Combinatorial geometries in algebra and topology

Emanuele Delucchi

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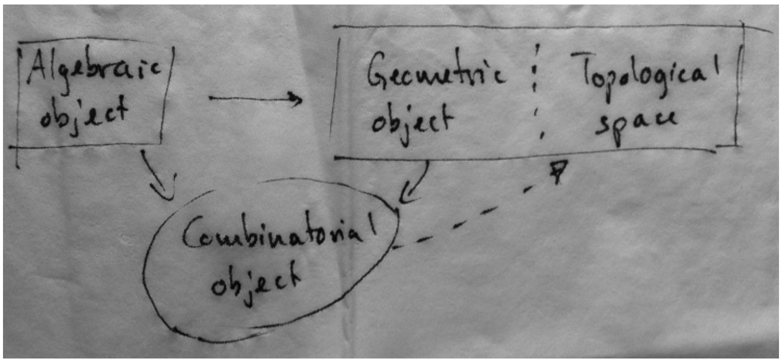
Introduction

The purpose of this introduction is twofold: first, to explain the unifying theme linking the material of this thesis; second, to give an overview of its contents.

The first is best illustrated by a quick sketch.



This napkin could come from any of Pisa's bars, Berkeley's coffee shops, Binghamton's diners or Bremen's Kaffeehäuser - and this ubiquity entails a lack of precision. For instance, one may ask what is meant by 'geometric object', or what exactly is considered 'combinatorial'.

In Part I and II of this thesis we will focus on two different interpretations of the latter, pertaining to the particular cases where the algebraic objects of this thesis are real or complex vector spaces. In Part III we will substitute a finitely generated integer lattice for the vector space structure, entering a subject where the enticing current problem is yet to *find* a suitable combinatorial description. Let us start to make things more concrete.

Example 0.0.1.

Algebraic object: an $m \times n$ matrix A with entries in a field $\mathbb{K}$.

Geometric object: the set of hyperplanes given as orthogonal complements to the columns of A in $\mathbb{K}^m$.

Combinatorial structure: the collection of linearly independent sets of columns of A or, equivalently, the collection of all intersections of the hyperplanes.

We see a common combinatorial structure underlying both the algebraic and the geometric object: it is the associated *combinatorial geometry* (or *'matroid'*).

From our point of view, the interest in understanding the combinatorial structure of such an example lies in the development of new tools for the study of both the algebra and the geometry.

A basic question then is how much information about the algebra or the geometry is encoded in the combinatorial structure. A very general answer for the geometric part can be given by the dashed arrow on the napkin: usually one can only hope to recover some of the topological data, rather than the full geometric information.

It is this topological interpretation that we focus on in most of the thesis. The motivations and the specifics of the topological questions we will address are very varied. Therefore, we refrain from a 'lump-sum' treatment here and prefer to introduce and contextualize the different topics at the beginning of the corresponding chapters.

Outline of the thesis

We will start with Chapter 1, where we introduce "combinatorial giemetries" or "matroids", the combinatorial backbone of the book, and review some of their theory. We will also present some facts about arrangements of hyperplanes, as an instance where we can 'recover' at least some of the topological data of the original geometrical configuration from matroid theory.

Chapter 1 as well as Chapters 2 and 6 are designed to highlight the structural parallelism of the theories - i.e., to convey the idea that matroids, oriented matroids and complex matroids can be viewed as 'the same theory, with different structure sets', depending on the peculiarity of the vector space structure associated to a fixed field.

In this respect, Chapter 2 introduces oriented matroids as 'matroids with structure set $\{-1, 0, 1\}$', explaining how this structure set gives a natural combinatorialization of real numbers and of convexity and orthogonality in Euclidean space. This is the groundwork for the remainder of Part I, where the combinatorics of oriented matroids is used to explore a recent result by Salvetti and Settepanella on the minimality of arrangement complements. A result obtained by Randell and, independently, by Dimca and Papadima shows that every arrangement complement has the homotopy type of a minimal CW-complex, i.e. with number of cells matching the Betti numbers of the complement. For arrangements with hyperplanes defined over the reals, so called "complexified arrangements", Salvetti and Settepanella [87] give an explicit computation of the minimal complex.

Now, in the complexified case the homotopy type of the complement is known to be determined by the associated oriented matroid (see Theorem 2.3.4, due to Salvetti).

