Combinatorics of covers of complexified hyperplane arrangements

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Abstract. This is a survey of combinatorial models for covering spaces of the complement of a complexified hyperplane arrangement. We obtain a unified picture of the subject, and a generalization of various known results, by exploiting the toolkit of homotopy colimits for combinatorial applications developed by Welker, Ziegler and Živaljević.

Introduction

A *cover of an arrangement* is a topological cover of the space obtained by removing a finite set of hyperplanes from a complex, finite-dimensional vector space.

The study of combinatorially defined complexes modeling covers of arrangements has a story that goes back to the beginnings of the topological theory of hyperplane arrangements, and arises in the context of finite real reflection groups, where one can consider the set of hyperplanes ('mirrors') fixed by the reflections in the group. In 1971 Brieskorn [16] conjectured the complement of the complexification of this set of hyperplanes to be an aspherical space (we then say that this is a $K(\pi, 1)$-*arrangement*).

Brieskorn's conjecture was settled by a general theorem of Deligne [29], who proved that the complexification of any real arrangement of linear hyperplanes whose chambers are simplicial cones is $K(\pi, 1)$. The idea was to prove contractibility of the universal covering space of the arrangement's complement, and the method involved designing a cell complex that, under certain conditions, models the universal cover of the arrangement's complement.

The $K(\pi, 1)$ problem for hyperplane arrangements, i.e., the problem of deciding whether the property of being $K(\pi, 1)$ is determined by the combinatorics of the lattice of intersections of the hyperplanes, is still open and in the focus of active research. The construction and the study of different models for the universal covering space of arrangement complements has been one of the main strategies used to attack this problem. Alternative approaches have been successfully exploited - most notably the idea to reduce the problem to a lower dimensional situation by linear fibrations (that led to the concept of supersolvable arrangements [76, 37]), the use of fibrations onto the complex torus [54], or a mix of the different techniques [21]. For a general reading on the $K(\pi, 1)$ problem for arrangements we point to the survey of Falk and Randell [38, 39].

Among more recent topics in arrangement theory are the study of local system cohomology on arrangement complements and of the topology of the Milnor fiber. In both these subjects, the homology of certain covering spaces plays an important role (see e.g. [27] and [30]).

For general complex arrangements not much is known. Björner and Ziegler [11] described a simplicial model for the complement of a complex arrangement, but no description of the covering space is at hand. After previous partial results of Nakamura [53], the case of finite complex reflection arrangements was recently settled by Bessis [3], who described a model for the universal cover of the orbit space and showed its contractibility using the theory of Garside groups and Garside categories, thus proving the $K(\pi, 1)$ conjecture for this class of arrangements (for more details see Remark 5.15 and Section 6).

In this survey we present a unified view on the different combinatorial models for covers of complexified real arrangements.

We put the subject into the framework of the theory of diagrams of spaces and homotopy colimits for combinatorial applications, as developed by Welker, Ziegler and Živaljević in [81]. Diagrams of spaces have already been fruitfully exploited to study the *link* of hyperplane arrangements (i.e., the space defined by the union of the hyperplanes) [82, 77, 46]. In our context, these techniques allow for instance to link the two main classes of complexes we will be dealing with, namely the Salvetti-type models W_ρ (Definition 3.1) and the Garside-type models U_ρ (Definition 5.1). Each of these types of models generalizes some known constructions, that we will explain. We thus obtain a unified picture of the subject. Moreover, this language allows us to apply the known techniques for the study of the homotopy type of diagram of spaces.

We will use some facts from the covering theory of groupoids. Also, we will meet along our way the notion of oriented systems (with a corresponding covering theory) as introduced by Paris [59]. We hope that the chosen notation and the explanations will succeed in clarifying the interplay among the different notions of "cover", nevertheless avoiding confusion.

We will begin our exposition by recalling some definitions and facts that are nowadays standard in arrangement theory. In Section 2 we introduce diagrams of spaces and their homotopy colimits and state some basic facts about them.

Then, in Section 3 we will present a first type of diagram models and study their homotopy colimits. For every topological cover of the given arrangement we construct a diagram which homotopy colimit is isomorphic to the given covering space, and can be written as the order complex of an explicitly described poset. These models are called of *Salvetti type* because the model of the identical cover is actually isomorphic to the complex introduced by Salvetti in [67]. Specializing to the universal covering of arrangements of linear hyperplanes we recover naturally the simplicial complex obtained by Luis Paris in [60]. Moreover, we will mention here the work of Charney and Davis on Artin groups [22, 23], also pointing to an application of it given by Charney and Peifer [24] in the context of affine reflection arrangements.

In fact, Paris constructed topological models for arbitrary covers of linear arrangements [59]. In Section 4 we first explain this construction. Then, we describe a stratification of it which nerve is isomorphic to the poset obtained from the diagram model of the corresponding cover, thereby showing that the diagram models offer a compact and handy description (in fact, as order complexes of posets) of Paris' models.

Restricting our attention from affine to linear real arrangements, Section 5 introduces another type of diagram models generalizing a construction that arose in the context of Garside groups [14, 5, 25]. We call them therefore of *Garside type*. As an application, we then explain how Deligne's argument can be reformulated in view of this type of models. The closing section is about possible further applications and directions of work.

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1. Notations and recalls

1.1. Arrangements

We will denote by $\mathcal{A}$ a collection of affine hyperplanes in $\mathbb{R}^d$, also called a *real arrangement*. Our considerations will restrict to the case where the arrangement is locally finite (i.e., every point of $\mathbb{R}^d$ is contained in at most finitely many hyperplanes) and essential (i.e., the minimal intersections of hyperplanes are points).

The classical reference on arrangements of hyperplanes is the textbook of Orlik and Terao [56], and for the combinatorics of real arrangements in terms of oriented matroids we point to [10]. Let us here only recall the facts that we will need.

The closed strata that are determined by $\mathcal{A}$ in $\mathbb{R}^d$ are the *faces* of $\mathcal{A}$. The *support* of a face F is the set $\mathrm{supp}(F)$ of all hyperplanes containing F. The set of faces of $\mathcal{A}$ is partially ordered by reverse inclusion, so that for any two faces

F_1, F_2 we have $F_1 \geq F_2$ if and only if $F_1 \subseteq F_2$: this defines the poset of faces of the arrangement, denoted by $\mathcal{F}(\mathcal{A})$.

The minimal elements of $\mathcal{F}(\mathcal{A})$ are the connected components that are cut out in $\mathbb{R}^d$ by $\mathcal{A}$ and are usually called *chambers*, or *regions*. Given two chambers C_1, C_2 of $\mathcal{A}$, one may choose a point in the interior of each chamber and consider the line segment spanned by these points. The hyperplanes that are met by this segment *separate* C_1 from C_2; the set of all hyperplanes separating C_1 from C_2 is denoted by $S(C_1, C_2)$. Two chambers are said to be *adjacent* if they are separated by only one hyperplane. If the arrangement is linear, we define the *opposite* of a chamber C to be the unique chamber $-C$ so that $S(C, -C) = \mathcal{A}$. If C is any chamber and F any face of $\mathcal{A}$, we will denote by C_F the unique chamber that contains F in its closure, and that is not separated from C by any of the hyperplanes that contain F - i.e, such that $C_F < F$ and $S(C_F, C) \cap \operatorname{supp}(F) = \emptyset$. The set of all regions of $\mathcal{A}$ will be written $\mathcal{T}(\mathcal{A})$ and can be given different partial orderings, depending on the choice of a base element. Once a *base chamber* $B \in \mathcal{T}$ is fixed, an associated partial order $\prec_B$ can defined by setting $C_1 \preccurlyeq_B C_2$ if and only if $S(B, C_1) \subseteq S(B, C_2)$. This gives rise to the *poset of regions of* $\mathcal{A}$ *with base* B (introduced in [35]), that we will denote by $\mathcal{T}_B(\mathcal{A})$.

The *arrangement graph* $G(\mathcal{A})$ has $\mathcal{T}(\mathcal{A})$ as set of vertices, and it is constructed by putting two opposite oriented edges between each pair of vertices that represent adjacent chambers. As an example, see the left side of Figure 1 for a picture of $G(\mathcal{A})$ when $\mathcal{A}$ consists of two lines in the plane. A directed path in the arrangement graph is called *positive*; it is called also *minimal* if it does not "cross" twice any hyperplane.

The *complexification* of the arrangement $\mathcal{A}$ is the set $\mathcal{A}_{\mathbb{C}}$ of the complex hyperplanes obtained by considering the same (real) defining equations as for the elements of $\mathcal{A}$. We will be interested in the topology of the complement of the complexification (sometimes just called *the arrangement's complement*)

$$\mathcal{M}(\mathcal{A}) := \mathbb{C}^d \setminus \bigcup \mathcal{A}_{\mathbb{C}}.$$

1.2. Posets

We give a short review of some basic facts and notations about partially ordered sets (or, for short, *posets*). For a careful exposition of the subject see [75]. Given two elements x, y in a poset $\mathcal{P}$ we denote by $x \vee y$ their unique least upper bound (or *join*) and by $x \wedge y$ their unique maximal lower bound (or *meet*), if these exist. A poset where the meet and the join exist for every pair of elements is called a *lattice*. Given two posets $\mathcal{P}$ and $\mathcal{Q}$, we will partially order their disjoint union $\mathcal{P} \coprod \mathcal{Q}$ by letting $x \geq y$ if and only if either both $x, y \in \mathcal{P}$ and $x \leq y$ in $\mathcal{P}$, or $x, y \in \mathcal{Q}$ and $x \leq y$ in $\mathcal{Q}$. The main topological object associated to a poset $\mathcal{P}$ is its *order complex* $\Delta(\mathcal{P})$, that is the simplicial complex of the totally ordered subsets of $\mathcal{P}$. It is clear that if $\mathcal{P}$ has a unique minimal element $\hat{0}$, then the order complex $\Delta(\mathcal{P})$ will be a cone with apex $\hat{0}$, and thus in particular contractible. The analogous statement holds of course when $\mathcal{P}$ has a unique maximal element.

1.3. The Salvetti complex

We introduce the tool that will allow us to link the topology of $\mathcal{M}(\mathcal{A})$ to the combinatorics of the real arrangement. Let us begin with the abstract definition.

Definition 1.1. Let $\mathcal{S}(\mathcal{A})$ be the set of all pairs (F, C) with $F \in \mathcal{F}$, $C \in \mathcal{T}$ and $C < F$. We give this set a partial order by setting $(F_1, C_1) > (F_2, C_2)$ if and only if $F_1 > F_2$ in $\mathcal{F}$ and $C_2 = (C_1)_F$. The (simplicial version of the) *Salvetti complex* is

$$\mathrm{Sal}(\mathcal{A}) := \Delta(\mathcal{S}(\mathcal{A})).$$

The importance of this object lies in the following fundamental theorem, that was proved by Mario Salvetti by constructing an explicit homotopy equivalence.

Theorem 1.2 (Theorem 1 of [67]**).** *For every real arrangement $\mathcal{A}$, the geometrical realization of* $\mathrm{Sal}(\mathcal{A})$ *can be embedded into the arrangement's complement, and is a strong deformation retract of $\mathcal{M}(\mathcal{A})$.*

There is another way to look at this complex. Indeed, the poset $\mathcal{S}(\mathcal{A})$ satisfies the conditions given in [6] for a general poset to be actually the poset of cells of a regular CW-complex. Thus, $\Delta(\mathcal{S}(\mathcal{A}))$ is the barycentric subdivision of a regular CW-complex that we will call $\mathrm{Sal}(\mathcal{A})$ as well.

Remark 1.3. An explicit construction of the CW-version of the Salvetti complex is the following. Start with a geometric realization of the arrangement graph, and take it as the 1-skeleton of the CW-complex. The attaching of the higher dimensional cells $[F, C]$ is defined recursively by saying that the 1-skeleton of $[F, C]$ consists of the positive minimal paths that start at C and end at the chamber opposite to C with respect to $\mathrm{supp}(F)$; a cell $[G, K]$ is then contained in the boundary of $[F, C]$ if and only if the 1-skeleton of $[G, K]$ is a directed subgraph of the 1-skeleton of $[F, C]$ (see [67]).

Example 1.4. As an example we consider the arrangement of two lines passing through the origin of $\mathbb{R}^2$. The picture illustrates the arrangement graph and two 2-cells with their boundary.

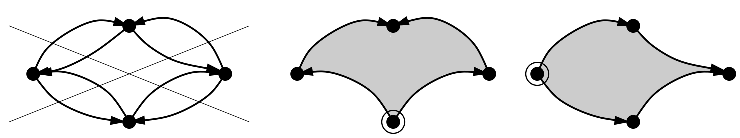


FIGURE 1. The arrangement of two lines in the plane with its arrangement graph, and two 2-cells of the cellular version of the associated Salvetti complex.

Those are the cells $[F, C]$ where F is the only codimension-2 face, namely the origin, and C is the chamber associated to the marked vertex. Note the boundary consisting of the positive minimal paths. The full complex has one such 2-cell for every chamber. $\triangle$

1.4. The arrangement groupoid

A *groupoid* is a category where every arrow is invertible. This notion was first introduced by Brandt [15] as a generalization of the concept of group that he developed in his study of quadratic forms. According to [19], the use of groupoids in topology goes back to Reidemeister [65]. Let us mention also the work of Gabriel and Zisman [40], who explain and exploit the functorial relations between topological spaces, simplicial sets, and the associated groupoids. More recent textbooks exploiting groupoids in topology were written by Higgins [44] and Brown [17].

One of the classical features of groupoids is their nice covering theory, that parallels the theory of topological spaces. As this is a very classical topic, we will sketch the definitions and state the results we need; proofs and complements can be found in the elementary approach to the topic by Brown [18], while the book by Gabriel and Zisman [40] provides a more advanced treatment of the subject, together with its homological implications. For connections with homotopy of diagrams of spaces, see [19, 20]. For the basics about categories we refer to [49].

