique map $\xi : \mathcal{B} \cap (E \setminus \{a\})^d \longrightarrow \mathbb{C}$ that satisfies all requirements. Now take a basis of the form $(a, x_2, \dots, x_d)$ and choose $e \in E \setminus \{a\}$: $(\xi 2)^*$ implies that the choice of e does not affect the value of $\frac{1}{\delta_D(e)}\xi(e, x_2, \dots, x_r)$ (D the cocircuit complementar to $< x_2, \dots, x_d >$). Therefore we can define:

$$\xi(a, x_2, \dots, x_r) := \frac{\delta_D(a)}{\delta_D(e)}\xi(e, x_2, \dots, x_d).$$

This determines uniquely the extension of ξ to the whole set E. We now show that this ξ satisfies $(\xi 2)$.

Let $(e, x_2, \dots, x_d)$, $(f, x_2, \dots, x_d) \in \mathcal{B} \cap (E \setminus \{a\})^d$ be two ordered bases of $\underline{\mathfrak{M}}$ and let γ_C denote an oriented circuit with support $C \subset \{e, f, x_2, \dots, x_d\}$. We want to show that

$$\xi(e, x_2, \ldots, x_r) = -\frac{\gamma_C(f)}{\gamma_C(e)}\xi(f, x_2, \ldots, x_d).$$

There are three possible cases:

1. $\boldsymbol{a \notin \{e, f, x_2, \ldots, x_d\}}$**.** In this case ($\xi$2) holds by induction hypotesis.

2. $\boldsymbol{a = e}$ ***or*** $\boldsymbol{a = f}$**.** W.l.o.g. let $a = f$. By orthogonality of circuits and cocircuits we have $\gamma_C(a)\delta_D(a) + \gamma_C(e)\delta_D(e) = 0$. This means $\frac{\delta_D(e)}{\delta_D(a)} = -\frac{\gamma_C(a)}{\gamma_C(e)}$, and the claim follows then from (ξ2)*, which we have already shown.

3. $\boldsymbol{a = x_i}$ ***for an*** $\boldsymbol{i \in \{2, \ldots, d\}}$ (w.l.o.g. let $a = x_2$). Again, we distinguish two cases:

   - $\boldsymbol{a \in C}$**.** In particular, $(e, f, x_3, \ldots, x_d)$ is a basis, and by case 2 above we know:

   $$\begin{cases} \xi(a, e, x_3, \ldots, x_d) = -\frac{\gamma_C(f)}{\gamma_C(a)}\xi(f, e, x_3, \ldots, x_d) \\ \xi(a, f, x_3, \ldots, x_d) = -\frac{\gamma_C(a)}{\gamma_C(e)}\xi(e, f, x_3, \ldots, x_d) \end{cases}$$

   Recalling that a was assumed to be x_2, this means:

   $$\begin{aligned} \xi(e, a, x_3, \ldots, x_d) &= -\frac{\gamma_C(f)}{\gamma_C(a)}\xi(e, f, x_3, \ldots, x_d) \\ &= \frac{\gamma_C(f)}{\gamma_C(e)}\xi(a, f, x_3, \ldots, x_d) \\ &= -\frac{\gamma_C(f)}{\gamma_C(e)}\xi(f, a, x_3, \ldots, x_d). \end{aligned}$$

   - $\boldsymbol{a \notin C}$**.** In this case, we consider the cocircuit D of $\mathfrak{M}$ complementar to the flat determined by the closure $< f, e, x_3, \ldots, x_d >$. Choose $y \in D \setminus \{a\}$. Then $(e, y, x_3, \ldots, x_d)$ and $(f, y, x_3, \ldots, x_d)$ are ordered bases of $\mathfrak{M}$, and by induction hypotesis we have $\xi(e, y, x_3, \ldots, x_d) = -\frac{\gamma_C(f)}{\gamma_C(e)}\xi(f, y, x_3, \ldots, x_d)$.
   Now it suffices to consider an orientation δ_D of D, and compute:

   $$\begin{aligned} \xi(f, a, x_3, \ldots, x_d) &= \frac{\delta_D(y)}{\delta_D(a)}\xi(f, y, x_3, \ldots, x_d) \\ &= -\frac{\delta_D(y)}{\delta_D(a)}\frac{\gamma_C(y)}{\gamma_C(f)}\xi(e, y, x_3, \ldots, x_d) \\ &= -\frac{\delta_D(y)}{\delta_D(a)}\frac{\gamma_C(y)}{\gamma_C(f)}\frac{\delta_D(a)}{\delta_D(y)}\xi(e, a, x_3, \ldots, x_d) \\ &= -\frac{\gamma_C(y)}{\gamma_C(f)}\xi(e, a, x_3, \ldots, x_d). \end{aligned}$$

Once we have (χ2), an analogue argumetation as in lemmas 2.4.5, 2.4.6, 2.4.7 shows that the basis orientation determines $\mathfrak{M}$ and completes the proof.□

Thus we can associate to each $\mathbb{C}$-matroid a basis orientation in a natural way, and each basis orientation determines a $\mathbb{C}$-matroid. We conclude the section stating the connection with complex chirotopes as defined in section (2.4).

Remark 2.5.3 *Consider a $\mathbb{C}$-matroid $\mathfrak{M}$. The map ξ constructed in Proposition (2.5.2) satisfies ($\vartheta 2$). This way we show that for each $\mathbb{C}$-matroid $\mathfrak{M}$ we can in a natural way construct a chirotope ϑ such that $\Gamma(\mathfrak{M}) = \Gamma^\vartheta$.*

Proof: Let $(x_1, \ldots, x_d), (y_1, \ldots, x_d)$ be two ordered bases of $\underline{\mathfrak{M}}$. Consider $\gamma_C \in [C]$, C being the circuit contained in $\{x_1, y_1, \ldots, y_d\}$, and an orientation δ_D of the cocircuit D determined by the complement of the flat $< x_2, \ldots, x_d >$. By construction of ξ we have for all i:

$$\begin{aligned}&\xi(y_i, x_2, \ldots, x_d)\xi(y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_d)\\ &= \frac{\delta_D(y_i)}{\delta_D(x_1)}\xi(x_1, \ldots, x_d)\Big(- \frac{\gamma_C(y_i)}{\gamma_C(x_1)}\Big)\xi(y_1, \ldots, y_d).\end{aligned}$$

Orthogonality of circuits and cocircuits implies

$$\sum_{e \in C \cap D} \gamma_C(e)\delta_D(e) = 0. \tag{2.3}$$

For $e \notin C$ we have $\xi(y_1, \ldots, y_{i-1}, e, y_{i+1}, \ldots, y_d) = 0$ and similar for $e \notin D$ is $\xi(e, x_2, \ldots, x_d) = 0$. Hence, extending the sum over all elements we have:

$$\begin{aligned}&\sum_{i=1}^{d} \xi(y_i, x_2, \ldots, x_d)\xi(y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_d)\\ &= \Bigg(\underbrace{\sum_{y_i \in C \cap D} -\frac{\gamma_C(y_i)}{\gamma_C(y_i)}\frac{\delta_D(x_1)}{\delta_D(x_1)}}_{=1 \ by \ (2.3)}\Bigg)\xi(x_1, \ldots, x_d)\xi(y_1, \ldots, y_d)\\ &= \xi(x_1, \ldots, x_d)\xi(y_1, \ldots, y_d),\end{aligned}$$

that is, ($\vartheta 2$).□

Basis signature

As in the previous section, we state the results for the signed-circuit description of a $\mathbb{C}$-matroid. With these we will be able to define $\mathbb{C}$-matroids in terms of basis orientations.

Remark: Recall $G_d(\mathbb{C}^n)$, the Grassmann variety of the n-dimensional subspaces of $\mathbb{C}^n$. Such a subspace can be uniquely determined by the images $\{v_1, \ldots, v_n\}$ of the projections of the standard coordinate vectors of $\mathbb{C}^n$ on the subspace, thus we can associate to each point of the variety an arrangement $\{H_1, \ldots, H_n\}$ of hyperplanes in $(\mathbb{C}^n)^*$ by setting $H_i = \text{Ker}(v_i)$. The equations defining $G_d(\mathbb{C}^n)$ as subvariety of $(\bigwedge^d \mathbb{C}^n)/\mathbb{C}^*$ force the coordinates of each $\mathcal{P} \in (\bigwedge^d \mathbb{C}^n)/\mathbb{C}^*$ to satisfy the requirements for being a $\mathbb{C}$-chirotope, and the underlying matroid of the so obtained $\mathbb{C}$-matroid is the matroid of the intersection lattice of $\mathcal{P}$. Using the basis signature presented in this setion we can encode the information of the signs by choosing an appropriate direction on each edge of the basis graph and suitably labelling the edges of the resulting directed graph with real numbers $0 \leq x < 2\pi$ (following the method described in the proof of theorem 3.4.2). To determine the directions of the edges one must introduce an arbitrary ordering on the set of hyperplanes. Once this is fixed, the structure of the no-broken-circuit complex and the $\mathbb{C}$-matroid determine uniquely the labels of the edges. This way the stratification determined by $\mathbb{C}$-matroids is really a refinement of the one given in [10] and, in analogy with the real case, it can be described as a labelling of the basis graph.