Chapter 3 gives an alternative explicit proof of this minimality result, where the construction of the minimal complex does not rely on any operation in the ambient space but instead involves only the oriented matroid data. As a byproduct we prove a nice structural result about no-broken-cicuit sets arising from certain total orderings of the hyperplanes of the arrangement (Theorem 3.4.16). In turn, Chapter 4, based on joint work with Simona Settepanella [43], studies Salvetti and Settepanella's main tool for the construction of their minimal complex: the 'polar orderings'. We give a description thereof in terms of (extensions of) the associated oriented matroid. Inspired by the treatment of the braid arrangement in [87] we then introduce a combinatorially defined class of arrangements satisfying a condition that allows for the minimal complex to be computed relatively easily. We give a complete characterization of this class in rank 2 and we show that it contains all supersolvable arrangements.

Part II is dedicated to the case where the matrix A in Example 0.0.1 has complex entries and, accordingly, the hyperplanes may not have real defining forms.

In view of the rich and successful theory of oriented matroids - which, as we have hinted to, encodes the homotopy type of complexified arrangements and offers handy computational tools for dealing with their topology - in the last 20 years the quest for a theory of 'matroids with complex structure' has been pursued by many authors. In Chapter 5 we review the story of the subject, analyzing past contributions and stating a number of still open problems in the field.

Then Chapter 6, based on joint work with Laura Anderson [4][1], offers our take on this topic: the idea is to consider the "real structure set" $\{-1, 0, 1\}$ as an incarnation of the set $S^0 \cup \{0\}$, where S^0 is the unit sphere in $\mathbb{R}$. Then, it lies at hand to take as "complex structure" the set $S^1 \cup \{0\}$, where S^1 is the unit sphere in $\mathbb{C}$. It turns out that, with this choice of signs, a theory of complex matroids can be developed mainly along the lines of oriented matroids, the differences beween the two giving valuable insight into the peculiarities of the complex case. Our theory includes orthogonality, duality, and axiomatizations via phirotopes, phased circuits and dual pairs. What is still missing is a corresponding concept of "phased vector". That the obvious choice does not work, is shown by Example 5.1.1.

After having considered matroids over real and complex vector spaces, in Part III we turn to a different situation - one where the question about the 'right' combinatorial invariant is still wide open. Instead of elements of a vector space we consider elements of a finite dimensional lattice, i.e., vectors with integer coordinates, taken as elements of $\mathbb{Z}^d$. In this case, there is more than just the data about (rational) linear dependency; namely, the arithmetic data about the index of the sublattices generated by sets of elements.

Motivation for considering this setup comes from recent interest in toric arrangements, viewed as objects that link the study of partition functions, properties of zonotopes and the theory of Dahmen-Micchelli spaces of splines. This field of investigation gained wide popularity especially following the monograph by De Concini and Procesi on the subject [36].

In Chapter 7 we lay out in more detail some of the aforementioned motivation and give a short account of the recent work of Luca Moci about combinatorial (enumerative) invariants of toric arrangements. With Chapter 8, based on joint work with Giacomo D'Antonio [34], we offer a contribution to the development of this young theory by describing a combinatorial model for the homotopy type of toric arrangements which can be viewed as a "toric

[1] Parts of this publication was also used in the 'preliminary' chapters.

analogue" of Salvetti's complex, and by computing a presentation of the fundamental group of the complement of toric arrangements.

Acknowledgments

The work which makes up this thesis could not have existed without the people whom I met along the road in the last years. I would especially like to thank my coauthors Laura Anderson, Simona Settepanella and Giacomo D'Antonio, who substantially contributed to the work presented here. In general, my work also greatly benefited from frequent conversations with Mario Salvetti and Thomas Zaslavsky.

In the last years I had the privilege to work in the stimulating atmosphere of different institutions. In particular, I thank the mathematics department at the University of Pisa, the department of mathematical sciences at Binghamton University (SUNY) and the department of mathematics and computer science at the University of Bremen. Also, I would like to express my gratitude to the Swiss National Science Foundation for funding my stay in Pisa and to the Mathematical Sciences Research Institute of Berkeley for its hospitality and support during the Fall semester 2007.

My recent research grew out of the experiences and the interests I developed during my doctoral years. Here, a grateful thought goes also to my PhD advisor Eva-Maria Feichtner for her support and advice even after my promotion.