Let $\mathcal{Q}$ be a groupoid and consider an $x \in \mathrm{Ob}(\mathcal{Q})$ (we will use latin lowercase letter for objects, and Greek lowercase letters for morphisms). The set of endomorphisms $\mathrm{End}(x)$ has a natural group structure that does not depend on the choice of x; this group is called the *object group* of $\mathcal{Q}$ and will be denoted by $\pi\mathcal{Q}$ for reasons that will become clear later. The source and target object of a morphism ω will be indicated by $\mathrm{start}(\omega)$ and $\mathrm{end}(\omega)$, respectively. The *star* of the object x is the set

$$\mathrm{St}(x) := \{\omega \in \mathrm{Mor}(\mathcal{Q}) \mid \mathrm{start}(\omega) = x\}$$

of all morphisms of $\mathcal{Q}$ that start in x. The groupoid $\mathcal{Q}$ is called *connected* if for every $x, y \in \mathrm{Ob}(\mathcal{Q})$ there is a morphism ω with $x = \mathrm{start}(\omega)$ and $y = \mathrm{end}(\omega)$.

Definition 1.5. A *morphism of groupoids* is a functor

$$\rho : \mathcal{Q}' \to \mathcal{Q}$$

between groupoids. If $\mathcal{Q}$ is connected, then ρ is called a *covering* if, for every $z \in \mathrm{Ob}(\mathcal{Q}')$, the induced map

$$\rho_z : \mathrm{St}(z) \to \mathrm{St}(\rho(z))$$

is bijective. Given a $\alpha \in \mathrm{Mor}(\mathcal{Q})$ and any $z \in \rho^{-1}(\mathrm{start}(\alpha))$, the *lift of α at z* is the morphism $\rho_z^{-1}(\alpha)$, and will be written $\alpha^{\langle z \rangle}$ when the covering ρ is understood.

Example 1.6. The groupoid described in Example 1.14 is a cover of the groupoid of Example 1.11: the bijection can be checked directly. At the end of Example 1.15 we sketch the proof that the groupoid of Example 1.15 is a cover of the one defined in Example 1.12. $\triangle$

If ρ is a covering of groupoids as above, the object group $\mathrm{End}(z) = \pi\mathcal{Q}'$ is mapped isomorphically by ρ to a subgroup of $\mathrm{End}(\rho(z)) = \pi\mathcal{Q}$ that is called the *characteristic group* of the covering.

The following result is classical.

Theorem 1.7 (see 9.4.3 of [18]**).** *Let $\mathcal{Q}$ be a connected groupoid, H a subgroup of $\pi\mathcal{Q}$, and choose a base object $x \in \mathrm{Ob}(\mathcal{Q})$. Consider the groupoid $\mathcal{Q}'$ defined by setting $\mathrm{Ob}(\mathcal{Q}') := \{H\omega \mid \omega \in \mathrm{St}(x)\}$ and where the morphisms between $H\omega_1$ and $H\omega_2$ correspond to morphisms α from $\mathrm{end}(\omega_1)$ to $\mathrm{end}(\omega_2)$ in $\mathcal{Q}$ such that $H\omega_1\alpha = H\omega_2$.*

The functor $\rho : \mathcal{Q}' \to \mathcal{Q}$ mapping $H\omega$ to $\mathrm{end}(\omega)$ is a covering of groupoids with characteristic group H.

Definition of the arrangement groupoid. Consider the free category on the arrangement graph (see [49, section II.4] for the definition), whose morphisms correspond to directed paths in $G(\mathcal{A})$.

Example 1.8. Take as an example the 1-dimensional arrangement given by the zero point inside the real line, that we will call $\mathcal{A}_1$. This arrangement has clearly two chambers A, B, and its arrangement graph consists of two vertices joined by two directed edges: the edge a directed from A to B, and the edge b directed from B to A (see Figure 5). The free category on it has two objects A, B, and the sets of morphisms are

$$\mathrm{Mor}(A,A) = \{(ab)^n \mid n \in \mathbb{N}_{\geq 0}\}$$
$$\mathrm{Mor}(A,B) = \{(ab)^n a(ba)^m \mid m,n \in \mathbb{N}_{\geq 0}\} = \{(ab)^n a \mid n \in \mathbb{N}_{\geq 0}\}.$$

and analogously for $\mathrm{Mor}(B,B)$ and $\mathrm{Mor}(B,A)$. △

Returning to the general situation, let R denote the smallest equivalence relation compatible with morphism composition and that identifies every two morphisms that come from positive minimal paths with same beginning and target. We might then build the quotient category $\mathcal{G}^+ := \mathtt{Free}(G(\mathcal{A}))/R$, called the category of positive paths.

It is clear that $\mathrm{Ob}(\mathcal{G}^+(\mathcal{A})) = \mathcal{T}(\mathcal{A})$. In general, the equivalence relation is such that any two chambers C_1, C_2 determine an equivalence class of positive minimal paths starting at C_1 and ending at C_2; we will write $(C_1 \to C_2)$ for any morphism representing this class.

Example 1.9. In the previous example, the relation is empty. To see a case where it actually plays a role, let $\mathcal{A}_2$ be the arrangement of two lines considered in Example 1.4 and depicted in Figure 1 together with its arrangement graph. The vertex set of $G(\mathcal{A}_2)$ is $\{C_0, C_1, C_2, C_3\}$ (say, in counterclockwise order in Figure 1) and we may label the edges $e_{i,i\pm 1}$, where the edge $e_{i,j}$ is directed from the vertex C_i to the vertex C_j (the indexing is taken modulo 4). The set of morphisms from C_i to C_j in the free category $\mathtt{Free}(G)$ is the set of directed paths in G starting at C_i and ending in C_j. The positive minimal paths in G are either single-edge paths or paths of length two of the form

$$e_{i,i+1}e_{i+1,i+2} \quad \text{or} \quad e_{i,i-1}e_{i-1,i-2}.$$

For any fixed i, the two above paths share the same start and the same target. Let us represent a directed path in G with the corresponding word in the alphabet $\{e_{i,i\pm 1}\}_{i=0,\dots,3}$. The set of morphisms from C_i to C_j in $\mathcal{G}^+$ is obtained from the set of directed paths from C_i to C_j after identification of any two paths represented by words that can be transformed into one another by a sequence of substitutions of the form $e_{i,i+1}e_{i+1,i+2} \leftrightarrow e_{i,i-1}e_{i-1,i-2}$.

To describe the set of morphisms of $\mathcal{G}^+$ in this case, consider any directed path in the arrangement graph starting, say, at C_0, and let us parse it following the orientation of the edges. The first two letters of ω either describe a loop (in which case the third letter represents an edge starting at C_0) or a positive minimal path. In the latter case, two situations may occur. If the second and third letter define a directed loop in the graph, then we can apply two "substitutions" as in Figure 2 to see that this three-edges path is equivalent to one which makes a loop based at C_0, and thus ω is equivalent to a path which third vertex is again C_0.

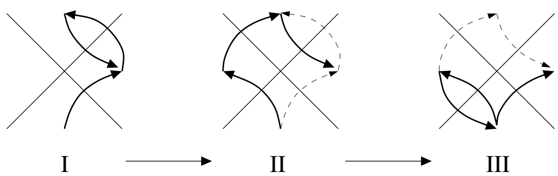


FIGURE 2

If the second and third letter define a positive minimal path, then we are already in the situation of Figure 2.II and one substitution suffices to show that ω is equivalent to a path which third vertex is C_0. So in any case we know that ω is equivalent to a loop followed by a directed path ω' still starting at C_0 but two edges shorter than ω. By induction we see that $\mathrm{Mor}_{\mathcal{G}^+}(C_0, C_0)$ is the free commutative monoid with generators $e_{0,1}e_{1,0}$ and $e_{0,3}e_{3,0}$, and

$$\mathrm{Mor}_{\mathcal{G}^+}(C_0, C_j) = \{\alpha(C_0 \to C_j) \mid \alpha \in \mathrm{Mor}_{\mathcal{G}^+}(C_0, C_0)\}.$$

$\triangle$

Let us again return to the general construction. We can now state the definition of the arrangement groupoid.

Definition 1.10. The *arrangement groupoid* $\mathcal{G}(\mathcal{A})$ is obtained from the category of positive paths $\mathcal{G}^+(\mathcal{A})$ by groupoid completion, i.e., adding formal inverses to every morphism.

The arrangement $\mathcal{A}$ being often understood, we will sometimes just write $\mathcal{G}$ for $\mathcal{G}(\mathcal{A})$.

Example 1.11 (Example 1.8 continued). We already described the objects and morphisms of $\mathcal{G}_1^+ := \mathcal{G}^+(\mathcal{A}_1)$ for the arrangement of one point in the real line. The associated arrangement groupoid is obtained by formally adding an element $a^{-1} \in \mathrm{Mor}(B, A)$ such that $a^{-1}a = aa^{-1} = id$ in $\mathrm{Mor}(A, A)$, and an analogous element $b^{-1} \in \mathrm{Mor}(A, B)$. Thus we have $abb^{-1}a^{-1} = id$ in $\mathrm{Mor}(A, A)$, which justifies the notation $(ab)^{-1} := b^{-1}a^{-1}$. Then, in the arrangement groupoid $\mathcal{G}_1 := \mathcal{G}(\mathcal{A}_1)$, we have

$$\mathrm{Mor}_{\mathcal{G}_1}(A,A) = \{(ab)^n \mid n \in \mathbb{Z}\} \text{ (i.e., the group } \mathbb{Z})$$
$$\mathrm{Mor}_{\mathcal{G}_1}(A,B) = \{(ab)^n x \mid n \in \mathbb{Z},\, x = a \text{ or } b^{-1}\}$$

△

Example 1.12 (Example 1.9 continued). Let us now consider again the arrangemet $\mathcal{A}_2$. As in the previous example, adding formal inverses to every $e_{i,\pm 1}$ in the positive category $\mathcal{G}^+(\mathcal{A}_2)$ that we described in Example 1.9, we can for instance see that, in the arrangement groupoid $\mathcal{G}_2 := \mathcal{G}(\mathcal{A}_2)$, $\mathrm{Mor}_{\mathcal{G}_2}(C_0, C_0)$ is the free abelian group on two generators $e_{o,1}e_{1,0}$ and $e_{o,3}e_{3,0}$. △

Remark 1.13. The arrangement groupoid was first defined by Deligne [29, (1.25)]. See also the work of Paris [61] for more on the construction. As a word of caution it has to be pointed out that in [29] this object is defined under the assumption that the arrangement is simplicial, thereby obtaining 'by default' some properties that are not granted in the general case, such as the faithfulness of the natural functor $\mathcal{G}^+ \to \mathcal{G}$ that turns out to be a crucial property in view of asphericity of the complement (see [29, 69, 60] and our Section 5.1). Note that our two examples indeed enjoy this property.

Coverings of the arrangement groupoid. From the definition of $\mathcal{G}(\mathcal{A})$ and from Remark 1.3 we see that indeed $\pi\mathcal{G}(\mathcal{A}) = \pi_1(\mathcal{M}(\mathcal{A}))$. So the same subgroups characterize the coverings of $\mathcal{M}(\mathcal{A})$ and the coverings of $\mathcal{G}(\mathcal{A})$.

If we apply Theorem 1.7 to the arrangement groupoid, we obtain coverings $\rho : \mathcal{G}_\rho \to \mathcal{G}$. The objects of $\mathcal{G}_\rho$ represent (right cosets of) paths on the arrangement graph. Therefore we will freely switch between the interpretation of them as objects (written with latin letters) or as morphisms in $\mathcal{G}$ (written with Greek letters). Moreover, universal covering will be denoted by a hat on the corresponding object. So $\hat{\rho}$ for the universal covering morphism and $\widehat{\mathcal{G}}$ for $\mathcal{G}_{\hat{\rho}}$.

Example 1.14. Consider the arrangement groupoid $\mathcal{G}_1$ of examples 1.8 and 1.11. Choose A as the base point and let $\widehat{\mathcal{G}}_1$ be the groupoid given by

$$\mathrm{Ob}(\widehat{\mathcal{G}}_1) := \{v_k\}_{k\in\mathbb{Z}}, \quad \mathrm{Mor}_{\widehat{\mathcal{G}}_1}(v_i, v_j) := \{\mu_{i,j}\},$$

$$\hat{\rho}: \quad v_i \mapsto \begin{cases} A & i \text{ even} \\ B & i \text{ odd} \end{cases}$$

$$\mu_{i,j} \mapsto \begin{cases} (ab)^{q(i,j)} a^{p(i,j)} & i \text{ even, } i<j \\ (ab)^{q(i,j)+1} b^{-p(i,j)} & i \text{ even, } i>j \\ (ba)^{q(i,j)} b^{p(i,j)} & i \text{ odd, } i<j \\ (ba)^{q(i,j)+1} a^{-p(i,j)} & i \text{ odd, } i>j \end{cases}$$

Where we used the following notation:

$$p(i,j) = \begin{cases} 0 & i-j \text{ even} \\ 1 & i-j \text{ odd} \end{cases} \quad q(i,j) := \frac{j-i-p(i,j)}{2}.$$

This groupoid can also be obtained from obtained from the free category over the directed infinite path by identifying every two morphisms with same beginning and endpoint.