Definition 2.5.1 (signed-circuit case) *Let $\mathfrak{M}$ be a rank d $\mathbb{C}$-matroid on the ground set E, given by the signature $\mathfrak{C}$ of its circuits. An orientation of the set $\mathcal{B}_o$ of ordered bases of the underlying matroid $\mathcal{M}$ is a map $\psi : \mathcal{B}_o \to U$ satisfying following conditions:*

$(\psi 0)$ $\psi \not\equiv \natural$

$(\psi 1)$ $\forall (x_1, \ldots, x_d) \in E^d \; \forall \sigma \in S_d :$
$\psi(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(r)}) = \frac{(1-sign(\sigma)}{2}\pi \dot{+} \psi(x_1, x_2, \ldots, x_d)$

$(\psi 2)$ $\forall (e, x_2, \ldots, x_d), (f, x_2, \ldots, x_d) \in \mathcal{B}_o :$
$\psi(e, x_2, \ldots, x_d) = (\pi \dot{+} (C(e) \dot{-} C(f))) \dot{+} \psi(f, x_2, \ldots, x_d)$
for the unique signedcircuit with $\underline{C} \in \{e, f, x_2, \ldots, x_d\}$.

Remark: The condition $(\psi 2)$ can be replaced by the following 'dual' statement:

$(\psi 2)^*$ $\forall (e, x_2, \ldots, x_d), (f, x_2, \ldots, x_d) \in \mathcal{B}_o :$
$\psi(e, x_2, \ldots, x_d) = (\pi \dot{+} (D(f) \dot{-} D(e))) \dot{+} \psi(f, x_2, \ldots, x_d),$
for the unique cocircuit D with $\underline{\text{D}}$ complementar to $< x_2, .., x_d >$.

Proposition 2.5.2 (signed-circuit case) *Consider a $\mathbb{C}$-matroid $\mathfrak{M}$ given with a circuit orientation Γ. Let $\mathcal{B}$ denote the set of bases of $\underline{\mathfrak{M}}$, and $(b_1, \ldots, b_d)$ a distinuished ordered basis of $\underline{\mathfrak{M}}$.*
There is a unique map $\psi : \mathcal{B} \to U$ satisfying $(\psi 0)$, $(\psi 1)$, $(\psi 2)^*$ and such that $\psi(b_1, \ldots, b_d) = 0$. This map satisfies $(\psi 2)$ too and this way determines $\mathfrak{M}$.

Remark 2.5.4 (signed-circuit case) *Consider a $\mathbb{C}$-matroid $\mathfrak{M}$. The map ψ constructed in Proposition (2.5.2) satisfies $(\omega 2)$. This way we show that for each $\mathbb{C}$-matroid $\mathfrak{M}$ we can in a natural way construct a $\mathbb{C}$-chirotope ω such that $\mathfrak{C}(\mathfrak{M}) = \mathfrak{C}^{\vartheta}$.*

Definition 2.5.1 (basis-signature approach) *A $\mathbb{C}$-matroid is a pair (E, ψ), where E is a finite set and $\psi : \mathscr{P}(E) \to U$ is a map satisfying $(\psi 0) - (\psi 2)$ and such that $\psi^{-1}(S^1)$ is the set of bases of a matroid.*

2.6 Reorientation

We now formalize the concept of reorientation for $\mathbb{C}$-matroids, giving the defintion for both the 'signed' and the 'oriented' interpetation.
Consider a signed set $R \in (U \setminus \natural)^E$ and recall from section 2.1 the operation $\dot{+}$. For $A \in U^E$ we let ${}_RA \in U^E$ denote the signed set defined by

$$ {}_RA(e) := R(e)\dot{+}A(e) \quad for\ all\ e \in E. $$

Similarly, taking $\rho \in \mathbb{C}^{*E}$, for $\alpha \in \mathbb{C}^E$ define ${}_\rho\alpha \in \mathbb{C}^E$ such that

$$ {}_\rho\alpha(e) := \rho(e)\alpha(e) \quad for\ all\ e \in E. $$

Definition 2.6.1 *Given a $\mathbb{C}$-matroid $\mathfrak{M}$ and a signed set $R \in (U \setminus \natural)^E$, we denote by ${}_R\mathfrak{M}$ the* ***reorientation of*** $\mathfrak{M}$ ***by R****, a $\mathbb{C}$-matroid defined by the set of signed circuits*

$$ {}_R\mathfrak{C} := \mathfrak{C}({}_R\mathfrak{M}) = \{{}_RC \,|\, C \in \mathfrak{C}(\mathfrak{M})\}. $$

It is easy to see that ${}_R\mathfrak{C}$ satisfies $(C0) - (C3)$.

Definition 2.6.1 (oriented-circuits case) *For a given $\mathbb{C}$-matroid $\mathfrak{M}$ and $\rho \in \mathbb{C}^{*E}$, we denote by ${}_\rho\mathfrak{M}$ the* ***reorientation of*** $\mathfrak{M}$ ***by*** ρ*, a $\mathbb{C}$-matroid defined by the set of oriented circuits*

$$ {}_\rho\Gamma := \Gamma({}_\rho\mathfrak{M}) = \{{}_\rho\gamma \,|\, \gamma \in \Gamma(\mathfrak{M})\}. $$

${}_\rho\Gamma$ is easily seen to satisfy $(\Gamma 0) - (\Gamma 3)$, and it is clear that $arg(\rho(e)) = R(e)$ for all e implies ${}_R\mathfrak{M} = {}_\rho\mathfrak{M}$. Hence we will sometimes use the intuitive notation ${}_\rho\mathfrak{C}$ to denote $\mathfrak{C}({}_\rho\mathfrak{M})$.

Reorientation of ω (resp. ϑ)

The idea behind following two definitions is, given ρ (resp. R) and a chirotope ϑ (resp. ω) as above, to construct 'reorientations' ${}_\rho\vartheta$ (resp. ${}_R\omega$) in such a way that the $\mathbb{C}$-matroid structure induced by the chirotopes satisfies

$$ {}_\rho(\mathfrak{M}^\vartheta) = \mathfrak{M}^{({}_\rho\vartheta)}, \text{ resp. } {}_R(\mathfrak{M}^\omega) = \mathfrak{M}^{({}_R\omega)}. $$

Definition 2.6.1 *Given a $\mathbb{C}$-chirotope $\omega : E^d \to U$ and $R \in (U \setminus \natural)^E$, we define the* ***reorientation of*** ω ***by*** R *as the map*

$$ \begin{array}{rccl} {}_R\omega : & E^d & \to & U \\ & (e_1, \dots, e_d) & \mapsto & R(e_1)\dot{+}\cdots\dot{+}R(e_d)\dot{+}\omega(e_1, \dots, e_d) \end{array} $$

Definition 2.6.1 (oriented case) *Given a complex chirotope $\vartheta : E^d \to \mathbb{C}$ and $\rho \in \mathbb{C}^{*E}$, we define the* ***reorientation of*** ϑ ***by*** ρ *as the map*

$$ \begin{array}{rccl} {}_\rho\vartheta : & E^d & \to & \mathbb{C} \\ & (e_1, \dots, e_d) & \mapsto & \rho(e_1)\cdots\rho(e_d)\vartheta(e_1, \dots, e_d) \end{array} $$

Recalling the constructions of section 2.4, an easy computation (that we leave to the reader) shows that ${}_\rho(\mathfrak{M}^\vartheta) = \mathfrak{M}^{({}_\rho\vartheta)}$, resp. ${}_R(\mathfrak{M}^\omega) = \mathfrak{M}^{({}_R\omega)}$.

Reorientation of ψ (resp. ξ)

As in the previous subsection, given ρ (resp.R), we want to define the reorientation ${}_\rho\xi$ (${}_R\psi$) of a base orientation (signature) in such a way that the $\mathbb{C}$-matroid with basis orientation ${}_\rho\xi$ be the same as the reorientation by ρ of the $\mathbb{C}$-matroid $\mathfrak{M}$ with basis orientation ξ (and similarly for the 'signed' case). The idea is the same as in the case of chirotopes, and therefore we state the definitions whitout further comments.