But many other people, in even more places, offered precious advice in many conversations whose focus transcended mathematics more often than not. Without even attempting to compile a comprehensive list, I would like to mention Giovanni Gaiffi and Filippo Callegaro in Pisa; at Binghamton: Ross Geoghegan, Matt Brin, Fernando Guzmàn, Diane Vavrichek, Lucas Sabalka, Marco Varisco; in Bremen: Eva-Maria Feichtner, Dmitry Kozlov and Jean-Marie Droz; around the world: Daniel Cohen, Corrado De Concini, Graham Denham, Priyavrat Deshpande, Michael Falk, Jon McCammond, Luca Moci, Vivien Ripoll, Masahiko Yoshinaga, Sergey Yuzvinsky, Max Wakefield, Rade Zivaljević and many others.

When it came to balance mathematics in life (mostly through music or a good laugh), I could count on many friendships who lasted despite my frequent geographical shifts. This makes them even more precious, and makes me feel even more privileged. So, I should thank Alessandro, Andrea, Benedetto, Cindy, Fabrizio, Francesca, Markus, Niccolò, Paolo, Sebastiano, Stefano, Ulisse... - and all others too.

For the constant love, support and patience during these years - and ever before - I thank my family.

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Contents

1 Combinatorial geometries

In their book “On the foundations of combinatorial theory”, Crapo and Rota qualified the term *matroid* as “ineffably cacophonic” [33, §1.5]. Alas, it has become nowadays quite standard. Even if the terminology evolved in a different way than Crapo and Rota might have desired, their wishes were fulfilled in what concerns the idea of the book: that a unified treatment of the subject would have been fruitful for the different fields of mathematics involved. As a tribute to this idea, our title reflects the name suggested by Crapo and Rota.

It is outside of the scope of this short introduction to carefully review the history of the subject (we refer the historically interested reader to [66]). Here we will present some basics, along with some applications of algebraic and topological flavor. The structure of this chapter will hopefully convey the perspective through which the remainder of the material in the book should be viewed. For proofs and further comments on our review of matroids we refer to any textbook, for example [77], with the exception of the rather new ‘modular elimination axiomatization’ of Theorem 1.2.8, published in [42]. For a first (perhaps not standard) definition we turn to a 1987 paper by Gel’fand, Goresky, MacPherson and Serganova [59].

The *hypersimplex* Δ^k_{n-k} is the convex polyhedron in $\mathbb{R}^n$ spanned by the vectors $e_J := \sum_{j\in J} e_j$, where $e_1,\ldots,e_n$ is the standard basis of $\mathbb{R}^n$ and $J \subseteq \{1,\ldots,n\}$ is of cardinality k.

Definition 1.0.2 (Theorem 4.1 of [59])**.** A *matroid of rank k on n elements* is given by a polyhedron $\mathbf{P}$ such that the vertices of $\mathbf{P}$ are vertices of Δ^k_{n-k} and such that every edge of $\mathbf{P}$ is parallel to $e_i - e_j$ for some $1 \le i, j \le n$.

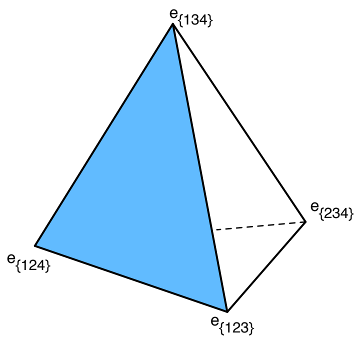


Figure 1.1: The hypersimplex Δ^4_1 with a matroid $\mathbf{P}$ (shaded).

One may define a matroid on any finite ground set of elements E by letting $n := |E|$ and choosing a bijection between the standard basis vectors of $\mathbb{R}^n$ and the elements of E. For an account of the problem of considering matroids on infinite ground sets (and a suggestion of solution thereof) see [28].

1.1 Cryptomorphisms

Some reader may (rightfully) have wondered about the connection between Definition 1.0.2 and the objects of the introductory Example 0.0.1. Indeed, matroid theory can be (and has been) approached from many different points of view, and it is precisely this variety of interpretations that makes the theory so rich. There is even a peculiar word that is customarily used to describe these equivalences.

Definition 1.1.1. Two definitions D_1, D_2 are said to be *cryptomorphic* if there is a bijective correspondence between the class of objects defined by D_1 and the class of objects defined by D_2. This correspondence is called *cryptomorphism.*

Case study: matrices

The roots of matroid theory are commonly traced back to a paper of Hassler Whithey [94], where abstract properties of linear dependence among points in vector spaces (thus the word "matroid") were linked to abstract dependence properties in graphs. To illustrate the situation we consider the case of linear dependency.