It is easy now to see that this is a covering of the groupoid $\mathcal{G}_1$. Indeed, $\mu_{i,j}$ ranges clearly over the morphisms exiting from v_i, and conversely every such morphism can be written as $\mu_{i,j}$ for an adequate choice of i and j. △

Example 1.15. The universal covering groupoid $\widehat{\mathcal{G}}_2 := \widehat{\mathcal{G}}(\mathcal{A}_2)$ of the arrangement groupoid of Example 1.9 and 1.12 is the following

$$\mathrm{Ob}(\widehat{\mathcal{G}}_2) = \{v_{i,j} \mid i,j \in \mathbb{Z}\}, \quad \mathrm{Mor}_{\widehat{\mathcal{G}}_2}(v_{i,j}, v_{k,l}) = \{\mu_{i,j,k,l}\} \text{ (a singleton).}$$

The covering map is defined on objects as

$$\hat{\rho}(v_{i,j}) := C_{r(i,j)}$$

where $0 \leq r(i,j) \leq 3$ and $r(i,j) \equiv p(i) - p(j) + 2p(i)p(j) \pmod 4$ with $p(i) := 0$ if i is even, $p(i) := 1$ if i is odd, for every $i,j \in \mathbb{Z}$. To define $\hat{\rho}$ on morphisms it is useful to think of every $v_{i,j}$ as corresponding to the integer point (i,j) in the real plane. Let us also think of every edge of the integer grid in $\mathbb{R}^2$ as being directed following the increase of the coordinates value. So every node (i,j) is the source of two edges: one oriented in x-direction, that we will label by $e_{r(i,j),r(i+1,j)}$, and one in y-direction, labeled $e_{r(i,j),r(i,j+1)}$ (remember from Example 1.9 that also the labeling $e_{i,j}$ is taken modulo 4).

Given any morphism $\mu_{i,j,k,l}$, consider any path of minimal length in the integer grid joining (i,j) to (k,l). Then $\hat{\rho}(\mu_{i,j,k,l})$ is the morphism of $\mathcal{G}_2$ represented by the word read along this path, where labels of edges that are traversed against their orientation should be taken with a negative exponent.

Stated otherwise, this groupoid can be constructed from the free category on the graph obtained by directing "north" and "south" the edges of the 2-dimensional integer grid, by identifying every two morphisms with same beginning and endpoint.

To see that this is indeed the universal covering groupoid of $\mathcal{G}_2$, we have to look, for every pair $i,j \in \mathbb{Z}$, at the stars of $v_{i,j}$ and $C_{r(i,j)}$. We have clearly

$$\mathrm{St}_{\widehat{\mathcal{G}}_2}(v_{i,j}) = \{\mu_{i,j,k,l} \mid k,l \in \mathbb{Z}\}, \quad \mathrm{St}_{\mathcal{G}_2}(C_{r(i,j)}) = \bigcup_{h=1}^{3} \mathrm{Mor}(C_{r(i,j)}, C_{r(i,j)+h}).$$

It is now clear that $\hat{\rho}$ induces a bijection between these sets. For instance, any $\mu_{i,j,k,l}$ is mapped to an element of $\mathrm{Mor}(C_{r(i,j)}, C_{r(i,j)+h})$ for $h = r(k,l) - r(i,j)$. Conversely, by comparing Example 1.12 we see that every morphism of $\mathcal{G}_2$ starting at $C_{r(i,j)}$ can be 'unwrapped' as a path on the integer grid starting at (i,j). The given morphism is then the image of $\mu_{i,j,k,l}$, where (k,l) is the endpoint of the unwrapped path. △

2. Homotopy colimits for combinatorial applications

The theory of homotopy colimits of diagrams of spaces comes from homological algebra and category theory. It was developed by Quillen, Bousfield, Kan and others (see for example [63], [12]), and has now reached remarkable extension and depth. Some orientation (and a good introduction to the general subject) may be found in the paper by Hollender and Vogt [45] and in the book by Goerss and Jardine [42] and its bibliography.

The work of Reiner M. Vogt [78] leaves the complete generality of diagrams over categories and focuses on what can be said if one makes more and more assumptions on the target category (never going beyond the properties satisfied by the category of topological spaces) and on the index category, requiring it to be *small* and, as a further restriction, *directed* (for example, Vogt derives the explicit form of Definition 2.2).

In our work we will take the latter and more combinatorial point of view, which was adopted by Welker, Ziegler and Živaljević in [81], where a useful toolkit for applications of homotopy colimits in discrete mathematics is developed. We will recall the main definitions and some basic results; for a more complete account of the theory we refer to [81] and the recent textbook by Kozlov [47, Chapter 15], which provides a readable and self-contained introduction to the subject.

Definition 2.1. A *diagram of spaces* is a covariant functor $\mathscr{D} : \mathcal{I} \to \mathtt{Top}$ from a small index category to the category of topological spaces and continuous maps.

In our setting, $\mathcal{I}$ will always be given by some poset $\mathcal{P}$. Indeed, a poset can be thought of as a small category with at most one arrow between any two objects, where an arrow from $p \in \mathcal{P}$ to $q \in \mathcal{P}$ actually exists if and only if $p \geq q$ in $\mathcal{P}$. From now on we shall only consider diagrams over posets. In order to simplify notation, it is common to write $\mathscr{D}_p$ for the space $\mathscr{D}(p)$, and $f_{p,q}$ for the map $\mathscr{D}(p > q)$, if the diagram is understood.

A *morphism* from a diagram $\mathscr{D}$ over the poset $\mathcal{P}$ to a diagram $\mathscr{E}$ over $\mathcal{Q}$ is a pair $(\mu, (\alpha_p)_{p\in\mathcal{P}})$, where $\mu : \mathcal{P} \to \mathcal{Q}$ is a morphism of posets, and $(\alpha_p)_{p\in\mathcal{P}}$ is a family of continuous maps $\alpha_p : \mathscr{D}(p) \to \mathscr{E}(\mu(p))$ (indexed by elements of $\mathcal{P}$) that commute with the diagram maps.

Definition 2.2 (Compare (5.10) of [78]**).** Given a diagram of spaces $\mathscr{D} : \mathcal{P} \to \mathtt{Top}$, the *homotopy colimit* of $\mathscr{D}$ is defined by

$$\mathrm{hocolim}\mathscr{D} := \coprod_{p\in\mathcal{P}} \Delta(\mathcal{P}_{\leq p}) \times D_p \Big/ \sim$$

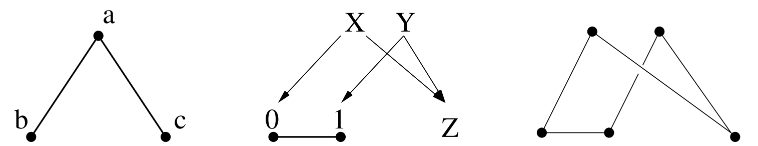
where the relation $\sim$ is given, for $p > q$, by identifying along the maps

$$\Delta(\mathcal{P}_{\leq q}) \times D_p \hookrightarrow \Delta(\mathcal{P}_{\leq p}) \times D_p, \quad \Delta(\mathcal{P}_{\leq q}) \times D_p \overset{(id\times f_{p,q})}{\longrightarrow} \Delta(\mathcal{P}_{\leq q}) \times D_q.$$

Example 2.3. Consider the poset $\mathcal{P}$ with three elements a, b, c ordered by $a > b$, $a > c$ and b, c incomparable, as in Figure 3.

Consider now the diagram $\mathscr{D}$ with $D_a := \{X, Y\}$, two points, $D_b = [0, 1]$, the unit interval, and $D_c = \{Z\}$, a single point. The map $f_{a,b}$ sends X and Y to the extremes 0 and 1 in $[0, 1]$, while $f_{a,c}$ maps everything to Z.

Let us consider the three terms of the disjoint union appearing in the definition of hocolim$\mathscr{D}$. The order complexes $\Delta(\mathcal{P}_{\leq b})$ and $\Delta(\mathcal{P}_{\leq c})$ consist both of a



The poset $\mathcal{P}$ A diagram $\mathscr{D}$ over $\mathcal{P}$ hocolim$\mathscr{D}$

Figure 3

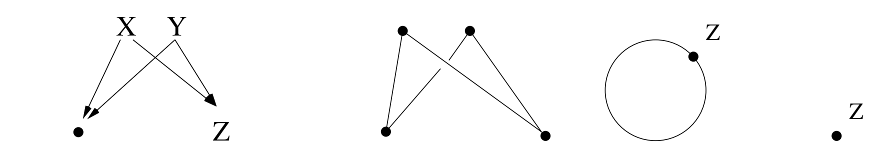
single point. Thus, $\Delta(\mathcal{P}_{\leq b}) \times D_b = b \times [0,1]$ and $\Delta(\mathcal{P}_{\leq c}) \times D_c$ is still a single point. On the other hand, $\Delta(\mathcal{P}_{\leq a})$ consists of the segments $b - a$ and $a - c$, joined at a, so that the corresponding term in the distinct union consists of two disjoint segments $b \times X - a \times X - c \times X$ and $b \times Y - a \times Y - c \times Y$. The equivalence relation identifies $b \times X$ with $b \times 0 \subset b \times [0,1]$, $b \times Y$ with $b \times 1 \subset b \times [0,1]$, and both the points $c \times X$, $c \times Y$ with Z. The result is a subdivision of the circle S^1 with 5 vertices, as can be seen in the Figure 3. $\triangle$

As mentioned before, homotopy colimits were designed to enjoy many naturality properties. The following functorial property is particularly useful

Lemma 2.4 (The Homotopy Lemma, see Proposition 3.7 of [81]**).** *Consider a morphism $\phi := (id, (\alpha_p)_{p \in \mathcal{P}}) : \mathscr{D} \to \mathscr{E}$ between two diagrams over the same poset $\mathcal{P}$. If every α_p is a (weak) homotopy equivalence, then the induced map* hocolim$\mathscr{D} \to$ hocolim$\mathscr{E}$ *is a (weak) homotopy equivalence.*

That this is not granted with usual colimits is precisely the reason why one has to introduce homotopy colimits.

Example 2.5. Consider the diagram $\mathscr{D}$ of the previous example. By taking the colimit of it, the points 0, 1 and Z are identified, so that colim$\mathscr{D}$ is obtained by identifying the endpoints of the unit interval, and thus is homotopy equivalent to a circle. Now consider the diagram $\mathscr{D}'$ that is defined the same way as $\mathscr{D}$ except for



The modified diagram $\mathscr{D}'$ hocolim$\mathscr{D}'$ colim$\mathscr{D}$ colim$\mathscr{D}'$

Figure 4

the fact that D_b' is now a single point as well, so that the contraction $D_b \to D_b'$ is clearly a homotopy equivalence. Now it easy to see that colim$\mathscr{D}'$ is a single point. On the other hand hocolim$\mathscr{D}'$ is, as the reader will verify, still a subdivision of S^1. $\triangle$

However, the colimit and the homotopy colimit of a diagram of space do agree in some cases, as stated in the following lemma.

Lemma 2.6 (The Projection Lemma, see Lemma 4.5 of [81]**).** *Let $\mathscr{D}$ denote a diagram of spaces on a poset $\mathcal{P}$. If all diagram maps $\mathscr{D}(p > q)$ are closed cofibrations, then the natural map* $\mathrm{hocolim}\mathscr{D} \to \mathrm{colim}\mathscr{D}$ *induces a homotopy equivalence.*

A *diagram of posets* is a diagram $\mathscr{D} : \mathcal{P} \to \mathtt{Pos}$ from a small index category (that in our work will be given as above by a poset $\mathcal{P}$) to the category $\mathtt{Pos}$ of partially ordered sets and order-preserving maps.

In this situation, we also can define the *poset limit* $\mathrm{Plim}\mathscr{D}$ of the diagram of posets $\mathscr{D}$. This is a poset with set of elements

$$\mathrm{Plim}\mathscr{D} := \bigcup_{p\in\mathcal{P}} \{p\} \times \mathscr{D}(p),$$

and order relations defined by

$$(p_1, q_1) \geq (p_2, q_2) :\Leftrightarrow \begin{cases} p_1 \geq p_2 \quad \text{and} \\ f_{p_1,p_2}(q_1) \geq q_2 \text{ in } \mathscr{D}(p_2) \end{cases}$$

where f_{p_1,p_2} as usual stands for the diagram map associated to the order relation $p_1 \geq p_2$.

To such a diagram of posets one can associate a diagram of spaces $\Delta(\mathscr{D}) : \mathcal{P} \to \mathtt{Top}$ with spaces $\Delta(\mathscr{D})_p$ defined to be the order complex $\Delta(\mathscr{D}(p))$, and maps $\Delta(f_{p,q}) : \Delta(\mathscr{D}(p)) \to \Delta(\mathscr{D}(q))$ induced by $f_{p,q}$ for all $p \geq q$.

Lemma 2.7 (The Simplicial Model Lemma). *Let $\mathscr{D}$ be a diagram of posets. Then the homotopy colimit of $\Delta(\mathscr{D})$ is homotopy equivalent to the order complex of the poset limit of $\mathscr{D}$:*

$$\mathrm{hocolim}\Delta(\mathscr{D}) \simeq \Delta(\mathrm{Plim}\mathscr{D}).$$

Proof. See [1], note after Corollary 2.11. □

3. Salvetti-type diagram models

We now introduce the first type of diagram models. The result of this section is summarized in Theorem 3.7, where it is proved that the homotopy colimit of the diagrams that we are going to introduce indeed model every covering space of the complement of a complexified arrangement. The fact that we will deal with diagrams of posets will allow us to actually write the covering spaces as order complexes of posets.

Definition 3.1. Given a cover of the arrangement groupoid $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$, we define a diagram of posets $\mathscr{D}_\rho : \mathcal{F} \longrightarrow \mathtt{Pos}$ with

$$\mathscr{D}_\rho(F) := \{v \in \mathrm{Ob}\,(\mathcal{G}_\rho) \mid \rho(v) < F\},$$

endowed with the trivial order relation defined by setting $v_1 \leq v_2$ if and only if $v_1 = v_2$, and maps being inclusions

$$\begin{array}{rccc} f^{\rho}_{F_1,F_2} := \mathscr{D}_\rho(F_1 > F_2): & \mathscr{D}_\rho(F_1) & \longrightarrow & \mathscr{D}_\rho(F_2) \\ & v & \longmapsto & \mathrm{end}\,(\rho(v) \to \rho(v)_{F_2})^{\langle v \rangle}. \end{array}$$

The Simplicial Model Lemma 2.7 allows to write $\mathrm{hocolim}\Delta(\mathscr{D}_\rho)$ as $\Delta(\mathrm{Plim}\mathscr{D}_\rho)$. Because this will be the main object of our attention for this section, let us from now set $W_\rho := \Delta(\mathrm{Plim}\mathscr{D}_\rho)$. The simplicial complex W_ρ has vertex set

$$\{(F, v) \in \mathcal{F} \times \mathrm{Ob}(\mathcal{G}_\rho) \mid \rho(v) < F\}$$

and the simplexes are chains with respect to the partial order

$$(F_1, v_1) \geq (F_2, v_2) \Leftrightarrow \left\{ \begin{array}{ll} 1) & F_1 \geq F_2, \\ 2) & v_2 = \mathrm{end}(\rho(v_1) \to \rho(v_1)_{F_2})^{\langle v_1 \rangle}. \end{array} \right.$$

Remark 3.2. A chain in $\mathrm{Plim}\mathscr{D}_\rho$ is given by a chain ϕ in $\mathcal{F}$ and an object v of the groupoid such that $\rho(v) \leq \max\phi$ in $\mathcal{F}$. Everything else can be reconstructed as above. Since all chains are of this form, we can encode each simplex of W_ρ by $\Delta(\phi, v)$.