Definition 2.6.1 *Given a basis signature $\psi : E^d \to U$ and $R \in (U \setminus \natural)^E$, we define the* ***reorientation of*** ψ ***by*** R *to be the map*

$$\begin{array}{rccl} {}_R\psi : & E^d & \to & U \\ & (e_1, \ldots, e_d) & \mapsto & R(e_1)\dot{+}\cdots\dot{+}R(e_d)\dot{+}\psi(e_1, \ldots, e_d). \end{array}$$

Definition 2.6.1 (oriented case) *Given a basis orientation $\xi : E^d \to \mathbb{C}$ and $\rho \in \mathbb{C}^{*E}$, we define the* ***reorientation of*** ξ ***by*** ρ *to be the map*

$$\begin{array}{rccl} {}_\rho\xi : & E^d & \to & \mathbb{C} \\ & (e_1, \ldots, e_d) & \mapsto & \rho(e_1)\cdots\rho(e_d)\xi(e_1, \ldots, e_d). \end{array}$$

This ends our short exposition of a first try on a theory of matroids with continuous set of signs. In the next chapter we will make use of this and some algebraic construction to relate this model to complex arrangements via the Orlik-Terao algebra.

Chapter 3

A Complex Version of the Cordovil Algebra

Just as done by Cordovil in [6], we define a commutative algebra associated with a $\mathbb{C}$-matroid $\mathfrak{M}$. For this algebra we show that there is a short exact sequence induced by the operations of contraction and deletion of an element of the matroid. As an application of this result (and of the techniques developed for its proof) we consider the case where the matroid arises from a complex arrangement $\mathcal{A}$. We then discuss the relation of this algebra with the Orlik-Terao algebra of $\mathcal{A}$ (as defined in [13]) and its dependence on the intersection lattice of $\mathcal{A}$ (i.e. on the underlying matroid $\underline{\mathfrak{M}}$).

3.1 The broken circuit complex

Consider a matroid $\mathcal{M}$ with an (arbitrary) ordering of its elements (w.l.o.g we may think of a matroid on the ground set $[n]$), and let $\mathcal{C}(\mathcal{M})$ be its set of circuits. This ordering allows to distinguish a minimal element in each circuit $C \in \mathcal{C}(\mathcal{M})$: we denote such an element by $\mu(C)$.

A *broken circuit* is a set of the form $C \setminus \mu(C)$ for a circuit $C \in \mathcal{C}(\mathcal{M})$ with $|C| > 1$ (here and in the following we slightly abuse notation and denote the singleton $\{x\}$ by x, as usual in the literature).

A *no broken circuit set* of a matroid is an independent set which contains no broken circuit.

At this point we fix some notation: for a matroid $\mathcal{M}$ let $IND_k(\mathcal{M})$ denote the family of indipendent sets of of $\mathcal{M}$ with cardinality k, and set $IND(\mathcal{M}) := \bigcup_{k\in[n]} IND_k(\mathcal{M})$. The *rank* of a subset $X \subseteq [n]$ is the cardinality of an inclusion-maximal independent set contained in X. By $NBC_k(\mathcal{M})$ we denote the family of no broken circuit sets of cardinality k in $\mathcal{M}$, and we set $NBC(\mathcal{M}) := \bigcup_{k\in[n]} NBC_k(\mathcal{M})$.

Remark 1 *$IND(\mathcal{M})$ and $NBC(\mathcal{M})$ are abstract simplicial complexes on the ground set $[n]$, with $NBC(\mathcal{M}) \subseteq IND(\mathcal{M})$.*

Remark 2 *Given a matroid $\mathcal{M}$ on the set $[n]$ and an $x \in [n]$ we have*

$$NBC(\mathcal{M}) = NBC(\mathcal{M} \setminus x) \uplus NBC(\mathcal{M}/x), \tag{3.1}$$

where $\uplus$ denotes disjoint union.

Proof : W.l.o.g. assume x=n. The claim becomes clear by writing out the definitions: we have then $NBC(\mathcal{M} \setminus n) = \{X \subseteq [n-1] | X \in NBC(\mathcal{M})\}$ and $NBC(\mathcal{M}/n) = \{X \subseteq [n] | X \cup \{n\} \in NBC(\mathcal{M})\}$. □

Recall the *closure operator* $< \cdots >_{\mathcal{M}}$ of a matroid, which is defined on subsets X of the ground set as

$$<X>_{\mathcal{M}} := \{x \in [n] \,|\, rank(X \cup x) = rank(X)\}.$$

Now consider an $X \in IND(\mathcal{M})$, choose $x \in <X>_{\mathcal{M}} \setminus X$, and let $C(X, x)$ denote the (unique) circuit of $\mathcal{M}$ contained in $X \cup x$. To each $X \in IND(\mathcal{M})$ we can now associate the set

$$EA(X) := \{x \in <X>_{\mathcal{M}} \mid x = \mu(C(X,x)) \text{ and } C(X,x) \neq \{x\}\},$$

called the set of the *externally active elements* of X.

Remark 3 *It is an immediate consequence of the definitions that for all $X \in IND(\mathcal{M})$:*

$$EA(X) \neq \emptyset \iff X \in NBC(\mathcal{M}).$$

If $EA(X) \neq \emptyset$, let $\alpha(X)$ denote the minimal externally active element of X. We have then $\alpha(X) = \mu(C(X, \alpha(X)))$.

Proposition 3.1.1 *Consider a matroid $\mathcal{M}$ and an $X \in IND(\mathcal{M})$. For $x \in C(X, \alpha(X)) \setminus \alpha(X)$ define $X_x := (X \setminus x) \cup \alpha(X)$.*
Then $EA(X_x) \subset EA(X) \setminus \alpha(X)$.

Proof: Take $\beta \in EA(X_x)$, and write α for $\alpha(X)$. The choice of β implies $\beta \neq \alpha$. It remains to show that $\beta \in EA(X)$.

Case 1: $\alpha \notin C(X_x, \beta)$. Since β is by definition in $EA(X_x)$, we have $\beta = \mu(C(X_x, \beta))$. The uniqueness of $C(X, \beta)$ implies $C(X_x, \beta) = C(X, \beta)$ and in particular $\beta = \mu(C(X, \beta))$. Hence, $\beta \in EA(X)$.

Case 2: $\alpha \in C(X_x, \beta)$. First note that, since $\alpha \neq \beta$, here $\beta < \alpha$. Then apply axiom C4 (elimination) to α, $C(X, \alpha)$ and $C(X_x, \beta)$ to get a signed circuit C' with $\beta \in C' \subseteq C(X_x, \beta) \cup C(X, \alpha)) \setminus \alpha$. This implies $C' = C(X, \beta)$ and $\beta = \mu(C')$. The claim follows. □

We introduce a directed graph associated to $NBC(\mathcal{M})$.

Let $\mathcal{M}$ be a matroid with an arbitrary ordering of its ground set. We define the directed graph $G = (V, E)$ as follows:

$$\begin{aligned} V &:= IND(\mathcal{M}) \\ E &:= \left\{ \overrightarrow{II'} \,\middle|\, I' = I \setminus \{x\} \cup \alpha, \ where\ \alpha = \alpha(I),\ x \in C(I, \alpha) \setminus \{\alpha\} \right\} \end{aligned}$$

Note that this graph is not connected, and each component consists of equicardinal indipendent sets. In particular, the connected component of G that contains the bases of $\mathcal{M}$ is a 'directed refinement' of the basis graph $G_{\mathcal{B}}$ of $\mathcal{M}$.

This graph has following interesting feature that will become essential in section 3.4:

Lemma 3.1.2 *Consider a matroid $\mathcal{M}$ with an arbitrary ordering of its ground set, and the graph $G = (V, E)$ constructed above. For every pair of vertices $I, I' \in V$ there is at most one directed path from I to I'.*

Proof: The proof of this lemma is quite technical. The interested reader will find it in [7].□

Final remark: In the case of a $\mathbb{C}$-matroid $\mathfrak{M}$ the considerations of this section hold for the underlying matroid $\underline{\mathfrak{M}}$. Thus we define:

$$IND(\mathfrak{M}) := IND(\underline{\mathfrak{M}}),$$
$$NBC(\mathfrak{M}) := NBC(\underline{\mathfrak{M}}).$$

3.2 The Orlik-Terao algebra $OT(\mathcal{A})$

Let V be a vector space of dimension d over some field $\mathbb{K}$ and consider an arrangement $\mathcal{A}$ of n hyperplanes $H_1, \ldots, H_n$ in V. Each $H \in \mathcal{A}$ can be represented as the kernel of a linear form, denoted by $\phi_H \in V^*$, which is defined up to a constant. Associated to $\mathcal{A}$ we have the *defining polynomial* of the arrangement, defined as $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \phi_H$.

Aomoto proposed the study of the graded vector space

$$AO(\mathcal{A}) := \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B}\ independent}} \mathbb{K} Q(\mathcal{B})^{-1},$$

where we call a subarrangement $\mathcal{B} \subseteq \mathcal{A}$ *independent* if the codimension of $\bigcap_{H \in \mathcal{B}} H$ equals $|\mathcal{B}|$. He shows that in the case $\mathbb{K} = \mathbb{R}$ the dimension of $AO(\mathcal{A})$ equals the number of chambers in the complement of the arrangement in V. In order to prove this claim for a general field, in [13] Orlik and Terao introduce a $\mathbb{K}$-algebra isomorphic to $AO(\mathcal{A})$ as a graded vector space, but as an algebra it has the advantage to be closed under mutiplication. Here we state the basic definition and two results about this algebra. For the proofs and further precisations see [13].