Let $v_1, \ldots, v_n$ be the columns of a matrix M over an arbitrary field and consider the following data sets:

1. the set $\mathbf{B}(M)$ of all $B \subseteq \{1, \ldots, n\}$ such that $\{v_b : b \in B\}$ are bases for the column space of M
2. $\mathbf{V}(M) := \{\text{supp}(x)^1 : x \in \ker(M)\}$
3. the set $\mathbf{C}(M)$ of minimal nonempty elements of $\mathbf{V}(M)$
4. $\mathbf{V}^*(M) := \{\text{supp}(x) : x \in \text{row}(M)\}$
5. the set $\mathbf{C}^*(M)$ of minimal nonempty elements of $\mathbf{V}^*(M)$.

For each of these data sets, in matroid theory we find a set of combinatorial axioms satisfied by that data set: $\mathbf{B}(M)$ satisfies the *basis axioms*, $\mathbf{V}(M)$ and $\mathbf{V}^*(M)$ satisfy what we call *vector axioms*, $\mathbf{C}(M)$ and $\mathbf{C}^*(M)$ satisfy the *circuit axioms.* The axioms will be stated in Section 1.1.

Two important points:

1. Each of these data sets associated to M determine each of the other data sets. The *matroid* of M is defined to be the information about M encoded by any one of these data sets. The definitions coming from the various data sets are *cryptomorphic.*

[1] Here the *support* of a vector is the set of indices of its nonzero coordinates.

2. The sets $\mathbf{V}(M)$ and $\mathbf{V}^*(M)$ arise from a vector space $\ker(M)$ and its orthogonal complement $\mathrm{row}(M)$. Given the matrix M, there is a matrix N whose row space is $\ker(M)$ and whose kernel is $\mathrm{row}(M)$, and thus $\mathbf{V}(N) = \mathbf{V}^*(M)$ and $\mathbf{C}(N) = \mathbf{C}^*(M)$.

In general

With the particular case of matrices in mind, we are ready to give some alternative abstract definitions of a matroid, and to illustrate their equivalence.

Definition 1.1.2.

1. A family $\mathbf{B} \subseteq 2^E$ of subsets of E is the set of *bases of a matroid* M if and only if $\mathbf{B} \neq \emptyset$ and

($\mathbf{B}1$) given $B_1, B_2 \in \mathbf{B}$ and $e \in B_1 \setminus B_2$, there is $f \in B_2 \setminus B_1$ such that $(B_1 \setminus e) \cup f \in \mathbf{B}$ *(the Basis Exchange Axiom).*

2. A family $\mathbf{V} \subseteq 2^E$ is the set of *vectors* of a matroid on the ground set E if and only if $E \in \mathbf{V}$ and

($\mathbf{V}1$) if $X_1, X_2 \in \mathbf{V}$ then $X_1 \cup X_2 \in \mathbf{V}$

($\mathbf{V}2$) if $X \in \mathbf{V}$ and $\{Y_1, \ldots, Y_k\}$ is the set of maximal elements of $\mathbf{V}$ properly contained in X, then the sets $X - Y_1, \ldots, X - Y_k$ partition X.

3. A family $\mathbf{C} \subset 2^E$ is the set of *circuits* of a matroid on the ground set E if and only if $\emptyset \notin \mathbf{C}$ and

($\mathbf{C}1$) if $C_1, C_2 \in \mathbf{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$ *(Incomparability).*

($\mathbf{C}2$) if $C_1, C_2 \in \mathbf{C}$ are distinct and there is an element $e \in E$ with $e \in C_1 \cap C_2$, then there is $C_3 \in \mathbf{C}$ with $C_3 \subseteq (C_1 \cup C_2) \setminus e$ *(Elimination).*

It is easy to check from the definition that all bases of a matroid have the same size ([77, Lemma 1.2.1]). Thus we can define the *rank* of a matrix to be the size of any basis.

Definition 1.1.3 (Rank)**.** Let M be a matroid on the ground set E with set of bases $\mathbf{B}$, and let $A \subseteq E$. Define the *rank* of A to be

$$\mathrm{rank}(A) := \max\{|A \cap B| \mid B \in \mathbf{B}\}.$$

The notion of rank defines a closure operator on E:

Definition 1.1.4 (Closure)**.** Let M be a matroid on the ground set E. Given $A \subset E$ define

$$\mathrm{cl}(A) := \max\{A' \subseteq E \mid A \subseteq A', \mathrm{rank}(A) = \mathrm{rank}(A')\}.$$

The function $\mathrm{cl} : E \to E$ is the *closure operator* of M.