Remark 3.3. If ρ is the identical cover, then $W := W_{id}$ is exactly the simplicial version of the Salvetti complex (the proof is carried out in [31, Proposition 4.1.2]).

Remark 3.4. For any covering $\rho : \mathcal{G}_\rho \to \mathcal{G}$, W_ρ is the barycentric subdivision of a CW-complex ${W_\rho}^{CW}$ having a d-cell $[F, v]$ for every $v \in \mathrm{Ob}(\mathcal{G}_\rho)$ and every $F \in \mathcal{F}$, $F \geq \rho(v)$, with $\mathrm{codim}(F) = d$. In fact,

$$[F, v] := \bigcup_{\max(\phi)=F} \Delta(\phi, v)$$

is the barycentric subdivision of a closed $\mathrm{codim}(F)$-ball.

Vertices of this complex are of the form $[\rho(v), v]$, and thus correspond bijectively to elements of $\mathrm{Ob}(\mathcal{G}_\rho)$. We will therefore identify vertices of ${W_\rho}^{CW}$ with objects of $\mathcal{G}_\rho$.

The cell $[F, v]$ is attached to those vertices v' that can be written as $v' = \mathrm{end}(\rho(v) \to \rho(v)_{F'})^{\langle v \rangle}$ with $F' < F$.

Note also that if $\mathcal{A}$ is central and P is the maximal element of $\mathcal{F}$, then any $[P, v]$ is the barycentric subdivision of the zonotope of the arrangement.

3.1. Covering maps

The functor $\rho : \mathcal{G}_\rho \to \mathcal{G}$ naturally induces a morphism of diagrams $\lambda : \mathscr{D}_\rho \longrightarrow \mathscr{D}$. In turn, by functoriality the morphism λ induces a map between the homotopy colimits, and thus a map of simplicial complexes

$$\Lambda_\rho : W_\rho \longrightarrow W.$$

In fact, Λ_ρ is a simplicial extension of λ: the simplex $\Delta(\phi, v)$ of W_ρ is mapped to $\Delta(\phi, \rho(v))$ in W. The previous considerations can be followed step by step to see that a morphism $\eta : \mathcal{G}_{\rho_1} \to \mathcal{G}_{\rho_2}$ between two covers $\rho_i : \mathcal{G}_{\rho_i} \to \mathcal{G}$ induces a map $\Lambda_\eta : W_{\rho_1} \to W_{\rho_2}$.

We now prove that groupoid coverings indeed induce topological coverings. In the remainder of this section we shall slightly abuse notation and write W_ρ for the *geometric realization* of the simplicial complex $\Delta(\mathrm{Plim}\mathscr{D}_\rho)$ (for the definition of the geometric realization of simplicial complexes see e.g. Spanier [73, Chapter 3]). This shall not cause confusion because the topological properties of a simplicial complex are indeed defined via its geometrical realization.

Proposition 3.5. *Let $\mathcal{A}$ be a locally finite real arrangement. For every covering of groupoids $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$, the induced map $\Lambda_\rho : W_\rho \to W$ is a topological cover of W.*

Proof. First, one sees that the base space is connected and locally arcwise connected because W is finite dimensional and locally finite.

Now take $P \in W$ and Let X be an open neighborhood of P that does not contain any vertex of W (except P if P happens to be a vertex). Let σ be the smallest dimensional simplex of W containing P, and let U be the star of σ. We have to show that each component of the preimage ${\Lambda_\rho}^{-1}(X \cap W)$ is mapped homeomorphically to $X \cap W$. For this, it is enough to show that ${\Lambda_\rho}^{-1}(U)$ is a disjoint union of copies of U, each of which is mapped identically to U by Λ_ρ.

In view of Remark 3.2, $\sigma = \Delta(\tilde{\phi}, \widetilde{C})$ for a chain $\tilde{\phi} \subset \mathcal{F}$ and a chamber $\widetilde{C} < \max(\tilde{\phi})$. Defining $\tilde{F} := \max(\tilde{\phi})$, we can write U and its preimage as

$$U = \bigcup_{\substack{\phi \supseteq \tilde{\phi} \\ C \in R(\phi)}} \Delta(\phi, C), \qquad {\Lambda_\rho}^{-1}(U) = \bigcup_{\substack{\phi \supseteq \tilde{\phi} \\ \rho(v) \in R(\phi)}} \Delta(\phi, v),$$

where $R(\phi)$ is the set of all chambers $C < \max(\phi)$ such that $C_{\tilde{F}} = \widetilde{C}$.

For every vertex $v' \in \rho^{-1}(\widetilde{C})$ we want to distinguish the subcomplex of U spanned by all vertices that can be attained from v' by the lift of a positive minimal path. To every such v' we thus associate the subcomplex

$$W_{v'} := \bigcup_{\substack{\phi \supseteq \tilde{\phi} \\ C \in R(\phi)}} \Delta(\phi, v(C, v'))$$

where, for $w \in \mathrm{Ob}(\mathcal{G}_\rho)$ and $C \in \mathrm{Ob}(\mathcal{G})$, we denote by $v(C, w)$ the object of $\mathcal{G}_\rho$ corresponding to the morphism $w(C \to \rho(w))^{-1}$ of $\mathcal{G}$, i.e., “the object mapping to C from which w can be reached by a positive minimal path”.

The proof now consists in the following three facts that amount to easy calculations with the morphisms of the involved groupoids. We list them and refer to [31, Proposition 4.3.2] for the complete argument.[1]

Fact 1: *For any $v_1 \neq v_2 \in \rho^{-1}(\widetilde{C})$, $W_{v_1} \cap W_{v_2} = \emptyset$.*

[1]Actually, the proof of [31] uses only the properties of *positive paths*. This makes the argument somewhat more involved, but makes sure that the arguments remain valid even restricting to the so-called “positive complexes” that will become important later.

Fact 2:

$$\coprod_{\tilde{v}\in\rho^{-1}(\widetilde{C})} W_{\tilde{v}} = \Lambda_\rho{}^{-1}(U).$$

Fact 3: *Fix $\tilde{v} \in \rho^{-1}(\widetilde{C})$. Then $\Lambda_\rho : W_{\tilde{v}} \to U$ is a homeomorphism.* (Note that it is enough to prove bijectivity on the vertex set).

□

3.2. The fundamental group

We want to compare the fundamental group of $W_\rho \simeq \operatorname{hocolim}\Delta(\mathscr{D}_\rho)$ with the object group of the corresponding groupoid. We begin by studying the structure of the low dimensional skeleta of the CW-complex $W_\rho{}^{CW}$ of which W_ρ is the barycentric subdivision (see Remark 3.4).

1-skeleton. Between two vertices $[\rho(v_1), v_1]$ and $[\rho(v_2), v_2]$ there is an edge if and only if the two corresponding chambers $\rho(v_i)$, $\rho(v_j)$ are separated by only one face F. Thus, there must be representatives of v_i and v_j that differ only by an edge that "crosses" F: either v_2 represents the concatenation of v_1 with the lift of $(\rho(v_1) \to \rho(v_2))$ at v_1, or v_1 represents the concatenation of a representative of v_2 with the lift $(\rho(v_2) \to \rho(v_1))^{\langle v_2\rangle}$. In the first case, we have $(F, v_1) > (\rho(v_1), v_1)$ and $(F, v_1) > (\rho(v_2), v_2)$ in $\operatorname{Plim}\mathscr{D}_\rho$, and thus we take the element (F, v_1) of $\operatorname{Plim}\mathscr{D}_\rho$ to represent an edge $[F, v_1]$ directed from $[\rho(v_1), v_1]$ to $[\rho(v_2), v_2]$ in the 1-skeleton of $W_\rho{}^{CW}$. The second case is treated analogously and produces an edge $[F, v_2]$ 'directed' away from $[\rho(v_2), v_2]$.

2-skeleton. Fix $v \in \operatorname{Ob}(\mathcal{G}_\rho)$ and $F \in \mathcal{F}$ with $\operatorname{codim}(F) = 2$ (w.l.o.g. $\rho(v) < F$). The vertices in the boundary $\partial[F, v]$ are those of the form $\left[C, \operatorname{end}(\rho(v) \to C)^{\langle v\rangle}\right]$, where C is any chamber adjacent to F. Let us label the vertices in circular order as $[\rho(v_i), v_i]$, $i = 0, \ldots, 2k-1$, and assume w.l.o.g. $v = v_0$, $C_0 := \rho(v_0)$.

Now consider $v_i \neq v_j$, and suppose that in $\partial[F, v]$ an edge between $[\rho(v_i), v_i]$ and $[\rho(v_j), v_j]$ exists. This means that $\rho(v_i)$ is adjacent to $\rho(v_j)$, and that there is an F_1 with $F > F_1 > \rho(v_i)$ and $F > F_1 > \rho(v_j)$, such that this edge can be written as $[F_1, \tilde{v}]$. To determine whether $\tilde{v} = v_i$ or v_j (which gives the 'direction' of the edge as above), recall that the fact that $(F, v_0) > (F_1, \tilde{v})$ in $\operatorname{Plim}\mathscr{D}_\rho$ implies $\rho(\tilde{v}) = \rho(v_0)_{F_1} = (C_0)_{F_1}$ - so $\tilde{v} = v_i$ if $\rho(v_i) = (C_0)_{F_1}$, i.e., if $\rho(v_i)$ is on the same side of F_1 as C_0.

Summarizing, in $\partial[F, v]$ we have then one edge for each codimension 1 face F_1 incident to F, and this edge is oriented away from the vertex that projects to the chamber on the same side as $\rho(v)$ w.r.t. F_1. We may also view $\partial[F, v]$ as a subgraph of G_ρ: it consists of the lift at v of the two positive minimal paths from $\rho(v)$ to its opposite chamber with respect to F. It follows that the cover $\mathcal{G}_\rho$ can be obtained from the graph G_ρ in the same way as $\mathcal{G}$ from G. In other words, $[F, v]$ provides a homotopy between two positive minimal paths in G_ρ.

It is not difficult now to compare the relations given by cells in ${W_\rho}^{CW}$ and the relation defining $\mathcal{G}_\rho$ and see that the following result holds. The proof is carried out in detail in [31, Section 4.4.3]

Theorem 3.6 (Proposition 4.4.3 of [31]**).** *Let $\mathcal{A}$ be a locally finite real arrangement. For every covering $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ of the arrangement groupoid, we have an isomorphism*

$$\pi\mathcal{G}_\rho \simeq \pi_1(\mathrm{hocolim}\Delta(\mathscr{D}_\rho))$$

Higher skeleta and MH-complexes. We have seen that the 2-cells are attached to the 1-skeleton in the very same way as in the original construction of the Salvetti complex from the arrangement graph. The reader is invited to check that also every higher dimensional cell $[F, v]$ is attached so that its 1-dimensional skeleton consists of the lift at v of all positive minimal paths from $\rho(v)$ to its opposite chamber with respect to F.

This is exactly the structure of the *Metrical Hemisphere complexes* (or MH-complexes) studied by Salvetti in [69], where the influence of the graph structure on the homotopical properties of these complexes is carefully explained. In particular the classical work of Gabriel and Zisman [40] is applied to give a very deep insight into the link between the homotopy of the complexes and the category of paths on the graph. The treatment starts from the full generality, and proceeds adding more and more restrictions as the proofs require them. For the case in which the MH-complex models the universal cover of a central arrangement (and thus agrees with $W_{\hat{\rho}}^{CW}$), Salvetti recovers, and puts into this broader context, Deligne's theorem about asphericity of simplicial arrangements [29].

3.3. Classification of the covers

Theorem 3.7. *For any topological cover $r : X \to \mathrm{Sal}(\mathcal{A})$ of the Salvetti complex of a locally finite real arrangement $\mathcal{A}$, there exists a cover of the arrangement groupoid $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ such that the homotopy colimit of the associated diagram of spaces $\Delta(\mathscr{D}_\rho)$ is isomorphic to X as a covering space of* $\mathrm{Sal}(\mathcal{A})$.

Proof. Let $\varphi : \pi\mathcal{G}(\mathcal{A}) \to \pi_1(\mathrm{hocolim}\Delta(\mathscr{D}))$ be the isomorphism of Theorem 3.6. Since $\mathrm{hocolim}\Delta(\mathscr{D}) \simeq \mathrm{Sal}(\mathcal{A})$, we can consider the preimage $U := \varphi^{-1}(r_\star(\pi_1(X)))$ of the fundamental group of X in $\pi\mathcal{G}(\mathcal{A})$.