Definition 3.2.1 *Let $\mathbb{K}$ be a field and $\mathcal{A}_{\mathbb{K}}$ be an arrangement of hyperplanes in a $\mathbb{K}$-vector space V, with $\phi_H \in V^*$ associated to each H. Consider the set of generators $u_{\mathcal{A}} := \{u_H\}_{H\in\mathcal{A}_{\mathbb{K}}}$, and let $\mathscr{J}(\mathcal{A})$ denote the ideal in the commutative free algebra $\mathbb{K}[u_{\mathcal{A}}]$ generated by the following elements (where we write $u_{\mathscr{B}}$ instead of $\prod_{H\in\mathscr{B}} u_H$):*

1. *$u_{\mathcal{B}}$ for all dependent subarrangements $\mathcal{B} \subseteq \mathcal{A}$,*
2. *$\sum_{H\in\mathcal{H}} \xi_H u_{\mathcal{H}\setminus H}$, with $\xi_H \in \mathbb{K}$, if $\sum_{H\in\mathcal{H}} \xi_H \phi_H = 0$.*

The Orlik-Terao algebra of $\mathcal{A}_{\mathbb{K}}$ is defined as

$$OT(\mathcal{A}_{\mathbb{K}}) := \mathbb{K}[u_{\mathcal{A}}]/\mathscr{J}(\mathcal{A}).$$

A subset $C \subseteq \mathcal{A}$ is called a *circuit* if it is a minimal dependent set, i.e. C is dependent but $C \setminus H$ is independent for all $H \in C$. We set $\mathscr{C}(\mathcal{A}) := \{C \subseteq \mathcal{A} \,|\, C \text{ is a circuit}\}$.

Proposition 3.2.2 *The ideal $\mathscr{J}(\mathcal{A})$ of definition 3.2.1 is generated by the elements*

1. *u_H^2 for each $H \in \mathcal{A}$,*
2. *$\sum_{i=1}^k \xi_i u_{C\setminus H_i}$ if $\sum_{i=1}^k \xi_i \phi_{H_i} = 0$, for each circuit $C = \{H_1, \ldots, H_k\}$ and with $\xi_i \in \mathbb{K}$.*

An (arbitrary) indexing $\mathcal{A} = \{H_1, \ldots, H_n\}$ induces a total order on $\mathcal{A}$ and allows to define the concept of a broken circuit, i.e., of a $\mathcal{B} \subset \mathcal{A}$ such that there is an $\tilde{H} \in \mathcal{A}$ with $\tilde{H} < \min \mathcal{B}$ and $\mathcal{B} \cup \tilde{H} \in \mathscr{C}(\mathcal{A})$. A *no broken circuit* is a subarrangement that does not contain any broken circuit. We denote by $NBC(\mathcal{A})$ the set of no broken circuits of the arrangement $\mathcal{A}$ and let ***nbc***$(\mathcal{A}) := \{u_{\mathcal{B}} \in OT(\mathcal{A}) |\ \mathcal{B} \in NBC(\mathcal{A})\}$. Note that this definitions coincides with the one given in Section 3.1: indeed we have $NBC(\mathcal{A}) = NBC(\mathcal{M}_{\mathcal{A}})$ for the matroid $\mathcal{M}_{\mathcal{A}}$ with ground set $\{H_1, \ldots, H_n\}$, determined by the $\mathbb{K}$-linear dependences of the ϕ_{H_i}.

Proposition 3.2.3 *The set **nbc**$(\mathcal{A})$ is a linear basis for $OT(\mathcal{A})$.*

3.3 A commutative algebra for complex matroids

3.3.1 Definition of the algebra $\mathscr{Q}(\mathfrak{M})$

Recall the the set of signs $U = S^1 \cup \{\natural\}$. Let J be a (not necessarily ordered) finite index set, and $S := \{s_j\}_{j \in J}$ a family of generators, indexed by J. Consider $\mathbb{C}[S]$, the free (commutative) $\mathbb{C}$-algebra on the generators $S \cup \{1\}$. We write $s_{\{j_1,\ldots,j_m\}}$ for $s_{j_1} s_{j_2} \cdots s_{j_m}$. Define a map from $\mathbb{C}^J$ into $\mathbb{C}[S]$ by

$$\begin{array}{cccc} \partial : & U^J & \longrightarrow & \mathbb{C}[S] \\ & A & \longmapsto & \sum_{x \in \underline{A}} e^{iA(x)} s_{\underline{A} \setminus x} \end{array}$$

Given a $\mathbb{C}$-matroid $\mathfrak{M}$ let J be the ground set of $\mathfrak{M}$, and let $\mathscr{I}(\underline{\mathfrak{M}})$ be the ideal of $\mathbb{C}[S]$ generated by the elements:

$$\begin{array}{cc} \partial(C) & \text{for all } C \in \mathfrak{C}(\mathfrak{M}), \\ s^2 & \text{for all } s \in S. \end{array}$$

Definition 3.3.1.1 *Given a $\mathbb{C}$-matroid $\mathfrak{M}$ with ground set S, we define a commutative $\mathbb{C} - algebra$ $\mathscr{Q}(\mathfrak{M})$ as:*

$$\mathscr{Q}(\mathfrak{M}) := \mathbb{C}[S]/\mathscr{I}(\mathfrak{M}).$$

Remarks:

1. The natural map from $\mathscr{P}(J)$ into $\mathbb{C}[S]$ defined by $X \mapsto s_X$ induces a map $X \mapsto [X]_{\mathscr{Q}(\mathfrak{M})} = \{s_X + q \mid q \in \mathscr{I}(\mathfrak{M})\}$ into the equivalence classes of $\mathscr{Q}(\mathfrak{M})$. When no confusion arises, we sometimes indicate the equivalence class $[X]_{\mathscr{Q}(\mathfrak{M})} \in \mathscr{Q}(\mathfrak{M})$ by the element s_X.

2. Extending the set of generators of the algebra with $\{1\}$ is the algebraic corresponding of the extension of the ground set by an additional loop that allows to define the concept of direct sum in classical matroid theory (see [14]).

3. In the following we can assume $\mathfrak{M}$ to be a simple $\mathbb{C}$-matroid, because if x is a loop or $x \parallel s$ for some other $s \in S$, we have $\mathscr{Q}(\mathfrak{M}) \cong \mathscr{Q}(\mathfrak{M} \setminus x)$.

4. For all $C \in \mathfrak{C}(\mathfrak{M})$ we have $[C]_{\mathscr{Q}(\mathfrak{M})} = 0$.
   Proof: Choose an $x \in \underline{C}$. Distributivity yields

   $$s_x \partial(C) = s_x \left(e^{iC(x)} s_{C \setminus x} + s_x \sum_{\substack{y \in \underline{C} \\ y \neq x}} e^{iC(y)} s_{(C \setminus y) \setminus x} \right) = e^{iC(x)} s_C + s_x^2 \Big(\cdots \Big).$$

   Recalling the definition of $\mathscr{I}(\mathfrak{M})$, we see that $\partial(C) = 0$ (recall rem. 2: $\mathfrak{M}$ simple, so $|C| > 1$) and $s_x^2 = 0$ in $\mathscr{Q}(\mathfrak{M})$. So $0 = s_C \in [C]_{\mathscr{Q}(\mathfrak{M})}$, and the claim follows. $\square$

Define $\mathscr{Q}_\ell(\mathfrak{M})$ as the submodule of $\mathscr{Q}(\mathfrak{M})$ generated by the elements $[X]_{\mathscr{Q}(\mathfrak{M})}$, where $X \in IND_\ell(\mathfrak{M})$. $\mathscr{Q}_0(\mathfrak{M})$ means then the submodule generated by 1, i.e. $\mathbb{C}$. For the remainder of this section we will fix a $\mathbb{C}$-matroid $\mathfrak{M}$ and write $\mathscr{Q}$ for $\mathscr{Q}(\mathfrak{M})$, if no confusion can occur.