Observation 1.1.5. The basis and circuit axioms are well-known to any student of matroids. The vector axioms are the obvious modification of Crapo's *axioms for flats* (first stated in [32], see also [77, §1.5.11]), which describe the set

$$\mathbf{F} := \{E - X : X \in \mathbf{V}\} = \{F \subseteq E : \mathrm{cl}(F) = F\}.$$

A detailed introduction to the axiomatics of matroids can be found in [77, Chapter 1 and 2].

There are sets satisfying the various axiom systems discussed above which do not arise from matrices. If a matroid arises from a matrix with coefficients in a field $\mathbb{K}$, it is called *realizable over* $\mathbb{K}$. However, just as for those arising from matrices, each set satisfying one of the axiom systems determines sets satisfying the other axiom systems. Thus we can refer to a *matroid with basis set* $\mathbf{B}$*, vector set* $\mathbf{V}$*, circuit set* $\mathbf{C}$.

To briefly state the cryptomorphisms:

- Given $\mathbf{P}$ the polytope of a matroid as in Definition 1.0.2, the set of indices of the nonzero coordinates of the vertices of $\mathbf{P}$ are the bases of a matroid on the ground set $\{1, \dots, n\}$.
- Given $\mathbf{B}$ the set of bases of a matroid on the ground set E, let $n := |E|$ and choose a bijection $f : E \to \{1, \dots, n\}$. Then
$$\mathbf{P} := \mathrm{conv}\{\sum_{i \in f(B)} e_i : B \in \mathbf{B}\}$$
is the polytope of a matroid of rank $k = \mathrm{rank}(E)$ on n elements.
- Given $\mathbf{B}$ the set of bases of a matroid, we say $A \subseteq E$ is *dependent* if it is not contained in a basis. The set $\mathbf{C}$ of all minimal dependent sets is the set of circuits of a matroid.
- Given $\mathbf{C}$ the set of circuits of a matroid, $\mathbf{V}$ is the set of all unions of elements of $\mathbf{C}$ (including the empty union).
- Given $\mathbf{V}$ the set of vectors of a matroid, we say that $A \subseteq E$ is a *basis* if A is maximal among sets not containing a vector. The set $\mathbf{B}$ of all bases is the set of bases of a matroid.

Duality

Definition 1.1.6. For $\mathbf{S} \subseteq 2^E$, we define $\mathbf{S}^\perp := \{A \subseteq E \mid \forall B \in \mathbf{S}\ |A \cap B| \neq 1\}$.

Theorem 1.1.7. *(cf. [77]) If M is a matroid with ground set E, basis set $\mathbf{B}$, vector set $\mathbf{V}$, and circuit set $\mathbf{C}$, then there is a matroid M^* with ground set E, basis set $\mathbf{B}^* := \{E \setminus X \mid X \in \mathbf{B}\}$, vector set $\mathbf{V}^* := V^\perp$, and circuit set $\mathbf{C}^*$ the set of minimal nonempty elements of V^*.*

If M is realized by a matrix with row space W, then M^* is realized by a matrix with row space $W^\perp$.

Definition 1.1.8. The matroid M^* in the statement of the previous theorem is called the *dual* to M. The sets $\mathbf{V}^*$ and $\mathbf{C}^*$ of the previous theorem are called the set of *covectors* resp. *cocircuits* of M.

We will make frequent use of the following basic fact. It follows immediately from our definitions, but we state it here for later reference.

Lemma 1.1.9 (Proposition 2.1.20 of [77])**.** *Let C be a circuit and D be a cocircuit of a matroid M. Then $|C \cap D| \neq 1$. In fact, the set*

$$\min\{D \subseteq E \mid D \neq \emptyset,\ |D \cap C| \neq 1 \text{ for all } C \in \mathbf{C}\},$$

where min *denotes inclusion-minimality, is the set of cocircuits of M.*

1.2 Modular elimination

Geometric lattices

A partially ordered set (or *poset*) is a pair $(P, \leq)$ consisting of a set P endowed with a partial order (i.e., a reflexive, antisymmetric and transitive binari relation) $\leq$. The *opposite* of the poset $(P, \leq)$ is the poset defined on the same set by the ordering $\leq^{\mathrm{op}}$ such that $x \leq y$ if and only $y \leq^{\mathrm{op}} x$ for all $x, y \in P$. If the ordering is understood we will write just P and P^{op} for a poset and its opposite.