Theorem 1.7 gives a cover $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ with $\rho_\star(\pi\mathcal{G}_\rho) = U$. Moreover, by Theorem 3.6 we have an isomorphism $\varphi_\rho : \pi\mathcal{G}_\rho \to \pi_1(\mathrm{hocolim}\Delta(\mathscr{D}_\rho))$. These isomorphisms come naturally from the inclusion ι of the graphs G_ρ as 1-skeletons of the CW-version of the homotopy colimits. Therefore the following diagrams commute

$$\begin{tikzcd} G_\rho \arrow[r, "\iota"] \arrow[d, "\rho"] & {W_\rho}^{CW} \arrow[d, "\Lambda_\rho"] \\ G \arrow[r, "\iota"] & W^{CW} \end{tikzcd} \qquad \begin{tikzcd} \pi\mathcal{G}_\rho \arrow[r, "\varphi_\rho"] \arrow[d, "\rho_\star"] & \pi_1(W_\rho) \arrow[d, "(\Lambda_\rho)_\star"] \\ \pi\mathcal{G} \arrow[r, "\varphi"] & \pi_1(W) \end{tikzcd}$$

and $r_\star(\pi_1(X)) = \varphi\rho_\star(\pi\mathcal{G}_\rho) = (\Lambda_\rho)_\star\varphi_\rho(\pi_1(W_\rho)) \cong (\Lambda_\rho)_\star(\pi_1(W_\rho))$. Hence, the cover $\Lambda_\rho : W_\rho \to \mathrm{Sal}(\mathcal{A})$ is isomorphic to $r : X \to \mathrm{Sal}(\mathcal{A})$. $\square$

Corollary 3.8. *Any cover $\rho : X \to \mathrm{Sal}(\mathcal{A})$ of the Salvetti complex can be written as the order complex of a poset, namely $\mathrm{Plim}\mathscr{D}_\rho$. The poset $\mathrm{Plim}\mathscr{D}_{id}$ is naturally isomorphic to the poset $\mathcal{S}(\mathcal{A})$ of cells of the Salvetti complex.*

Proof. Apply the Simplicial Model Lemma 2.7. $\square$

The following corollary generalizes [60, Theorem 3.7] (see also the definitions on [60, p. 164]) to affine arrangements.

Corollary 3.9. *Let $\hat\rho : \hat{\mathcal{G}} \to \mathcal{G}(\mathcal{A})$ denote the universal cover of $\mathcal{G}(\mathcal{A})$. Then $W_{\hat\rho} = \Delta(\mathrm{Plim}\mathscr{D}_{\hat\rho})$ is the universal cover of $\mathrm{Sal}(\mathcal{A})$.*

Proof. We prove universality. Take any cover $r : X \to \mathrm{Sal}(\mathcal{A})$; we have to show that there is a morphism of covers $m : W_{\hat\rho} \to X$. By the theorem, we know that there is a cover $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ with $W_\rho \cong X$ as a cover. Universality of $\hat{\mathcal{G}}$ implies the existence of a cover $\mu : \hat{\mathcal{G}} \to \mathcal{G}_\rho$, and this induces a morphism of diagrams $\lambda_\mu : \mathscr{D}_{\hat\rho} \to \mathscr{D}_\rho$. By functoriality, we have $\Lambda_\mu : W_{\hat\rho} \to W_\rho$, which gives the required morphism, as in the following diagram.

$$\begin{tikzcd}
\hat{\mathcal{G}} \arrow[rr, dashed] \arrow[dr, "\mu"] \arrow[dd, "\hat\rho"'] & & \mathscr{D}_{\hat\rho} \arrow[rr, dashed] \arrow[dr, "\lambda_\mu"] \arrow[dd, "\lambda_{\hat\rho}"'] & & W_{\hat\rho} \arrow[dr, "\Lambda_\mu"] \arrow[dd, "\Lambda_{\hat\rho}"'] \arrow[drr, bend left, "m"] & & \\
& \mathcal{G}_\rho \arrow[rr, dashed] \arrow[dl, "\rho"] & & \mathscr{D}_\rho \arrow[rr, dashed] \arrow[dl, "\lambda_\rho"] & & W_\rho \arrow[r, "\simeq"'] \arrow[dl, "\Lambda_\rho"] & X \arrow[dl, "r"] \\
\mathcal{G}(\mathcal{A}) \arrow[rr, dashed] & & \mathscr{D} \arrow[rr, dashed] & & W \arrow[r, equal] & \mathrm{Sal}(\mathcal{A}) \simeq \mathcal{M}(\mathcal{A}). &
\end{tikzcd}$$

$\square$

Example 3.10. Consider the arrangement given by one point $P \in \mathbb{R}$ (i.e., the arrangement $\mathcal{A}_1$ of examples 1.8, 1.11, 1.14). The space $\mathbb{R}$ is divided by P in two chambers A and B. It is easy to write down the face poset $\mathcal{F}_1 := \mathcal{F}(\mathcal{A}_1)$ and the arrangement graph $G(\mathcal{A}_1)$ as in Figure 5.

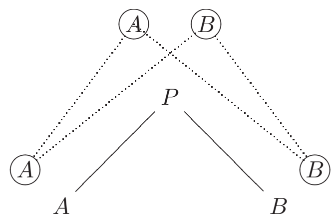
$\mathcal{A}_1$ and $G(\mathcal{A}_1)$ The poset of faces $\mathcal{F}(\mathcal{A}_1)$

FIGURE 5

The complexification of $\mathcal{A}_1$ is the arrangement given by a point in the complex plane. The complement $\mathcal{M}(\mathcal{A}_1)$ is then homotopy equivalent to S^1, hence its universal cover is $\mathbb{R}$. We will now see how the diagram models come to this conclusions.

First consider the diagram $\mathscr{D}$ on the poset $\mathcal{F}_1$. We have $\mathscr{D}(A) = \{A\}$, $\mathscr{D}(B) = \{B\}$, $\mathscr{D}(P) = \{A, B\}$. The diagram maps are in this case trivial, but let us explain where they come from:

$$\begin{aligned} \mathscr{D}(P > A): \quad & A \mapsto A_A = A \\ & B \mapsto B_A = A \\ \mathscr{D}(P > B): \quad & A \mapsto A_B = B \\ & B \mapsto B_B = B \end{aligned}$$



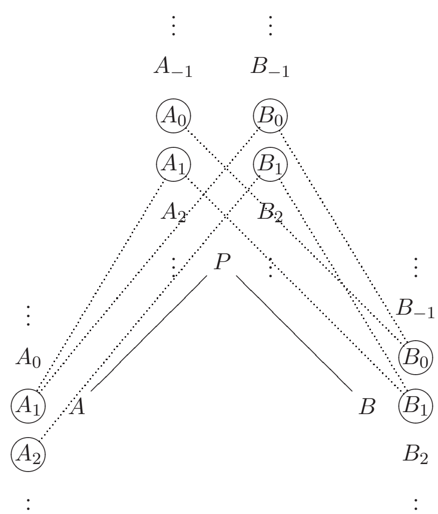
Note that the associated diagram of spaces $\Delta(\mathscr{D})$ is exactly the diagram of Example 3, where it is shown that $\mathrm{hocolim}\Delta(\mathscr{D}) \simeq S^1$.

We now have to look at the universal cover of $\mathcal{G}(\mathcal{A}_1)$. As we already pointed out, since $\mathrm{Sal}(\mathcal{A})$ has no 2-cells, the identification on $\mathtt{Free}(G(\mathcal{A}_1))$ is empty. Indeed, $\mathcal{G}(\mathcal{A}_1)$ is described in Example 1.11, and in Example 1.14 we computed its universal covering groupoid $\widehat{\mathcal{G}}_1$. Here we will slightly change notation and write $A_i := v_{2i}$ and $B_i := v_{2i+1}$, so that $\hat{\rho} : \hat{\mathcal{G}}_1 \to \mathcal{G}(\mathcal{A}_1)$ is defined by $\hat{\rho}(A_i) = A$, $\hat{\rho}(B_i) = B$ for all i and $G_{\hat{\rho}}(\mathcal{A}_1)$ is an infinite path $\ldots B_{-1} \to A_0 \to B_0 \to A_1 \to B_1 \ldots$

Writing down the diagram $\mathscr{D}_{\hat{\rho}}$ we have to keep in mind that the poset associated to an element $F \in \mathcal{F}_1$ has as many incomparable elements as there are objects in $\hat{\mathcal{G}}$ that project to a chamber adjacent to F. So we have $\mathscr{D}_{\hat{\rho}}(A) = \{A_i | \, i \in \mathbb{Z}\}$, $\mathscr{D}_{\hat{\rho}}(B) = \{B_i | \, i \in \mathbb{Z}\}$, $\mathscr{D}_{\hat{\rho}}(P) = \mathscr{D}_{\hat{\rho}}(A) \cup \mathscr{D}_{\hat{\rho}}(B)$. For the maps one has to take care of how paths are lifted. Let us work out some special case and write down the diagram in the same fashion as above:

$$\begin{aligned} &\mathscr{D}_{\hat{\rho}}(P > B): \\ &A_i \mapsto \mathrm{end}((A \to B)^{\langle A_i \rangle}) \\ &\qquad\qquad = B_i \\ &B_i \mapsto \mathrm{end}((B \to B)^{\langle B_i \rangle}) \\ &\qquad\qquad = B_i \end{aligned}$$

$$\begin{aligned} &\mathscr{D}_{\hat{\rho}}(P > A): \\ &A_i \mapsto \mathrm{end}((A \to A)^{\langle A_i \rangle}) \\ &\qquad\qquad = A_i \\ &B_i \mapsto \mathrm{end}((B \to A)^{\langle B_i \rangle}) \\ &\qquad\qquad = A_{i+1} \end{aligned}$$



By Lemma 2.7, we now only have to write down the poset $\mathrm{Plim}\mathscr{D}_{\hat{\rho}}$. The order relation is such that for $F_i \in \mathcal{F}_1$ and $v_i \in \mathrm{Ob}(\hat{\mathcal{G}}_1)$ we have $(F_1, v_1) \geq (F_2, v_2)$ if and only if $F_1 \geq F_2$ and $v_2 = \mathrm{end}(\rho(v_1) \to \rho(v_1)_{F_2})^{\langle v_1 \rangle}$.

In our case, this means that the dotted lines in the above picture are yet a piece of the Hasse diagram of $\mathrm{Plim}\mathscr{D}_{\hat{\rho}}$, which we can redraw in a more readable way as

$$\begin{tikzcd}[column sep=small]
 & (P,B_{-1}) \arrow[dl, dash, dashed] \arrow[dr, dash] & & (P,A_0) \arrow[dl, dash] \arrow[dr, dash] & & (P,B_0) \arrow[dl, dash] \arrow[dr, dash] & & (P,A_1) \arrow[dl, dash] \arrow[dr, dash] & & {} \arrow[dl, dash, dashed] \\
{} & & (A,A_0) & & (B,B_0) & & (A,A_1) & & (B,B_1) &
\end{tikzcd}$$

It is now clear that $\mathrm{hocolim}\mathscr{D}_\rho \cong \Delta(\mathrm{Plim}\mathscr{D}_{\hat{\rho}}) \simeq \mathbb{R}$, as required. $\triangle$

3.4. Reflection arrangements and Charney-Davis models

Suppose that $\mathcal{A}$ is the set of reflecting hyperplanes for a finite real refection group W; then W acts on $\mathcal{M}(\mathcal{A})$. The fundamental group $\pi_1(\mathcal{M}(\mathcal{A})/W)$ is the associated Artin group, as was proved by Brieskorn [16], and $\mathcal{M}(\mathcal{A})$ is aspherical (i.e., its homotopy groups are trivial in degree bigger or equal 2, as proved in [29]). Among other things, this means that the Salvetti complex $\mathrm{Sal}(\mathcal{A})$, and its "quotiented" version presented in [68] are finite $K(\pi, 1)$s for the Artin groups of finite type.

Ruth Charney and Michael W. Davis showed in [23] that this situation generalizes to many infinite Coxeter groups. The argument builds on previous work [22] of the same authors, who introduced a *"modified Deligne complex"* Φ ([22, (1.5)]) in order to describe, via the theory of complexes of groups, the universal cover of a space M associated to any linear reflection group W [22, see Theorem 1.5.1, Corollary 3.2.2, Proposition 3.2.3]. The space M can be obtained as the quotient (by the action of W) of the complement of the "reflection hyperplanes" associated to the action of W on a certain space (see [22, Section 2]). Also, M is conjecturally a $K(\pi, 1)$ space for the Artin group associated with W; in [22] this conjecture is proved true for two classes of reflection groups ("2-dimensional Artin groups" and "Artin groups of FC type"), by showing that Φ is contractible. In both cases this is achieved by proving that a suitable piecewise euclidean metric on Φ is CAT(0) [22, Section 4]. Then, in [23] the same authors describe a finite complex that is homotopy equivalent to M, thus providing finite $K(\pi, 1)$ complexes for the Artin groups for which the above conjecture holds, and therefore making the situation for finite reflection groups part of a more general picture. The finite complex is called *Salvetti complex* in [23, (1.2)].

One of the question raised by Charney and Davis' work in our context is whether this similarity can lead to any generalization of their technique - i.e., whether techniques of CAT(0) geometry used on Φ can be applied to the diagram models in the general case.

The first candidates for such a program could be the affine reflection arrangements. In this context, and building on the results of [22, 23], Charney and

Peifer proved the $K(\pi,1)$ conjecture for the affine braid arrangements by realizing [24, Section 3] the universal cover of these arrangements as the nerve of a certain covering by contractible subcomplexes of the "Bestvina Normal Form Complex" for the Artin group of (finite) type B_n (see [5, 25]) that we will encounter later in this survey (see Definition 5.12). This complex is contractible by either [29] (i.e., because it models the universal cover of the B_n arrangement) or [25] (i.e., using the Garside structure of the associated Artin group). At present, the $K(\pi,1)$ conjecture for affine real reflection arrangements is solved for the arrangements of type $\tilde{A}_n$ and $\tilde{C}_n$ (first by Okonek [54], type $\tilde{A}_n$ also by Charney and Peifer [24]) as well as for type $\tilde{B}_n$ (as proved by Callegaro, Moroni and Salvetti [21]).

4. Paris' topological models

We explain a construction of topological models for covers of complexified arrangements that is due to Luis Paris [59]. In this construction, the information on the fundamental group is encoded in so-called *oriented systems* rather than in groupoids, as is the case in our treatment. In later work [60], Paris himself gave a combinatorial stratification of his models for the universal cover of a linear arrangement.