Proposition 3.3.1.1 *Let $\mathfrak{M}$ be $\mathbb{C}$-matroid of rank r. We define recursively:*

$A_\ell := 0$ *for all* $\ell > r$,
$A_r := \{x \in \mathscr{Q}(\mathfrak{M}) \mid xy \in \mathscr{Q}_0 \text{ for all } y \in \mathscr{Q} \setminus \mathbb{C}\}$,
and for $\ell > 0$:

$$A_{r-\ell} := \left\{ x \in \mathscr{Q}(\mathfrak{M}) \;\middle|\; \begin{array}{l} xy \in A_r \oplus \cdots \oplus A_{r-(\ell-1)} \\ \text{for all} y \in \mathscr{Q} \setminus \mathbb{C} \end{array} \right\} \Big/ A_r \oplus \cdots \oplus A_{r-(\ell-1)}$$

Then $\mathscr{Q} = \bigoplus_{i\in\mathbb{N}} A_i$ and for all $i \in \mathbb{N}$ $A_i \cong \mathscr{Q}_i$, hence in particular $\mathscr{Q} = \bigoplus_{\ell\in\mathbb{N}} \mathscr{Q}_\ell$.

Proof:

$\mathscr{Q}_\ell = A_\ell$ is trivial for $\ell > r$.

$\mathscr{Q}_r = A_r$ is clear, because r is the maximal size of an indipendent set.

Let now $\ell < r$, and suppose for 'backwards induction' that $\mathscr{Q}_i = A_i$ for all $i > \ell$. We consider now a general $x \in \mathscr{Q}$, written as $x = \sum \xi_j [X_j]$ for $X_j \in IND(\mathfrak{M})$. We have:

$\mathscr{Q}_\ell \subseteq A_\ell$ since $x \in \mathscr{Q}_\ell$ means $X_j \in IND_\ell(\mathfrak{M})$ for all j, and therefore implies $x[I] \in \mathscr{Q}_{\ell+1} \oplus \cdots \oplus \mathscr{Q}_r$ for each $I \in IND(\mathfrak{M})$. We can then conclude that $xy \in A_{\ell+1} \oplus \cdots \oplus A_r$ for all $y \in \mathscr{Q} \setminus \mathbb{C}$, but $x \notin A_{\ell+1} \oplus \cdots \oplus A_r \Rightarrow x = 0$, because by induction hypotesis then $x \in \mathscr{Q}_\ell \cap \mathscr{Q}_{\ell+1} \oplus \cdots \oplus \mathscr{Q}_r = \{0\}$.

$A_\ell \subseteq \mathscr{Q}_\ell$ because $x \in A_\ell \setminus \{0\}$ implies on the one side $X_j \in \bigcup_{k\geq\ell} IND_k$ since $x[I] \in A_r \oplus \cdots \oplus A_{r-(\ell-1)} = \mathscr{Q} \oplus \cdots \oplus \mathscr{Q}_{r-(\ell-1)}$ for all $I \in \bigcup_{k>0} IND_{\ell+k}$ and, in particular, given j we have $X_j \cup \{e\} \in \bigcup_{k>\ell} IND_k$ for all $e \in E$ with $[X_j][e] \neq 0$ (note that there is always such an e). On the other side, $x \in A_\ell \setminus \{0\}$ means as above $x \notin \mathscr{Q}_{\ell+1} \oplus \cdots \oplus \mathscr{Q}_r$, followly $X_j \in \bigcup_{k\leq\ell} IND_k$ for all j. Putting all together we have then $X_j \in IND_\ell$ for all j.

3.3.2 The morphisms π_x and ι_x

We show now how the matroid operation of contracting an element induces an epimorphism between the corresponding algebras:

Proposition 3.3.2.1 *For every $x \in S$ there is a $\mathbb{C}$-modules epimorphism*

$$\pi_x : \mathscr{Q}(\mathfrak{M}) \to \mathscr{Q}(\mathfrak{M}/x)$$

which is uniquely determined by the following conditions:

$$\pi_x(s_I) := \begin{cases} s_{I\setminus x} & \text{if} \quad x \in I \\ 0 & \text{if} \quad x \notin I \end{cases} \quad \textit{for all } I \in IND(\mathfrak{M}).$$

Proof: Since $[X] = 0$ for every dependent set (remark 3), defining the values of π on $IND(\mathfrak{M})$ uniquely determines the function. It remains to show that π_x is well-defined: this amounts to prove that $\pi_x(\mathscr{I}) \subseteq \mathscr{I}$. Clearly $\pi_x(s^2) = \pi_x(s)^2 = 0$, followly we must consider elements of the form:

$$\pi_x(s_X \partial(C)), \text{ where } X \subseteq J \text{ and } C \in \mathfrak{C}(\mathfrak{M}).$$

W.l.o.g let $X \cap \underline{C} = \varnothing$ and $x \in X \cup \underline{C}$. Keeping in mind that $C \setminus x$ is a vector $(= \sum_{i=1}^m C_i')$ in $\mathfrak{M} \setminus x$, we can compute:

$$\pi_x(s_X \partial(C)) = \begin{cases} s_X \partial(C \setminus x) & \text{if } x \in C, \quad = 0 \text{ since } \partial(C) = \sum_{i=1}^m \partial(C_i') \\ s_{X\setminus x} \partial(C) & \text{if } x \in X, \quad = 0 \text{ since } x \notin C \Rightarrow C \in \mathfrak{C}(\mathfrak{M}/x) \end{cases}$$

This completes the proof.□

Corollary 3.3.2.2 *Consider a $\mathbb{C}$-matroid $\mathfrak{M}$ on the ground set J, and let $X \subseteq J$. Then:*

$$X \in IND(\mathfrak{M}) \Longleftrightarrow [X]_{\mathscr{Q}(\mathfrak{M})} \neq 0$$

Proof: The implication "$\Rightarrow$" is given by remark 4.

For the implication "$\Leftarrow$" we proceed by induction in $n = |X|$. Since $[\emptyset]_{\mathscr{Q}} = 1$, we can assume $|X| > 0$. Suppose by induction that the claim holds for all sets $Y \in IND(\mathfrak{M})$, and let $X \in IND_n(\mathfrak{M})$. Pick an $x \in X$: clearly $X \setminus x \in IND_{n-1}(\mathfrak{M}/x)$. Now $[X]_{\mathscr{Q}(\mathfrak{M})} = 0$ would imply

$$0 = \pi_x([X]_{\mathscr{Q}(\mathfrak{M})}) = [X \setminus x]_{\mathscr{Q}(\mathfrak{M}/x)},$$

which is a contradiction with the induction hypotesis. □

The next step is the definition of a morphism associated with the operation of deleting an element from the matroid.

Proposition 3.3.2.3 *For every $x \in J$ there is a morphism of $\mathbb{C}$-modules*

$$\iota_x : \mathscr{Q}(\mathfrak{M}) \to \mathscr{Q}(\mathfrak{M} \setminus x)$$

which is uniquely determined by the following conditions:

$$\iota_x(s_I) := s_I \qquad \textit{for all } I \in IND(\mathfrak{M}).$$

Proof: As in Proposition 3.3.2.1, we only have to show that $\iota_x(\mathscr{I}(\mathfrak{M} \setminus x)) \subseteq (\mathscr{I}(\mathfrak{M}))$. After Corollary 3.3.2 it is enough to consider an $I \in IND(\mathfrak{M} \setminus x)$ and a $C \in \mathfrak{C}(\mathfrak{M} \setminus x)$ and compute:

$$\iota_x(s_I \partial(C)) = s_I \partial(C) = 0 \text{ in } \mathscr{Q}(\mathfrak{M}).\square$$

3.3.3 The short exact sequence

In this section we introduce an *arbitrary* total ordering on the ground set J of a $\mathbb{C}$-matroid $\mathfrak{M}$ (w.l.o.g. let us suppose $J = [n]$). Then we can distinguish the no broken circuits of $\underline{\mathfrak{M}}$ and define:

$$\boldsymbol{nbc}_\ell(\mathfrak{M}) := \{[I]_{\mathscr{Q}(\mathfrak{M})} \,|I \in NBC_\ell(\mathfrak{M})\},$$
$$\boldsymbol{nbc}(\mathfrak{M}) := \bigcup_{\ell \in \mathbb{N}} \boldsymbol{nbc}_\ell(\mathfrak{M}).$$

Proposition 3.3.3.1 *Given a $\mathbb{C}$-matroid $\mathfrak{M}$ on the ground set $[n]$, the set $\boldsymbol{nbc}_\ell(\mathfrak{M})$ generates $\mathscr{Q}_\ell(\mathfrak{M})$.*

Proof: Recall the definitions of section (3.1). Consider $X \in IND_\ell(\mathfrak{M}) \setminus NBC_\ell(\mathfrak{M})$ and abbreviate $\alpha := \alpha(X)$. Since $\partial(C(X, \alpha)) = 0$, we can express $[X]_{\mathscr{Q}}$ as a linear combinaton of the elements $[X_x]_{\mathscr{Q}}$ where $X_x := (X \setminus x) \cup \alpha$ and $X \in \underline{C}(X, \alpha) \setminus \alpha$. By remark (3.1.1) we know that for all $x \in X$, $EA(X_x) \subset EA(X) \setminus \alpha(X)$. Now we can iterate this process, ending up with an expression of $[X]_{\mathscr{Q}}$ as linear combinaton of terms $[Y]_{\mathscr{Q}}$ with $EA(Y) = \emptyset$. By remark 2 in section 3.1, these are precisely the elements of $\boldsymbol{nbc}(M)$. $\square$

In this section we take an arbitrary $x \in [n]$ and consider the following sequence of graded $\mathbb{C}$-modules:

$$0 \longrightarrow \mathscr{Q}(\mathfrak{M} \setminus x) \xrightarrow{\iota_x} \mathscr{Q}(\mathfrak{M}) \xrightarrow{\pi_x} \mathscr{Q}(\mathfrak{M}/x) \longrightarrow 0. \tag{3.2}$$

It is clear from the definitions that $\pi_x \circ \iota_x = 0$: followly $Im(\iota_x) \subseteq Ker(\pi_x)$.