As a general reference on the combinatorics of posets and lattices we refer to [90, Chapter 3]. Here let us only recall that a *chain* J in a poset $(P, \leq)$ is any totally ordered subset of P; the *length* of the chain J is then $\ell(J) := |J| - 1$. Given $x \in P$ we write $P_{\geq x} = \{x' \in P \mid x' \geq x\}$ and $P_{\leq x} = \{x' \in P \mid x' \leq x\}$. The *length* of P is $\ell(P) := \max\{\ell(J) \mid J \text{ a chain of } P\}$, and for $x \in P$ write $\ell(x) := \ell(P_{\leq x})$. The poset P is *graded* if all maximal chains of P have the same length. In this case, there is a unique *rank function* $\mathrm{rank} : P \to \{0, \dots \ell(P)\}$ such that $\mathrm{rank}(x) = 0$ if x is a minimal element of P, and $\mathrm{rank}(y) = \mathrm{rank}(x) + 1$ if y covers x.

Given two elements $x, y \in P$, we say that y *covers* x, written $x \lessdot y$, if $x \leq y$ and $|P_{\geq x} \cap P_{\leq y}| = 2$.

If for any $x, y \in P$ the poset $P_{\geq x} \cap P_{\geq y}$ has a unique minimal element, this element is denoted $x \vee y$ and called the *join* of x and y. Analogously we call $x \wedge y$, or *meet* of x and y, the unique maximal element of $P_{\leq x} \cap P_{\leq y}$, if it exists. The poset P is called a *lattice* if meet and join are defined for every pair of elements of P. In particular, every finite lattice has a unique minimal element, called $\widehat{0}$, and a unique maximal element, called $\widehat{1}$. In any poset with a unique minimal element $\widehat{0}$, the elements a with $\widehat{0} \lessdot a$ are called *atoms*. The lattice L is called *atomic* if every $x \in L$ is $x = \bigvee A$ for some set A of atoms of L.

Definition 1.2.1. A graded (finite) lattice L is called *semimodular* if

$$\mathrm{rank}(x) + \mathrm{rank}(y) \geq \mathrm{rank}(x \wedge y) + \mathrm{rank}(x \vee y)$$

for all $x, y \in L$.

Definition 1.2.2. A finite lattice L is *geometric* if it is atomic and semimodular.

Given a matroid M, we now consider its set of *flats* $\mathbf{F}$, and order it by containment, so $F \leq G$ if $F \subseteq G$. This defines a poset $\mathbf{L}$ where meet and join are defined on $\mathbf{L}$ by

$$F \vee G = \mathrm{cl}(F \cup G), \quad F \wedge G = F \cap G$$

and therefore it is called the *lattice of flats* of the matroid M. In fact $\mathbf{L}$ is a geometric lattice - but even more is true.

Theorem 1.2.3. *[77, 1.7.5] A finite lattice* $\mathbf{L}$ *is geometric if and only if it is the lattice of flats of a matroid.*

Remark 1.2.4. The lattice of flats of a matroid does not characterize a matroid, the problem being that in general there could be elements e of the ground set E that either are not contained in any basis (such an e is called a *loop* of M, characterized by $\operatorname{rank}_M(e) = 0$), or are contained in every basis (such an e is a *coloop* of M, and fulfills $\operatorname{rank}_{M^*}(e) = 0$).

A matroid without loops or coloops is called *simple*. It is then true that two simple matroids are isomorphic if and only if their lattices of flats are.

The maximal proper flats of a matroid M are called *hyperplanes*. Thus, in a matroid of rank r, all hyperplanes have rank $r - 1$. One can check that the cocircuits of M are exactly the complements of hyperplanes of M.

Remark 1.2.5. There are two obvious "relatives" of the poset $\mathbf{L}$: namely the posets obtained by ordering $\mathbf{V}$ and $\mathbf{V}^*$ by containment. By [77, 2.1.6], set theoretic complementation gives an isomorphism of posets

$$\mathbf{V}^* \simeq \mathbf{L}^{\mathrm{op}}.$$

In particular, $\mathbf{V}$ and $\mathbf{V}^*$ are graded lattices - but not necessarily geometric.