We will start by giving the definition of Paris' oriented systems and outlining the parallels with the theory of groupoids. Then we will explain the construction of the topological models and conclude by giving a combinatorial stratification of them in the most general case, providing an explicit homotopy equivalence with the Salvetti-type diagram models that works for any cover of all complexified arrangements.

4.1. Oriented systems and their covers

Paris introduced the notion of *oriented system*, that we briefly recall.

Definition 4.1. An *oriented system* is a pair $(\Gamma, \sim)$ where Γ is an oriented graph and $\sim$ is an identification between paths of Γ such that

(1) $\alpha \sim \beta$ implies $\mathrm{start}(\alpha) = \mathrm{start}(\beta)$ and $\mathrm{end}(\alpha) = \mathrm{end}(\beta)$,
(2) $\alpha\alpha^{-1} \sim \mathrm{start}(\alpha)$ for every α,
(3) $\alpha \sim \beta$ implies $\alpha^{-1} \sim \beta^{-1}$,
(4) $\alpha \sim \beta$ implies $\gamma_1\alpha\gamma_2 \sim \gamma_1\beta\gamma_2$ for any γ_1, γ_2 with $\mathrm{end}(\gamma_1) = \mathrm{start}(\alpha)$ and $\mathrm{start}(\gamma_2) = \mathrm{end}(\alpha)$,

where the 'inverse' of an oriented path α is obtained by going along α in the reverse direction.

Given a real arrangement of hyperplanes $\mathcal{A}$, the natural way to associate to it an oriented system $(\Gamma(\mathcal{A}), \sim)$ is of course to take $\Gamma(\mathcal{A}) = G(\mathcal{A})$, the arrangement graph, and identify two paths if they are both positive minimal and they start at the same point v and end at the same point w.

Forgetting orientation of edges, one can view Γ as a 1-complex, and therefore consider its fundamental group $\pi_1(\Gamma)$. The conditions that were required in the definition of the equivalence relation on paths ensure that $\sim$ induces an equivalence relation on $\pi_1(\Gamma)$; therefore we can consider the quotient $\pi(\Gamma,\sim) := \pi_1(\Gamma)/\sim$, which is called by Paris the *fundamental group* of the oriented system $(\Gamma,\sim)$.

For oriented systems, Paris [59] introduced the following concept of a cover:

Definition 4.2. Given two oriented systems $(\Theta,\sim_\Theta)$ and $(\Psi,\sim_\Psi)$, a morphism of oriented graphs $\rho:\Theta\to\Psi$ is said to be a *cover* of $(\Psi,\sim_\Psi)$ if

(1) for every vertex v of Θ and every path α in Ψ with $\mathrm{start}(\alpha)=\rho(v)$ there is a unique path $\hat{\alpha}_v$ in Θ with $\rho(\hat{\alpha}_v)=\alpha$ and $\mathrm{start}(\hat{\alpha}_v)=v$. This path is called the *lift* of α at v.

(2) for any two paths α,β in Ψ with $\mathrm{start}(\alpha)=\mathrm{start}(\beta)=\rho(v)$, if $\alpha\sim\beta$ then $\hat{\alpha}_v\sim\hat{\beta}_v$.

At this point, the similarity with the theory of groupoids as sketched in the prologue is clear, and we summarize it.

Corollary 4.3. *Let $\mathcal{G}(\mathcal{A})$ denote the arrangement groupoid. We have immediately $\mathrm{Ob}(\mathcal{G}(\mathcal{A}))=V(\Gamma(\mathcal{A}))$, the set of vertices of Γ. Any path γ on Γ identifies an equivalence class of morphisms $[\gamma]$ of $\mathcal{G}$; moreover,*

$$\gamma_1\sim\gamma_2 \text{ in } (\Gamma,\sim) \text{ if and only if } [\gamma_1]=[\gamma_2]\in\mathrm{Mor}_{\mathcal{G}}(\mathrm{start}(\gamma_1),\mathrm{end}(\gamma_2)).$$

In particular, observe that $\pi(\Gamma(\mathcal{A}),\sim)\cong\pi\mathcal{G}(\mathcal{A})$.

Furthermore, the requirement on the star $\mathrm{St}(v)$ in the definition of covering of a groupoid translates naturally into condition (1) of Definition 4.2, where condition (2) takes account of the quotient to which we pass in defining the arrangement groupoid. Therefore, for any choice of a subgroup of $\pi(\Gamma(\mathcal{A}),\sim)\simeq\pi\mathcal{G}(\mathcal{A})$ we obtain a covering $(\Theta,\sim)$ (by Theorem 4.4) and a covering $\mathcal{G}_\rho$ (by Theorem 1.7) that may be compared as follows:

$$V(\Theta)=\mathrm{Ob}(\mathcal{G}_\rho)$$
$$(v,w)\in V(\Theta)\Leftrightarrow \text{there is } \alpha\in\mathrm{Mor}_{\mathcal{G}_\rho}(v,w) \text{ lifting an edge of } G(\mathcal{A}).$$

The following result is now evident.

Theorem 4.4 ([59]). *Consider an oriented system $(\Psi,\sim)$. For each subgroup H of $\pi(\Psi,\sim)$ there exists a cover of oriented systems $\rho:(\Theta,\sim)\to(\Psi,\sim)$ with $\rho_\star(\pi(\Theta,\sim))=H$.*

4.2. Topological covers associated to oriented systems

We fix from now on a cover of oriented systems $\rho:(\Theta,\sim)\to(\Gamma,\sim)$, the latter being the oriented system naturally associated to the arrangement graph, and describe the construction developed by Paris in [59] that associates a topological space to $(\Theta,\sim)$.

Remark 4.5. In this Section the faces F have to be considered as convex, relatively open subsets of $\mathbb{R}^d$, and we will write $|F|$ for the affine span of any face F. Also, given a chamber C and a face F, we will denote by $C_{|F|}$ the unique chamber of the arrangement $\mathcal{A}_{|F|}:=\mathrm{supp}(F)$ containing C. Thus, $C\subseteq C_{|F|}$ for every chamber C and every face F.

The key objects out of which the space is built are pieces of the form

$$M(C) := \bigcup_{F \in \mathcal{F}} (F + iC_{|F|}) \subset \mathbb{R}^d \oplus i\mathbb{R}^d = \mathbb{C}^d.$$

For every vertex v of Θ we define $N(v) := M(\rho(v))$, and the topological space associated to $(\Theta, \sim)$ is given by

$$N_\Theta := \coprod_{v \in V(\Theta)} N(v) \Big/ \approx \ .$$

The relation $\approx$ is defined pointwise by identifying two points $z \in N(v)$, $z' \in N(v')$ whenever (v, v') is an edge of Θ and they correspond to a unique point $z = z'$ in $M(\rho(v)) \cap M(\rho(v'))$ which real part lies on the same side as $\rho(v')$ with respect to the hyperplane separating $\rho(v)$ from $\rho(v')$.

Theorem 4.6 ([59])**.** *Given any cover $\rho : (\Theta, \sim) \to (\Gamma, \sim)$ of the oriented system associated to a linear arrangement $\mathcal{A}$, the space N_Θ with the map induced by ρ is a topological cover of $\mathcal{M}(\mathcal{A})$ with characteristic group $\pi(\Theta, \sim)$.*

Paris defines for any two vertices v, w of Θ a topological space $Z(v, w)$ as the interior of $\bigcup \overline{\rho(u)}$, where the overline denotes topological closure in $\mathbb{R}^d$ and the union is over all $u \in V(\Theta)$ that are reachable from both v and w by lifts of positive minimal paths of Γ. If (v, w) is an edge of Θ, then $Z(v, w)$ contains the real part of the set that is identified between $N(v)$ and $N(w)$. For general v and w, Paris shows the following fact that we state for later reference.

Lemma 4.7 (see Lemma 3.5 of [59]**).** *If $N(v) \cap N(w) \neq \emptyset$ in N_Θ, then $Z(v, w)$ is not empty. Indeed,*

$$N(v) \cap N(w) = M(\rho(v)) \cap M(\rho(w)) \cap (Z(v, w) + iV).$$

4.3. Combinatorial stratifications and diagram models

We now draw the link between Paris' construction and the diagram models. For this, we describe a stratification of N_Θ by contractible subsets with contractible intersections, and show that its nerve is isomorphic to $\mathrm{Plim}\mathscr{D}_\rho$. By the Nerve Lemma (see [7, 10.6(ii)] or [47, Theorem 15.21]) this proves directly $\mathrm{hocolim}\Delta(\mathscr{D}_\rho) \simeq N_\Theta$.

Remark 4.8. In comparing the two constructions one has to bear in mind that a vertex of an oriented system corresponds to an object of the corresponding covering groupoid, which is an equivalence class of paths in the arrangement graph. We will offer adequate explanations unless the context already makes clear which point of view is taken on the objects denoted by $v, w, \ldots$

Let us define, for any $F \in \mathcal{F}$ and any $C \in \mathcal{T}$, a set

$$M_C^F := \bigcup_{F' \leq F} F' + iC_{|F'|}.$$

Accordingly, for any $v \in V(\Theta)$ let $N_v^F := M_{\rho(v)}^F \subset N(v)$.

It is then clear that

$$M(C) = \bigcup_{F\in\mathcal{F}} M_C^F, \qquad N(v) = \bigcup_{F\in\mathcal{F}} N_v^F.$$

Lemma 4.9. *For any $F \in \mathcal{F}(\mathcal{A})$ and any $C \in \mathcal{T}(\mathcal{A})$, the space M_C^F is contractible. This implies contractibility of N_v^F for all $v \in V(\Theta)$, $F \in \mathcal{F}$.*

Proof. Fix $F \in \mathcal{F}$, $C \in \mathcal{T}$ and $z \in F+iC_{|F|}$. We show that the segment connecting z to any $x \in M_C^F$ lies fully in M_C^F.

If $x \in F+iC_{|F|}$ the claim is trivial by convexity. Choose therefore $F' \leq F \in \mathcal{F}$ such that $x \in F' + iC_{|F'|}$, and consider the segment

$$\gamma(t) := tz + (1-t)x, \qquad 0 \leq t \leq 1.$$

It is clear that $\Re(\gamma(t)) \in F'$ for $0 \leq t < 1$, and by assumption $\Re(\gamma(1)) \in F$.

For the imaginary part, note that $\Im(x) \in C_{|F'|}$ and $\Im(z) \in C_{|F|} \subset C_{|F'|}$ (see Remark 4.5), so $\Im(\gamma(t))$ is a straight line between two points of the convex set $iC_{|F'|}$. □

Lemma 4.10. *Given $v_1, v_2 \in V(\Theta)$ and $F_1, F_2 \in \mathcal{F}$, $N_{v_1}^{F_1} \subset N_{v_2}^{F_2}$ if and only if $(F_1, v_1) < (F_2, v_2)$ in $\mathrm{Plim}\mathscr{D}_\rho$.*

Proof. For the "only if"-part, suppose that $N_{v_1}^{F_1} \subset N_{v_2}^{F_2}$. Then clearly $N_{v_1}^{F_1} \subset N(v_1) \cap N(v_2)$ and by Lemma 4.7 we have that $N_{v_1}^{F_1}$, which by definition is a copy of $M_{\rho(v_1)}^{F_1}$, is indeed contained in $Z(v_1, v_2)$. In particular, we thus have that $\rho(v_1) \subset Z(v_1, v_2)$, and by the very definition of $Z(v_1, v_2)$, in the oriented system v_1 can be reached from v_2 by the lift of a minimal path $(\rho(v_2) \to \rho(v_1))$. In the corresponding covering groupoid every object is an equivalence class of paths, and these objects are the vertices of G_ρ. Thus we may state the above conclusion as a composition of (equivalence classes of) paths as:

$$\text{(i) } v_1 = v_2(\rho(v_2) \to \rho(v_1))^{\langle v_2 \rangle}$$

(recall that the v_i correspond to homotopy classes of paths, and as such may be concatenated with other paths). On the other hand, $N_{v_1}^{F_1} \subset N_{v_2}^{F_2}$ implies $M_{\rho(v_1)}^{F_1} \subset M_{\rho(v_2)}^{F_2}$, which is clearly equivalent to:

$$\text{(ii) } F_1 < F_2 \text{ and } \rho(v_1) = \rho(v_2)_{F_1}.$$

The sentences (i) and (ii) above constitute the definition of the order relation $(F_1, v_1) < (F_2, v_2)$ in $\mathrm{Plim}\mathscr{D}_\rho$.

For the "if"-part, suppose $(F_1, v_1) < (F_2, v_2)$ in $\mathrm{Plim}\mathscr{D}_\rho$, that means

$$\text{(a) } F_1 < F_2, \quad \text{(b) } \rho(v_1) = \rho(v_2)_{F_1}, \quad \text{(c) } v_1 = v_2(\rho(v_2) \to \rho(v_1))^{\langle v_2 \rangle}.$$

As above, (c) means that $N(v_1)$ and $N(v_2)$ intersect, and Lemma 4.7 describes this intersection. Since the (a) and (b) above ensure that $M^{F_1}_{\rho(v_1)} \subseteq M^{F_2}_{\rho(v_2)}$, to prove the inclusion $N^{F_1}_{v_1} \subset N^{F_2}_{v_2}$ we only have to show that $Z(v_1, v_2)$ contains all faces $F < F_1$.

The latter assertion means that no chamber C such that $C < F$ (i.e., $F \subset \overline{C}$) is separated from $\rho(v_1)$ by any hyperplane that separates $\rho(v_1)$ from $\rho(v_2)$. But $C < F$ implies $C < F_1$, and so C can be separated from $\rho(v_1)$ only by hyperplanes $H \in \operatorname{supp}(F_1)$ and, by construction, the set of hyperplanes separating $\rho(v_1)$ from $\rho(v_2)$ is contained in $\operatorname{supp}(F_2) \setminus \operatorname{supp}(F_1)$. This concludes the proof. □

We end by proving the announced proposition.