The very natural question at this point is, wether (1) is exact or not. Before to show, by induction on n, that this is in fact the case, we need to prove a consequence of the claim which will be applied to the induction hypotesis:

Remark 3.3.3.1 *The exactness of (1) for all matroids on at most n elements and each $x \in [n]$ implies that $\boldsymbol{nbc}(\mathfrak{M})$ is a basis of the module $\mathscr{Q}(\mathfrak{M})$.*

Proof: For $n = 0$ we have $\mathscr{Q}(\mathfrak{M}(\emptyset)) = \mathbb{C}$ and $\boldsymbol{nbc}(\mathfrak{M}(\emptyset)) = 1$. We then proceed by induction on n: consider an $n > 0$ and suppose that the claim holds for all matroids on a most $n-1$ elements. By proposition (3.3.3.1) we know that $\boldsymbol{nbc}(\mathfrak{M})$ is a generating set for $\mathscr{Q}(\mathfrak{M})$: then we only have to show that

$$\sum_{X\in NBC(\mathfrak{M})} \omega_X [X]_{\mathscr{Q}(\mathfrak{M})} = 0 \;\Rightarrow\; \forall X:\; \omega_X = 0. \tag{3.3}$$

With (3.1) we can write the left term in (3.3) as:

$$\underbrace{\sum_{X'\in NBC(\mathfrak{M}\setminus n)} \omega_{X'}[X']_{\mathscr{Q}(\mathfrak{M})}}_{\in Ker(\pi_n)} + \sum_{\substack{X''\in NBC(\mathfrak{M})\\ X''\cup\{n\}\in NBC(\mathfrak{M})}} \omega_{X''}[X'']_{\mathscr{Q}(\mathfrak{M})} = 0 \tag{3.4}$$

applying π_x to (3.4) we get

$$\sum_{\widetilde{X}\in NBC(\mathfrak{M}/n)} \omega_{\widetilde{X}\cup\{n\}} [\widetilde{X}]_{\mathscr{Q}(\mathfrak{M}/n)} = 0 \text{ in } \mathscr{Q}(\mathfrak{M}/n), \tag{3.5}$$

and by induction hypotesis we conclude $\omega_{\widetilde{X}\cup\{n\}} = 0$ for all $\widetilde{X} \in NBC(\mathfrak{M}/n)$. This means that the second sum in (3.4) vanishes. Now we can write

$$0 = \sum_{X'\in NBC(\mathfrak{M}\setminus n)} \omega_{X'}[X']_{\mathscr{Q}(\mathfrak{M})} = \iota_x\left(\sum_{X'\in NBC(\mathfrak{M}\setminus n)} \omega_{X'}[X']_{\mathscr{Q}(\mathfrak{M}\setminus n)}\right). \tag{3.6}$$

Exactness of the sequence implies in particular that ι_x is mono, therefore we conclude:

$$\sum_{X'\in NBC(\mathfrak{M}\setminus n)} \omega_{X'}[X']_{\mathscr{Q}(\mathfrak{M}\setminus n)} = 0. \tag{3.7}$$

This means, by induction hypotesis, that $\omega_{X'} = 0$ for all $X' \in NBC(\mathfrak{M}\setminus n)$.

Knowing that $NBC(\mathfrak{M}) = NBC(\mathfrak{M}\setminus n) \cup NBC(\mathfrak{M}/n)$, the implication in (3.3) follows. $\square$

With the tool provided by remark 3.3.3.1 we can now prove the following three lemmata:

Lemma 3.3.3.2 *If (1) is exact for every $\mathbb{C}$-matroid on at most $n-1$ elements, then the sequence of $\mathbb{C}$-modules*

$$\mathscr{Q}(\mathfrak{M}\setminus x) \xrightarrow{\iota_x} \mathscr{Q}(\mathfrak{M}) \xrightarrow{\pi_x} \mathscr{Q}(\mathfrak{M}/x) \longrightarrow 0 \tag{3.8}$$

is exact for every $\mathbb{C}$-matroid on the ground set $[n]$ and every $x \in [n]$.

Proof: It remains to prove that $Ker(\pi_x) \subseteq Im(\iota_x)$, and it is enough to do this for $x = n$. Consider an element $q \in Ker(\pi_n)$: by Proposition 3.3.3.1 we can write q as $\sum_{i=1}^{m} \omega_i [I_i]_{\mathscr{Q}(\mathfrak{M})}$, where the ω_i are in $\mathbb{C}$ and the I_i are in

$NBC(\mathfrak{M})$. With the same technique employed in the previous proof, we compute

$$\pi_n\Big(\sum_{i=1}^{m}\omega_i[I_i]_{\mathscr{Q}(\mathfrak{M})}\Big) = \sum_{\substack{I_{\tilde{i}}\in NBC(\mathfrak{M})\\ I_{\tilde{i}}\not\ni n}} \omega_{I_{\tilde{i}}}[I_{\tilde{i}}\setminus n]_{\mathscr{Q}(\mathfrak{M}/n)} = 0. \tag{3.9}$$

The $I_{\tilde{i}}\setminus n$ in this expression are no broken circuits, because n was the biggest element in $\mathfrak{M}$. Now the ground set of $\mathfrak{M}/n$ is $[n-1]$ and by hypotesis we can apply remark 3.3.3.1, saying that $\boldsymbol{nbc}(\mathfrak{M}/n)$ is a basis of $\mathscr{Q}(\mathfrak{M}/n)$: from (3.9) we conclude $\omega_{I_{\tilde{i}}} = 0$ for all no broken circuits of $\mathfrak{M}$ wich occour in our expansion of q, and don't contain n. So we can finally write

$$q = \sum_{i=1}^{m}\omega_i[I_i]_{\mathscr{Q}(\mathfrak{M})} = \sum_{\substack{I_{i'}\in NBC(\mathfrak{M})\\ I_{i'}\ni n}} \omega_{I_{i'}} \underbrace{[I_{i'}\setminus n]_{\mathscr{Q}(\mathfrak{M})}}_{=\iota_x([I_{i'}\setminus n]_{\mathscr{Q}(\mathfrak{M}\setminus n)})} \in Im(\iota_n), \tag{3.10}$$

and the claim follows. □

Lemma 3.3.3.3 *Suppose (1) exact for all the* $\mathbb{C}$*-matroids with at most* $n-1$ *elements, and let* $q\in\mathscr{Q}=\mathbb{C}\oplus\mathscr{Q}_1(\mathfrak{M})\oplus\cdots\oplus\mathscr{Q}_r(\mathfrak{M})$ *for a* $\mathbb{C}$*-matroid* $\mathfrak{M}$ *on the ground set* $[n]$*. We have:*

$$q\in\mathbb{C}\Longleftrightarrow\forall x\in[n]:\pi_x=0.$$

Proof: First pick a $q\in\mathscr{Q}_0=\mathbb{C}$: such an element has the form $q=\omega[\emptyset]_{\mathscr{Q}}$ for an $\omega\in\mathbb{C}$. From the definition of π_x, one has then $\pi_x(q)=0$ for all $x\in[n]$. Consider now $q\in\mathscr{Q}$ such that $\pi_x(q)=0$ for all $x\in[n]$. Lemma 3.3.3.2 implies $Ker(\pi_x)=Im(\iota_x)$ for all $x\in[n]$, and therefore

$$q=\bigcap_{x\in[n]}Ker(\pi_x)=\bigcap_{x\in[n]}Im(\iota_x)=\mathscr{Q}_0,$$

where the last equality follows from the definition of ι_x. □

Lemma 3.3.3.4 *Suppose (1) exact for all the* $\mathbb{C}$*-matroids with at most* $n-1$ *elements. For a* $\mathbb{C}$*-matroid* $\mathfrak{M}$ *on the ground set* $[n]$ *we have then the exact sequence:*

$$0\longrightarrow\mathscr{Q}(\mathfrak{M}\setminus x)\xrightarrow{\iota_x}\mathscr{Q}(\mathfrak{M})$$

Proof: Consider a $\mathbb{C}$-matroid $\mathfrak{M}$ on the ground set $[n]$. The case $n=1$ being trivial, let $n>1$. Take a pair x,y of elements of [n], $x\neq y$: by proposition 2.1.4 we have $\mathfrak{C}(\mathfrak{M}\setminus x/y)=\mathfrak{C}(\mathfrak{M}/y\setminus x)$ and therefore $\mathscr{Q}(\mathfrak{M}/y\setminus x)\backsimeq\mathscr{Q}(\mathfrak{M}\setminus x/y)$. Proposition 3.3.2.1 applied to the $\mathbb{C}$-matroid $\mathfrak{M}\setminus x$ gives

$$\hat{\pi}_y:\quad\mathscr{Q}(\mathfrak{M}\setminus x)\quad\longrightarrow\quad\mathscr{Q}(\mathfrak{M}\setminus x/y),$$

and proposition 3.3.2.3 applied to $\mathscr{Q}(\mathfrak{M}/y \setminus x)$ gives

$$\hat{\iota_x}: \quad \mathscr{Q}(\mathfrak{M} \setminus x/y) \quad \longrightarrow \quad \mathscr{Q}(\mathfrak{M}/y),$$

which is by assumption mono because $\mathscr{Q}(\mathfrak{M}/y \setminus x)$ has only $n-1$ elements.