The modular elimination axiom

Definition 1.2.6. Two elements A, B of a ranked lattice L are a *modular pair* if the inequality of Definition 1.2.1 is sharp, i.e., if

$$\operatorname{rank}(A) + \operatorname{rank}(B) = \operatorname{rank}(A \wedge B) + \operatorname{rank}(A \vee B).$$

Remark 1.2.7. In particular, we note

(1) two circuits A, B are a modular pair in $\mathbf{V}$ if and only if $\operatorname{rank}(A \vee B) = 2$, and

(2) two hyperplanes are a modular pair of flats if and only if their complements are a modular pair of cocircuits.

We take inspiration from Remark 1.2.7.(1) to give a definition of modularity for members of a collection of incomparable sets which, in the case where this collection is known to be the set of circuits of a matroid, reduces to Definition 1.2.6. We will need its full generality in the statement of Definition 6.1.3.

Definition 1.2.8. Given any family $\mathbf{S}$ of subsets of a set E, consider the set

$$U(\mathbf{S}) := \{\bigcup \mathbf{T} \mid \mathbf{T} \subseteq \mathbf{S}\}$$

partially ordered by inclusion - so, for $A, B \in U(\mathbf{S})$, $A \leq B$ if $A \subseteq B$.

If the members of $\mathbf{S}$ are incomparable (i.e., none is contained in another), then $U(\mathbf{S})$ is a lattice with $\widehat{0} = \emptyset$ where join and meet of any two elements $A, B \in U(\mathbf{S})$ are defined by $A \vee B := A \cup B$, $A \wedge B := \bigcup\{S \in \mathbf{S} | S \subseteq A \cap B\}$. This lattice is atomic by definition. We will say that two members of $\mathbf{S}$ are a *modular pair* if they are a modular pair in $U(\mathbf{S})$.

The definition of complex matroids in terms of phased circuits rests on the following strengthening of the axiomatization of matroids in terms of circuits.

Theorem 1.2.9 ([42]). *A collection* $\mathbf{C}$ *of incomparable nonempty subsets of a ground set* E *is the set of circuits of a matroid if and only if the Elimination property* ($\mathbf{C}2$) *of Definition 1.1.2 holds for all modular pairs* C_1, C_2 *of elements of* $\mathbf{C}$.

To prove the theorem it will be useful to define a general form of the 'elimination property' between sets. Given two elements C_1, C_2 of a collection $\mathbf{C}$ of incomparable subsets of a ground set E let

$$\mathscr{E}(C_1, C_2, \mathbf{C}): \text{ for all } e \in C_1 \cap C_2 \text{ there is } C_3 \in \mathbf{C} \text{ with } C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}.$$

Theorem 1.2.10 (Theorem 1 of [42]). *Let* $\mathbf{C}$ *be a collection of incomparable finite subsets of a set* E*. If* $\mathscr{E}(A, B, \mathbf{C})$ *for all modular pairs* $A, B \in \mathbf{C}$*, then* $\mathscr{E}(A, B, \mathbf{C})$ *for all pairs* $A, B \in \mathbf{C}$.

Proof. Take $A, B \in \mathbf{C}$ with $A \neq B$, $e \in A \cup B$ and let $Z := A \cup B = A \vee B$. We want to show that a $C \in \mathbf{C}$ exists with $e \notin C \subseteq Z$. The finiteness requirement on the cardinality of the elements of $\mathbf{C}$ ensures that $\ell(X) < \infty$ for all $X \in U(\mathbf{C})$. We will proceed by induction on $\ell(Z)$.

If $\ell(Z) = 2$ then A, B is a modular pair and we are done. Suppose now $\ell(Z) = n > 2$ and let J be a chain of maximal cardinality in $U(\mathbf{C})_{\leq Z}$. The chain J contains exactly one element $A' \in \mathbf{C}$ and at least an element Y with $A' \lneq Y \lneq Z$. If $e \notin A'$ we are done with $C := A'$. Else, since $U(\mathcal{C})$ is atomic, there is $B' \in \mathbf{C}$ with $A' \vee B' \leq Y$. Again, if $e \notin B'$ then $C := B'$ does it; otherwise $e \in A' \cap B'$ and we may apply the inductive hypothesis to the pair A', B' (because $Y < Z$ implies $\ell(Y) < \ell(Z)$), obtaining C as desired. □

Theorem 1.2.9 now follows immediately by restricting to finite sets E.