Proposition 4.11. *Let $\Theta \to \Gamma(\mathcal{A})$ be a cover of oriented systems, $\rho : \mathcal{G}_\rho \to \mathcal{G}$ the associated cover of the arrangement groupoid, and $\mathscr{D}_\rho$ the corresponding diagram. Then*

$$N_\Theta = \bigcup_{\substack{F \in \max(\mathcal{F}) \\ v \in \mathrm{Ob}(\mathcal{G}_\rho)}} N^F_v$$

is a covering by open, contractible subsets with empty or contractible intersections. Moreover, the nerve of this covering is the poset $\mathrm{Plim}\mathscr{D}_\rho$.

Proof. After the above preparations, we only have to show that $N^{F_1}_{v_1} \cap N^{F_2}_{v_2}$ is not empty if and only if it equals $N^{F_1 \wedge F_2}_{\tilde{v}}$, where $\tilde{v}$ represents the path obtained by concatenating any representative of v_1 with the lift $(\rho(v_1) \to \rho(v_1)_{F_1 \wedge F_2})^{\langle v_1 \rangle}$ (where the wedge is taken in $\mathcal{F}$).

For this, recall Lemma 4.7. It implies that if $z \in N(v_1) \cap N(v_2)$ and, say, $\Re(z) \in F$, then all pieces of $N(v_1)$ of the form $F' + iC'_F$ with $F' < F$ are also in the intersection. Therefore, if an intersection is nonempty, it contains some $N^{F'}_{v'}$. Now, by Lemma 4.10, this is equivalent with $(F', v') < (F_i, v_i)$ for $i = 1, 2$.

Again by Lemma 4.10, it follows that $N^{F_1 \wedge F_2}_{\tilde{v}} \subset N^{F_1}_{v_1} \cap N^{F_2}_{v_2}$ because in $\mathrm{Plim}\mathscr{D}_\rho$ we have $(F_1 \wedge F_2, \tilde{v}) = (F_1, v_1) \wedge (F_2, v_2)$. The reverse inclusion follows because with some $z' \in N^{F_1}_{v_1} \cap N^{F_2}_{v_2} \setminus N^{F_1 \wedge F_2}_{\tilde{v}}$ we would have a whole $N^{F'}_{v'}$ included in both sets, but for which $(F', v') \not< (F_1, v_1) \wedge (F_2, v_2)$, contradicting Lemma 4.10. □

Corollary 4.12. *Given any cover $\rho : (\Theta, \sim) \to (\Gamma(\mathcal{A}), \sim)$, the spaces N_Θ and W_ρ are homotopy equivalent and isomorphic as covers of $\mathcal{M}(\mathcal{A})$.*

Proof. Homotopy equivalence is obtained from Proposition 4.11 via the Nerve Lemma ([7, 10.6(ii)] or [47, Theorem 15.21]). By naturality we obtain the isomorphism as covering spaces. □

5. Garside-type diagram models

The introduction of diagrams of spaces into the picture allows us to take the next step, modifying the diagram spaces so to get a new type of combinatorial models, with different features. The name indicates that this construction is inspired by the theory of Garside groups, in a way that will be made precise in Section 5.2.

Definition 5.1. Let $\mathcal{A}$ be a linear arrangement of real hyperplanes. Given a covering $\mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ of the arrangement groupoid let U_ρ be the flag complex on the vertex set $\mathrm{Ob}(\mathcal{G}_\rho)$ such that a set $\{v_0, v_1, \ldots, v_d\} \subset \mathrm{Ob}(\mathcal{G}_\rho)$ is a simplex if and only if, for all $i < j$ and given representatives γ_i of v_i, γ_j of v_j, the path $\gamma_j \gamma_i^{-1}$ is positive minimal.

Remark 5.2. In the previous definition, as in what follows, we need to consider specific representatives of the classes of paths that are given by objects of the groupoids. As agreed in Section 1.4, we will write $v, w, \ldots$ for the objects of the groupoids (i.e., classes of paths) and use Greek lowercase letters for specific paths in the arrangement graph.

For the sake of this survey it will be enough to give a slightly different version of the diagram models than the original one in [31], also in order to reduce to a minimum the required new definitions. Also, note that throughout the whole section $\mathcal{A}$ *denotes a linear arrangement*, even if the construction can easily be modified to hold also for affine arrangements. For instance, in the case of affine braid arrangements the construction would specialize to the complex used by Charney and Peifer in [24] (see the discussion in Section 3.4). Indeed part of the motivation in introducing the Garside-type models was the strive toward a (still missing) possible generalization of the methods of [24].

Definition 5.3. Let a cover $\mathcal{G}_\rho$ of the arrangement groupoid be given, and fix a face $F \in \mathcal{F}$. For every path γ representing a $v \in \mathrm{Ob}(\mathcal{G}_\rho)$ such that $\mathrm{end}(\gamma) < F$ we consider the set

$$\{\gamma(C \to \mathrm{end}(\gamma))^{-1} \mid C \in \mathcal{T}(\mathcal{A}),\, S(C, \mathrm{end}(\gamma)) \cap \mathrm{supp}(F) = \emptyset\} \subset \mathrm{Ob}(\mathcal{G}_\rho).$$

There is a natural partial order on this set that is induced by $\preccurlyeq_{\mathrm{end}(\gamma)}$ (for the definition see Section 1); we call this poset $\mathcal{Q}_\rho^F(\gamma)$.

Moreover, if C is a chamber, then $\mathrm{supp}(C) = \emptyset$ and thus every $\mathcal{Q}_\rho^C(\gamma)$ is naturally isomorphic to the poset $\mathcal{T}_C(\mathcal{A})$. To emphasize this structure we will write $\mathcal{T}_\rho(\gamma)$ for $\mathcal{Q}_\rho^C(\gamma)$.

Definition 5.4. For any covering of groupoids $\rho : \mathcal{G}_\rho \to \mathcal{G}(\mathcal{A})$ we define a diagram of posets $\mathscr{G}_\rho : \mathcal{F} \to \mathtt{Pos}$ by setting

$$\mathscr{G}_\rho(F) := \coprod_{\mathrm{end}(v) < F} \mathcal{Q}_\rho^F,$$

$$\begin{array}{rrcl} \mathscr{G}_\rho(F_1 > F_2): & \mathscr{G}_\rho(F_1) & \hookrightarrow & \mathscr{G}_\rho(F_2) \\ & \gamma' \in \mathcal{Q}_{F_1}(\gamma) & \mapsto & \gamma' \in \mathcal{Q}_{F_2}(\gamma\alpha) \end{array}$$

where α is defined as the positive minimal path from $\mathrm{end}(\gamma)$ to $\mathrm{end}(\gamma)_{F_2}$. The right side is well defined because if $\gamma' = \gamma(C' \to C)^{-1}$, where $C = \mathrm{end}(\gamma)$ and $C' = \mathrm{end}(\gamma')$, then $\gamma' = (\gamma\alpha)((C' \to C)\alpha)^{-1}$.

An easy check shows that the maps are well-defined. Let us go on to the following theorem, that is now easy to prove in the context of diagram of spaces.

Theorem 5.5. *For every cover $\mathcal{G}_\rho \to \mathcal{G}$ we have a homotopy equivalence*

$$\mathrm{hocolim}\Delta(\mathscr{D}_\rho) \simeq \mathrm{hocolim}\Delta(\mathscr{G}_\rho).$$

Proof. It is evident that the minimal elements of $\mathcal{Q}_\rho^F(\gamma)$ are the v such that $\rho(v) < F$ in $\mathcal{F}$. Every one of these vertices belongs to a different connected component of $\Delta(\mathcal{Q}_\rho^F(\gamma))$ and is a cone point for that component, so that clearly $\Delta(\mathscr{G}_\rho(F))$ is homotopy equivalent to $\Delta(\mathscr{D}_\rho(F))$. It is easy to check that the map that sends every $\gamma \in \mathcal{Q}_\rho^F \subset \mathscr{G}_\rho(F)$ to $v := \min \mathcal{Q}_\rho^F \in \mathscr{D}_\rho(F)$ induces a morphism of diagrams. We then conclude by an application of the Homotopy Lemma 2.4. □

Now we can prove that the flag complexes defined at the beginning of this section indeed model the arrangement covers.

Theorem 5.6. *For every cover $\mathcal{G}_\rho \to \mathcal{G}$, we have*

$$U_\rho \simeq \mathcal{M}_\rho$$

Proof. After Theorem 5.5 it suffices to show that $\mathrm{hocolim}\Delta(\mathscr{G}_\rho) \simeq U_\rho$. For this, note that for any morphism $g = \mathscr{G}_\rho(F_1 > F_2)$, the image $g(\Delta(\mathscr{G}_\rho(F_1)))$ is a simplicial subcomplex of $\Delta(\mathscr{G}_\rho(F_2))$ (compare Definition 5.4).
In particular, $(\Delta(\mathscr{G}_\rho(F_2)), \Delta(\mathscr{G}_\rho(F_1)))$ is a NDR-pair and g is a closed cofibration.

So we are in the situation to apply the Projection Lemma 2.6, obtaining a homotopy equivalence $\mathrm{hocolim}\Delta(\mathscr{G}_\rho) \simeq \mathrm{colim}\Delta(\mathscr{G}_\rho)$. We are left with showing that the right hand side is the complex U_ρ. Indeed, every simplex is contained in (maybe more than) a $\Delta(\mathcal{T}_\rho(\gamma))$. The maps of the diagram are inclusions, so

$$\mathrm{colim}\Delta(\mathscr{G}_\rho) = \Bigg(\coprod_{\gamma \in Ob(\mathcal{G}_\rho)} \Delta(\mathcal{T}_\rho(\gamma)) \Bigg) \Big/ \sim$$

and we only have to check the identifications.

Of course $\omega_1 \in \mathcal{T}_\rho(\gamma_1)$ and $\omega_2 \in \mathcal{T}_\rho(\gamma_2)$ are identified if and only if $\omega_1 = \omega_2$ in $\mathcal{G}_\rho$. Given a chain $\sigma \in \Delta(\mathcal{T}_\rho(\gamma_1)) \cap \Delta(\mathcal{T}_\rho(\gamma_2))$ (which is then automatically of length $< n$), let us write $C_1 := \mathrm{end}(\gamma_1)$, $C_2 := \mathrm{end}(\gamma_2)$, $\gamma' := \min \sigma$, $A := \mathrm{end}(\gamma')$, $B := \mathrm{end}(\max(\sigma))$, as in Figure 6.

The case $C_1 = A = C_2$ is trivial, and if both $C_1 \neq A$ and $C_2 \neq B$ one may consider the poset $\mathcal{T}_\rho(\gamma')$ that contains the chain σ. Then it is enough to show the claim for the case $C_1 = A$, $C_2 \neq A$. In this case we may suppose that $\gamma_1 = \gamma_2\alpha$ with $\alpha = (A \to C_2)$ (all equalities of paths here are in fact equivalences in $\mathcal{G}_\rho$). We will argue by induction on the length of α, the case where $\ell(\alpha) = 0$ being trivial. If $\ell(\alpha) > 0$, let F denote the first face that is crossed by α and let $\epsilon = (C_2 \to C)$ denote the edge of α crossing F, so that $\alpha = \epsilon\tau$ for a positive minimal path τ.

Then clearly $F > C_2$, and by definition the hyperplane H supporting F does not separate A from B. Thus, we have that

$$\sigma \subset \mathcal{Q}_\rho^F(\gamma_2\epsilon\tau),$$

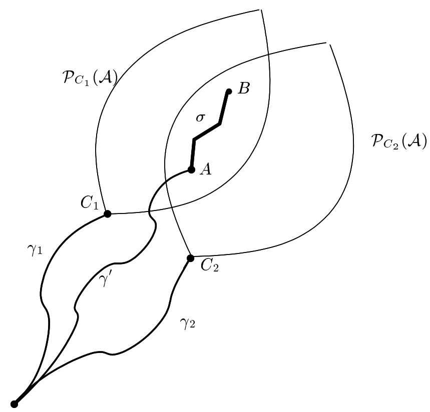


FIGURE 6. Figure for the proof of Theorem 5.5.

and σ is mapped identically to $\sigma' \subset \mathcal{Q}^C_\rho(\gamma_2\epsilon)$, the latter being equal to $\mathcal{T}_\rho(\gamma_2')$, where $\gamma_2' = \gamma_2\epsilon = \gamma_1(C \to A)^{-1}$. Therefore σ is identified with σ' in the colimit, and it remains to show that $\sigma' \subset \mathcal{T}_\rho(\gamma_2')$ is identified with $\sigma \subset \mathcal{T}_\rho(\gamma_1)$: but this follows now by induction, since $\ell((C \to A)) = \ell(\alpha) - 1$. □

Example 5.7. Consider the arrangement $\mathcal{A}_2$ of Example 1.4. The universal covering groupoid is computed in Example 1.15, and keeping those notations we may describe the complex $U_{\hat{\rho}}$ by identifying every vertex $v_{i,j}$ with the corresponding point (i,j) of the Cartesian plane, agreeing that $v_{0,0}$ projects to the base chamber we choose for the construction. Then it is clear that the morphisms that actually give edges of $U_{\hat{\rho}}$ are the $\mu_{i,j,k,l}$ of the form $\mu_{i,j,i+1,j}$, $\mu_{i,j,i,j+1}$, $\mu_{i,j,i+1,j+1}$. The 2-simplices are of the form $\{v_{i,j}, v_{i,j+1}, v_{i+1,j+1}\}$ or $\{v_{i,j}, v_{i+,j}, v_{i+1,j+1}\}$. Figure 7 shows a piece of this complex which is, in fact, a triangulation of the real plane. △

5.1. The 'Strong Lattice Property'

A linear real arrangement is called *simplicial* if its chambers are cones over simplexes. Brieskorn's conjecture was settled by Deligne, who showed that the complexification of every simplicial arrangement is $K(\pi, 1)$.

As an application of our construction, let us recast the proof of this result in view of the Garside-type diagram models.