These morphisms are the 'natural' ones: we get a commutative diagram:

$$\begin{array}{ccc} \mathscr{Q}(\mathfrak{M} \setminus x) & \xrightarrow{\iota_x} & \mathscr{Q}(\mathfrak{M}) \\ {\scriptstyle \hat{\pi_y}}\downarrow & & \downarrow{\scriptstyle \pi_y} \\ \mathscr{Q}(\mathfrak{M} \setminus x/y) & \xrightarrow[\hat{\iota_x}]{} & \mathscr{Q}(\mathfrak{M}/y) \end{array}$$

Now we can turn to the proof that ι_x is injective: let $p, q \in \mathscr{Q}(\mathfrak{M} \setminus x)$ and suppose $\iota_x(p) = \iota_x(q)$. By commutativity of the above diagram we have for all $y \in [n]$:

$$\hat{\iota}_x \circ \hat{\pi}_y(p) = \pi_y \circ \iota_x(p) = \pi_y \circ \iota_x(q) = \hat{\iota}_x \circ \hat{\pi}_y(q)$$

and, since $\hat{\iota}_x$ is mono, this means $\hat{\pi}_y(p) = \hat{\pi}_y(q)$. We have shown that the element $p - q$ lies in the kernel of $\hat{\pi}_y$ for all $y \in [n]$: lemma 3.3.3.3 implies then that it is an element of $\mathscr{Q}_0 = \mathbb{C}$. We conclude

$$0 = \iota_x(p) - \iota_x(q) = \iota_x(p - q) = p - q,$$

because $\iota_{x \restriction \mathscr{Q}_0} = id_{\restriction \mathscr{Q}_0}$ $\Box$

As conclusion of the work done in this section, we state

Theorem 1 *Given a $\mathbb{C}$-matroid $\mathfrak{M}$ on the ground set J and an $x \in J$, there is a split short exact sequence*

$$0 \longrightarrow \mathscr{Q}(\mathfrak{M} \setminus x) \xrightarrow{\iota_x} \mathscr{Q}(\mathfrak{M}) \xrightarrow{\pi_x} \mathscr{Q}(\mathfrak{M}/x) \longrightarrow 0.$$

Proof: It remains to show that the sequence splits: we do this by specifying an explicit section. Let $|J| = n$ and consider the ordering induced on J by a map $\phi : J \to \mathbb{N}_n$ with $\phi(x) = n$. With respect to this order we know by remark 3.3.3.1 that $\boldsymbol{nbc}_(\mathfrak{M} \setminus x)$ and $\boldsymbol{nbc}(\mathfrak{M}/x)$ are respectively bases of $\mathscr{Q}(\mathfrak{M} \setminus x)$ and $\mathscr{Q}(\mathfrak{M}/x)$. So we define the section $\sigma_x : \mathscr{Q}(\mathfrak{M}/x) \to \mathscr{Q}(\mathfrak{M})$ by

$$\sigma_x([I]_{\mathscr{Q}(\mathfrak{M}/x)}) := [I \cup n]_{\mathscr{Q}(\mathfrak{M})} \text{ for all } I \in NBC(\mathfrak{M}/x).$$

The map is then well-defined, and clearly $\pi_x \circ \sigma_x = id$. By (3.1) it is clear that

$$\mathscr{Q}(\mathfrak{M}) \backsimeq \iota_x(\mathscr{Q}(\mathfrak{M} \setminus x)) \oplus \sigma_x(\mathscr{Q}(\mathfrak{M}/x)).\Box$$

3.4 Connection with the Orlik-Terao algebra

In this section we consider a $\mathbb{C}$-matroid arising from a complex arrangement $\mathcal{A}_{\mathbb{C}}$ and discuss an interesting feature of $\mathscr{Q}(\mathfrak{M})$ in rapport to $OT(\mathcal{A}_{\mathbb{C}})$. This was already proved by Cordovil in [6] for the case of a vetor space over an ordered field: the generalization to the complex case bases on the algorithmicallity and constructiveness of the proofs of proposition 3.3.3.1 and remark 3.3.3.1 and this is also why we carried out these proofs very explicitly.

Theorem 2 *Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of hyperplanes in a vector space of dimension d over $\mathbb{C}$. This (arbitrarily) indexing of the H_i induces an ordering an the set of generators of the algebras $\mathscr{U} := OT(\mathcal{A})$ and $\mathscr{Q} := \mathscr{Q}(\mathfrak{M}(\mathcal{A}))$, so that we can consider their no broken circuit bases* ***nbc***$(\mathfrak{M}(\mathcal{A}))$ *and* ***nbc***$(\mathcal{A})$. *Given an $X \subset [n]$, suppose*

$$\begin{array}{lll} (1) & [X]_{\mathscr{U}} = \sum_{j=1}^{m} \xi_j [I_j]_{\mathscr{U}} & \textit{with } [I_j]_{\mathscr{U}} \in \textbf{nbc}(\mathscr{U}),\ \xi_j \in \mathbb{C}, \\ (2) & [X]_{\mathscr{Q}} = \sum_{j=1}^{m} e^{i\alpha_j} [I_j]_{\mathscr{Q}} & \textit{with } [I_j]_{\mathscr{Q}} \in \textbf{nbc}(\mathfrak{M}),\ \alpha_j \in S^1. \end{array}$$

Then we have: $\forall j:\ \alpha_j = arg(\xi_j)$.

Proof: Consider the directed graph $G = (V, E)$ associated with the matroid $\mathcal{M}(\mathcal{A})$ as in proposition 3.1.2. For an $X \in [n]$ let $P_1, \ldots, P_m$ a list of all maximal lenght directed paths in G that start at X. By 3.1.2 we know that this is a tree: following the idea of the proof of 3.1.1 we interprete this as a 'parsing tree' for the expansion of $[X]_{\mathscr{U}}$ (resp.$[X]_{\mathscr{Q}}$) in the no broken circuit basis. Indeed, starting on the 'root' X and following the direction of the edges (sort of breadth-first-search), the childs I_i of a vertex I (with appropriate coefficients) give a legal decomposition of $[I]_{\mathscr{U}}$ (resp.$[I]_{\mathscr{Q}}$) in a linear combination of the $[I_i]_{\mathscr{U}}$ (resp.$[I_i]_{\mathscr{Q}}$). This way we conclude that the leaves of this tree give a legal decomposition of X in the above sense. By maximality, the endpoint N_i of the path P_i must be a sink, and the only sinks in G are the no broken circuits: so we get a legal decomposition of $[X]$ in a linear sum of the no broken circuits $[N_i]$, which is unique by maximality of the paths (in effect these considerations give an alternative proof of 3.3.3.1, the finiteness of G implying the existence of a decomposition). The problem is that we don't know yet the coefficients involved in this linear sum. To this end we define two edge-labelling functions

$$\begin{array}{llcll} (1) & \varphi_{\mathscr{U}}: & E[G] & \longrightarrow & \mathbb{C} \\ (2) & \varphi_{\mathscr{Q}}: & E[G] & \longrightarrow & S^1\ (= \mathbb{R}/2\pi\mathbb{Z}), \end{array}$$

with the property that

(1) the coefficient of $[I_i]_{\mathscr{U}}$ is $\xi_i = \prod_{\vec{e} \in P_i} \varphi(\vec{e})$ and
(2) the coefficient of $[I_i]_{\mathscr{Q}}$ has the argument $\alpha_i = \sum_{\vec{e} \in P_i} \varphi(\vec{e})$,

where $P_i \subset E[G]$ denotes the (uniquely determined) path from X to I_i.
Definition of $\varphi_{\mathscr{Q}}$ and $\varphi_{\mathscr{U}}$:

> Let $\vec{AB}$ be an edge of G, $\{\alpha\} = B\backslash A$, $\{\beta\} = B\backslash A$ and $\underline{C} := C(A,\alpha) = \{\alpha, \beta, x_1, x_2, \ldots, x_m\}$.
>
> (1) By proposition 3.2.3 (and scaling if needed) we have $\xi_i \in \mathbb{C}$ such that :
> $u_{\underline{C}\backslash\alpha} + \xi_\beta u_{\underline{C}\backslash\beta} + \sum_{i=1}^m \xi_i u_{\underline{C}\backslash x_i} = 0$, i.e.
> $[\underline{C} \setminus \alpha]_{\mathscr{U}} = -\xi_\beta[\underline{C} \setminus \beta]_{\mathscr{U}} - \sum_{i=1}^m \xi_i[\underline{C} \setminus x_i]_{\mathscr{U}}$.
> Define $\varphi_{\mathscr{U}}(\vec{AB}) := -\xi_\beta$.
>
> (2) By definition of $\mathscr{I}(\mathfrak{M})$ we have :
> $s_{\underline{C}\backslash\alpha} + e^{iC(\beta)} s_{\underline{C}\backslash\beta} + \sum_{i=1}^m e^{iC(x_i)} s_{\underline{C}\backslash x_i} = 0$, i.e.
> $[\underline{C} \setminus \alpha]_{\mathscr{Q}} = -e^{iC(\beta)}[\underline{C} \setminus \beta]_{\mathscr{Q}} - \sum_{i=1}^m e^{iC(x_i)}[\underline{C} \setminus x_i]_{\mathscr{Q}}$, C being the circuit of $\mathfrak{M}(\mathcal{A})$ with support $\underline{C}$ and $C(\alpha) = 0$.
> Define $\varphi_{\mathscr{Q}}(\vec{AB}) := C(\beta) + \pi$.
>
> These functions have the required properties.

Now it suffices to note that for all directed edges $\vec{e} \in E[G]$ we have $\varphi_{\mathscr{Q}}(\vec{e}) = arg(\varphi_{\mathscr{U}}(\vec{e}))$: the claim then follows immediately from the above expression of α_i and ξ_i. □

3.5 Dependence on the lattice of flats

In this section we will consider two $\mathbb{C}$-matroids $\mathfrak{M}$ and $\mathfrak{M}'$ on the same ground set E and investigate the connection between $\mathscr{Q}(\mathfrak{M})$ and $\mathscr{Q}(\mathfrak{M}')$ in the case where $\underline{\mathfrak{M}} = \underline{\mathfrak{M}'}$.

We will make use of the following lemma:

Lemma 3.5.1 *Let $\varphi : \mathscr{K}_1 \to \mathscr{K}_2$ be a bijective morphism of algebras, $\mathscr{J}_1$ an ideal of $\mathscr{K}_1$, $\mathscr{J}_2$ an ideal of $\mathscr{K}_2$. Then φ induces an isomorphism of algebras $\varphi_{\restriction \mathscr{K}_1/\mathscr{J}_1} : \mathscr{K}_1/\mathscr{J}_1 \to \mathscr{K}_2/\mathscr{J}_2$ if and only if $\varphi_{\restriction \mathscr{J}_1} : \mathscr{J}_1 \to \mathscr{J}_2$ is bijective.*

Proof: The 'only if' part is trivial. For the other direction note that φ induces $\tilde{\varphi} : \mathscr{K}_1 \twoheadrightarrow \mathscr{K}_2/\mathscr{J}_2$. Then we have $Ker(\tilde{\varphi}) = \mathscr{J}_2$, and by a theorem of basic algebra φ induces an isomorphism $\mathscr{K}_1/Ker(\varphi) \tilde{\to} Im(\varphi)$. □

3.5.1 Reorientation gives isomorphic algebras

Suppose that $\mathfrak{M}'$ is a reorientation of $\mathfrak{M}$: we want to show that then $\mathscr{Q}(\mathfrak{M})$ is isomorphic to $\mathscr{Q}(\mathfrak{M}')$. More precisely, we will show:

Proposition 3.5.2 *Consider a* $\mathbb{C}$*-matroid* $\mathfrak{M}$ *on the ground set* S*. For each signed set* $R \in (U \setminus \natural)^E$ *we have*

$$\mathscr{Q}(\mathfrak{M}) \cong \mathscr{Q}({}_R\mathfrak{M}).$$

Proof: In this proof we will write $\mathscr{Q}$ for $\mathscr{Q}(\mathfrak{M})$ and $\mathscr{Q}'$ for $\mathscr{Q}({}_R\mathfrak{M})$. We consider the map φ_R, defined on the generators $s \in S$ as follows:

$$\begin{array}{rccc} \varphi_R: & \mathbb{C}[S] & \longrightarrow & \mathbb{C}[S] \\ & s & \longmapsto & e^{-iR(s)}s. \end{array}$$

φ_R is clearly a bijective homomorphism. Now by lemma 3.5.1 it suffices to show that φ_R induces an isomorphism between the ideals $\mathscr{I} := \mathscr{I}(\mathfrak{M})$ and $\mathscr{I}' := \mathscr{I}({}_R\mathfrak{M})$. By the definition of reorientation of a $\mathbb{C}$-matroid we can write $\mathfrak{C}(\mathfrak{M}) := \{C_1, \ldots, C_w\}$, $\mathfrak{C}({}_R\mathfrak{M}) := \{{}_RC_1, \ldots, {}_RC_w\}$. Now recall the convention $e^{\natural} := 0$ and consider a circuit C_ℓ with the corresponding generator

$$\partial(C_\ell) = \sum_{j=1}^{m_\ell} C_\ell(x_j^\ell) \prod_{\substack{k=1..m_\ell \\ k \neq j}} x_k^\ell.$$

(where we let $\{x_1^\ell, \ldots, x_{m_\ell}^\ell\}$ denote the support of Let C_ℓ, for all ℓ).

φ_R maps this expression to

$$\varphi_R(\partial(C_\ell)) = \sum_{j=1}^{m_\ell} e^{iC_\ell(x_j^\ell)} \prod_{\substack{i=1..m_\ell \\ i \neq j}} e^{-iR(x_i^\ell)} x_i^\ell$$

Setting $\lambda := \prod_{k=1..m_\ell} e^{iR(x_k^\ell)}$, we can rewrite this expression as

$$\frac{1}{\lambda} \sum_{j=1}^{m_\ell} e^{i(C_\ell(x_j^\ell) \dot{+} R(x_j^\ell))} \prod_{\substack{k=1..m_\ell \\ k \neq j}} x_k^\ell = \frac{1}{\lambda} \sum_{j=1}^{m_\ell} e^{i\,{}_RC_\ell(x_j^\ell)} \prod_{\substack{k=1..m_\ell \\ k \neq j}} x_k^\ell = \frac{1}{\lambda}\partial(C'_\ell).$$

This way we conclude that $\varphi_R(\mathscr{I}) = \frac{1}{\lambda}\mathscr{I}' = \mathscr{I}'$. The homomorphism $\tilde{\varphi}_R$, defined on generators as $\tilde{\varphi}_R(s) := e^{iR(s)}s$, is clearly the inverse of φ_R and, by the same argument as above, satisfies $\tilde{\varphi}_R(\mathscr{I}') = \mathscr{I}$. Therefore $\varphi_{R\restriction\mathscr{I}} : \mathscr{I} \to \mathscr{I}'$ is an isomorphism. Application of lemma 3.5.1 concludes the proof. □

3.5.2 *Affaire à suivre...*

The question of isomorphism between two algebras corresponding to $\mathbb{C}$-matroids that have the same lattice of flats but differ by more than a reorientation is more difficult and still open, even for the Orik-Terao algebra [12]

The main problem seems to be the lack of general methods in deciding the isomorphism problem for algebras given by 'polynomial algebra modulo ideal'. One strategy for finding examples of $\mathbb{C}$-matroids with same lattice of flats but non-isomorphic algebras is to set up systems of equations solvable in one of the algebras, but not in the other. Another promising way seems to be the study as to what extent the lattice of ideals determines the structure of the algebras. In particular the followig two questions seem interesting:

1. Given a $\mathbb{C}$-matroid $\mathfrak{M}$, describe explicitely the structure of the lattice of ideals of $\mathscr{I}(\mathfrak{M})$ and $\mathscr{Q}(\mathfrak{M})$ in terms of $\underline{\mathfrak{M}}$.

2. Let a polynomial algebra A and two subalgebras F_1, F_2 be given. What can be said about the structure of $A/F_1, A/F_2$ if the subalgebras are subject on the only condition of having the same lattice of ideals? What if there is a weak equivalence between the two lattices?

Trying to answer these questions exceeded the time setting allowed for this diploma thesis, and therefore we only refer to [5] as a basic reference on universal algebra that might be useful on approaching such problems.

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