Deligne's strategy has been to construct a contractible simplicial complex and then to show that under some technical assumptions this complex models the universal cover of the arrangement's complement. See also [61] for a reformulation of the argument. The first part of Deligne's proof establishes a crucial property of positive paths of simplicial arrangements. This property was given the name "property D" by Paris, who proved that it is indeed *equivalent* to the arrangement being simplicial (see [62] and, for an alternative proof, [31, Chapter 6]).

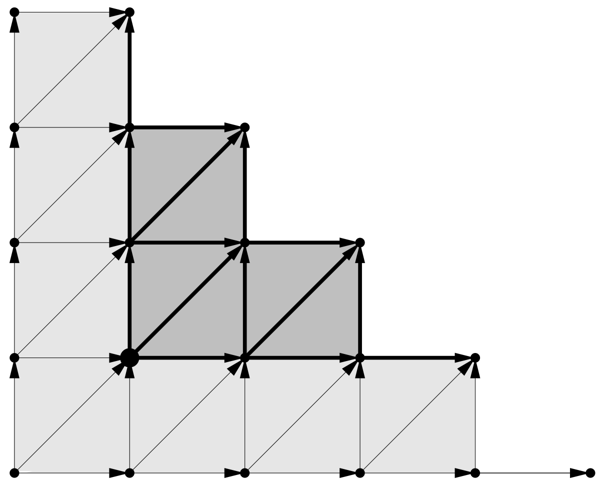


Figure 7. A part of the Garside-type universal cover complex for the arrangement $\mathcal{A}_2$ of Example 5.7, with the vertex $v_{0,0}$ represented by the thicker dot. The shaded part is a piece of the positive complex, namely $\widehat{U}_3$.

Lemma 5.8 ("Property D", see [60]). *Let $\mathcal{A}$ be a simplicial arrangement. For every $v \in \mathrm{Ob}(\hat{\mathcal{G}})$ representing a positive path on the arrangement graph, the following holds: there is a unique chamber C_v such that $\beta(C \to C_0)$ represents v for a positive path β if and only if $C \preccurlyeq_{C_0} C_v$.*

Once this is established, one can look at the part of the Garside-type universal cover complex $U_{\hat{\rho}}$ that is generated by the positive paths of length at most n - let us call it $\widehat{U}_n$ (see Figure 7). It is easy to see that Lemma 5.8 implies:

If the arrangement $\mathcal{A}$ is simplicial, then for every vertex v of $\widehat{U}_n \setminus \widehat{U}_{n-1}$,
($*$) v is the apex of a cone over a contractible subcomplex of $\widehat{U}_{n-1}$.

Thus, it is clear that $\widehat{U}_{n-1}$ is homotopy equivalent to $\widehat{U}_n$ by simply successively "pushing in" the cones. By induction, we obtain that the subcomplex of U generated by the positive paths is contractible. Moreover, one can show [60, Lemma 4.16] (but see also [29, 69]) that contractibility of the positive complex implies contractibility of the whole universal cover, and thus asphericity of the arrangement.

Question I. The natural (and open) question is to find a condition on the positive paths of a real arrangement that is weaker than property D but keeps the validity of statement ($*$) above.

Remark 5.9 (On the Lattice Property). It is a well known fact that an arrangement is simplicial if and only if its poset of regions is a lattice for every choice of a base region (see [9] - let us call this the "Strong Lattice Property"). In an attempt to generalize Deligne's argument, one might consider a weakening of this condition, i.e., requiring that the poset of regions be a lattice for at least one choice of base chamber ("Weak Lattice Condition"). Indeed, this condition is satisfied by all simplicial and all supersolvable arrangements, i.e., the two major known classes of aspherical arrangements, and by all hyperfactored arrangements (which are conjecturally $K(\pi,1)$; see [48]). However, an example exists that satisfies the Weak Lattice Condition but is *not* $K(\pi,1)$: it is the arrangement $\mathcal{A}_2$ of [36].

The Weak Lattice Property has however nice consequences with respect of the structure of the Garside type models, e.g. leading to a coarsening of the stratification by order complexes of the $\mathcal{T}_C$'s - see [31, Chapter 7] for further details.

5.2. Garside groups and Bestvina's complex

The name "Garside-type" comes from the analogy with the following construction that can be carried out when $\mathcal{A}$ is the reflection arrangement associated to a finite reflection group W. In this situation, W acts on $\mathcal{M}(\mathcal{A})$ and the fundamental group of $\mathcal{M}(\mathcal{A})/W$ is the associated *Artin group*. Among other nice properties, one has that Artin groups are *Garside groups* (see e.g. [41, 51, 28]).

Definition 5.10. A group G is a *Garside group* if there is a bounded, graded lattice $\mathcal{L}$ of finite height, with a labeling of the edges of its Hasse diagram in some alphabet S, so that G is the group of fractions of the monoid generated by S with relations that identify any two words that can be read along saturated chains of $\mathcal{L}$ with same begin- and endpoint. The labeling must satisfy some very important technical conditions, to ensure that the monoid actually embeds into its group of fractions. For every pair $x < y$ of the lattice $\mathcal{L}$, let $\lambda(x, y)$ denote the set of all words in S that can be read along any saturated chain starting at x and ending at y. All of these words are equal, and thus $\lambda(x, y)$ represents a single element, in the generated monoid. Then the conditions are that the labeling be

- *Balanced:* The sets $\{\lambda(\hat{0}, x) \mid x \in \mathcal{L}\}$ and $\{\lambda(x, \hat{1}) \mid x \in \mathcal{L}\}$ are equal.
- *Group-like:* For every two triples $x \leq y \leq z$, $x' \leq y' \leq z'$ of elements $\mathcal{L}$, if two of the corresponding pairs of labelings are equal, so is the third.

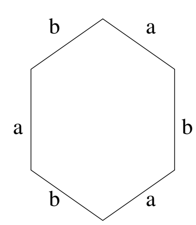


FIGURE 8.
$\{\lambda(\hat{0}, x) \mid x \in \mathcal{L}\} = \{a, b, ab, ba, aba, bab\} = \{\lambda(x, \hat{1}) \mid x \in \mathcal{L}\}$

Example 5.11. An instructive example is one of the Garside structures that lead to the Artin group of type A_2. In this case (as for every finite-type Artin group) the poset is the weak order of the corresponding Coxeter Group, with the natural labeling by simple roots. We depict the poset with an equivalent labeling in Figure 8. For details on the weak order and the labeling see [8]. △

We then introduce the following simplicial (flag) complex that was defined by Brady for braid groups [13], by Brady and Watt [14] and Bestvina [5] for finite type Artin groups, and was extended by Charney, Meier and Whittlesey to the more general context of Garside groups [25].

Definition 5.12 (Compare [25] **and Section 2.2 of** [5]**).** Given a Garside group G, let $X(G)$ denote the simplicial complex on the vertex set G obtained by declaring a subset $\{g_0, \ldots, g_d\} \subset G$ to be a simplex if for any $0 \leq i < j \leq d$ the element $g_i^{-1} g_j$ is an *atom* (i.e., any word that can be read (bottom-to-top) along some saturated chain in $\mathcal{L}$).

The following fact is proved as Theorem 3.1 of [25], but see also Theorem 3.6 of [5] and Theorem 6.9 of [13].

Theorem 5.13 ([25]**).** *For any Garside group G, the complex $X(G)$ is contractible.*

Remark 5.14. Returning to real reflection arrangements and Artin groups, the lattice $\mathcal{L}$ is the so-called *weak order* on the associated Coxeter group, with the natural labeling by standard generators (see [8] for definitions). Bestvina remarked that in this case $X(G)$ has a covering with contractible intersections which nerve is the universal cover of the Salvetti complex of $\mathcal{M}(\mathcal{A})$, and by the nerve Lemma [7, 10.6(ii)] one concludes homotopy equivalence [5, Section 2.2].

It is well-known that for any finite reflection arrangement the weak order of the associated reflection group is isomorphic to the poset of regions (by symmetry the choice of base chamber does not matter). Thus, one sees that the Garside-type universal cover models specialize to Bestvina's complex when the arrangement is the reflection arrangement of a finite real reflection group.

Remark 5.15. In addition to Remark 5.14, it has to be mentioned that all finite-type Artin groups can be presented as Garside groups also by partially ordering the associated reflection group by *reflection length* (see [13, 14]). This gives rise to the so-called *dual Garside structure* for these Artin groups (the word "dual" coming, so far, only from some enumerative properties of the two structures). One of the many interesting things about these orderings is that they can be defined also for finite groups of *unitary* reflections as classified by Shephard and Todd [72] (see [3, 4]). The lattices associated to the dual structures can all be described as posets of *generalized noncrossing partitions* ordered by refinement [66, 4], and appear in many different contexts. Bessis [3] shows that, in general, the Garside group obtained in this way turns out to be the fundamental group of the quotient of the complement of the associated reflection arrangement by the action of the reflection group (see Orlik and Terao [56] and Orlik and Solomon [55] for a combinatorial study of arrangements defined by unitary reflection groups). In the same work, Bessis was able to exploit this structure and give a proof of asphericity of all (real and unitary) finite reflection arrangements. Since every central arrangement in $\mathbb{C}^2$ is $K(\pi, 1)$ [56, Proposition 5.6], only the case of dimension strictly bigger than 2 must be handled. In all but one of those cases, Bessis shows that the universal covering space of the arrangement's complement is indeed homotopic to the associated complex $X(G)$, and thus contractible by Theorem 5.13. This holds for all *well-generated* groups, i.e., for all groups that can be generated by d reflections, where d is the dimension of an irreducible representation of the group (the dimension of the ambient space of the arrangement we are interested in). In the only left case the above argument must be refined, using the more general notion of Garside groupoid, also developed by Bessis in [2]. We will come to speak again about this topic in Section 6.

The natural question to ask is now whether these structures, and in particular the combinatorics of noncrossing partitions, can give rise to a corresponding presentation of the actual fundamental group of the arrangement's complement. Note that, for real reflection groups, this is the corresponding pure Artin group. For the type A_n some work was done by McCammond and Margalit [50] who gave presentations for the pure braid group that are inspired by the pictures of 'classical' noncrossing partitions. But the question is open and much work still needs to be done.

6. Applications and open ends

Spectral sequences and homology of covers. In the context of local system homology of arrangements attention has been paid to the computation of the homology of cyclic covers of arrangement complements, as they generalize in many ways the Milnor fibre (see the work of Cohen and Orlik [27] and, for a survey and the relevant bibliography, the paper by Suciu [74]). Therefore we want to point out that there exist spectral sequences that calculate the homology and cohomology of homotopy colimits of diagrams of spaces, thus offering an alternative to the spectral sequence approach described by Denham [30] and later generalized by Papadima and Suciu [58]. The original idea goes back to Segal [71], and a formulation for our combinatorial setting is given in [81]. The complexes resulting from the application of this spectral sequence to the general covers are described in detail in [31, Chapter 5], where also a direct derivation of this spectral sequence starting by a filtration by the skeleta of the underlying order complex can be found.

Minimality. The question whether $\mathcal{M}(\mathcal{A})$ has the homotopy type of a minimal CW-complex (i.e., one that has as many k-cells as there are generators of the k-th homology) was raised by Papadima and Suciu in [57]. An affirmative answer to this question was given by Dimca and Papadima [34] and, independently, Randell [64]. In the case of complexified arrangements, the question of minimality was studied by Yoshinaga [79] who in particular attempted to describe the attaching maps of the lift of a minimal CW-structure to the universal cover. Indeed, one point of interest of minimal complexes in this context is that the linearization of the equivariant chain complex obtained by their lift to the universal covering space is equivalent to the Aomoto complex – a well-known complex associated to arrangements and defined from the cohomology ring of the complement. This was first proved by Cohen and Orlik [26] and subsequently in increasing generality by other authors [34, 80, 58].

Recently, new explicit constructions were described to actually construct such a minimal CW-complex, at least when the arrangement is complexified. Both these works exploit discrete Morse theory in order to describe a collapsing of all 'superfluous' cells of the Salvetti complex.

Salvetti and Settepanella [70] introduce a new total ordering of the faces of the arrangement that they call *polar ordering* because it is obtained by lexicographically ordering polar coordinates of a distinguished point on every face. Given this ordering, an algorithm allows to construct the required discrete Morse vector field, and so to describe the collapsing that leads to the minimal CW-model. A closed formula for the boundary maps of the minimal complex is also given in [70]. In view of an easier computation of the polar ordering, it has to be pointed out that in fact it is not necessary to actually determine the polar coordinates. It was shown in [33] that the construction works also with a more general type of orderings, called *combinatorial polar orderings*, that are constructed from any valid sequence of "flippings" (see [10, Chapter 5]) along which a general position hyperplane can be "swept" through a generic section of the given arrangement.

The method used in [32] can be carried out entirely in terms of the intrinsic combinatorics of the associated oriented matroid. The data needed to construct the discrete Morse vector field is given by a maximal chain in the poset of regions of the arrangement (the tope poset of the associated oriented matroid). This construction exhibits a very direct correspondence between the no-broken-circuit sets of the matroid and the corresponding cells of the minimal complex. However, an explicit description of the attaching maps is still missing.

Question II. Does any interesting simplification appear by 'lifting' one of the described collapsing of the cells to the Salvetti-type universal cover complex?

Complex reflection groups. The most recent achievement about asphericity of arrangements is the result of David Bessis, who proved that every complex finite reflection arrangement is $K(\pi,1)$ [3]. As we explained in Remark 5.15, the method of Bessis involves the technique of Garside groups, and in particular, when the group is *well generated*, the universal covering space is modeled by a "complex analogue" of $X(G)$ (Definition 5.12). Among the complex reflection arrangements associated to well generated groups we find the complexification of the real reflection arrangements, and the translation of the argument of Bessis to that case turns out to use the so-called "dual Garside structure" instead of the "standard" one. It is then natural to ask whether, in the complexified case, a combinatorial way exists to prove the equivalence between Bessis' complex and $U_{\hat{\rho}}$. This problem amounts to a better understanding of the combinatorial relationship between the two Garside structures of finite type Artin groups, which still lacks a satisfactory explanation.

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