1.3 Arrangements

We have now to describe the geometric-topologic part of our introductory Example 0.0.1.

Let then A be a matrix (say, $d \times n$) with entries from a field $\mathbb{K}$. The columns $a_1, \ldots, a_n$ of A define linear forms

$$\alpha_i : x \mapsto x \cdot a_i$$

by taking scalar product, and hyperplanes

$$H_i := \ker \alpha_i.$$

The *arrangement of hyperplanes* defined by A is the set

$$\mathscr{A} := \{H_1, \ldots, H_n\}.$$

The *complement* of the arrangement is $\mathscr{M}(\mathscr{A}) := \mathbb{K} \setminus \bigcup \mathscr{A}$.

The poset $\mathcal{L}(\mathscr{A})$ of all subspaces of the form $X = \bigcap_{i \in I} H_i$ for a subset $I \subset \{1, \ldots, n\}$ (where we let the empty intersection be the whole $\mathbb{K}$) partially ordered by reverse containment is a geometric lattice (graded by the codimension). Thus, there is a matroid M on the ground set $\{1, \ldots, n\}$ associated to $\mathscr{A}$. For a fixed $H \in \mathscr{A}$, we call $\mathscr{A}' := \mathscr{A} \setminus \{H\}$ the *deletion* of H from $\mathscr{A}$ and $\mathscr{A}'' := \{H \cap H' : H' \in \mathscr{A}'\}$ the *contraction* of $\mathscr{A}$ to H. Then the matroid associated to $\mathscr{A}'$ resp. $\mathscr{A}''$ is $M \setminus i$ resp. M/i, where i is the index for which $H = H_i$ (see also Definition 1.5.1).

Question 1.3.1. What information about $\mathscr{M}(\mathscr{A})$ is encoded in the matroid M?

Of course one can ask for different sorts of 'information', depending on the field $\mathbb{K}$ one has chosen. One question one can always ask is about the *topology* of the complement.

Zaslavski's enumeration

If $\mathbb{K} = \mathbb{R}$, then $\mathscr{M}(\mathscr{A})$ consists topologically of a certain number of contractible connected components (called *chambers* or *regions*). So the only topological data is the number of regions.

In his Ph.D. thesis, Zaslavsky [97] obtains several enumerative results about the cells of the stratification of $\mathbb{R}^d$ given by $\mathscr{A}$ in terms of the combinatorics of the associated matroid. In particular, the number of regions is computed as an evaluation of a polynomial invariant of the matroid.

Definition 1.3.2. Let M be a matroid on the ground set E. Its *characteristic polynomial* is

$$\chi_M(t) := \sum_{X \subseteq E} (-1)^{|X|} t^{\operatorname{rank}(E) - \operatorname{rank}(X)}$$

Theorem 1.3.3 (See [97]). *The number of regions of a real hyperplane arrangement with matroid M is*

$$|\chi_M(-1)|$$

The proof of the theorem relies on following basic fact about characteristic polynomials of matroids of arrangements.

Remark 1.3.4. Let M be the matroid of an arrangement on the ground set E. If $e \in E$ is not a loop nor a coloop, then

$$\chi_M(t) = \chi_{M/e}(t) + \chi_{M \setminus e}(t).$$

If $\mathbb{K}$ is a finite field, e.g. $\mathbb{F}_p$ for a prime number p, then the complement $\mathscr{M}(\mathscr{A})$ consists of all (discrete) points that do not satisfy (mod q) the 'defining equations' given by the linear forms. The number of such points can be computed from the characteristic polynomial of the associated matroid as well. We give here the general statement of the theorem and refer to [7, §2] for an account of the history of this enumerative problem and a discussion of the theorem (in particular of the assumption on the size of p).

Theorem 1.3.5 (Theorem 2.2 of [7]). *If the arrangement $\mathscr{A}$ is defined over the integers and p is a large enough prime, then*

$$\chi_M(p) = |\mathbb{F}_p^n \setminus \bigcup \mathscr{A}|.$$

The Orlik-Solomon algebra

If $\mathbb{K} = \mathbb{C}$, then $\mathscr{M}(\mathscr{A})$ is a connected topological space, and the question about the topology becomes far more subtle. In this section we briefly describe a landmark result by Brieskorn [26] and Orlik and Solomon [75], referring to the excellent and very readable survey by Yuzvinsky [96] for further details and references.

Let a matroid M on the ground set $E = \{e_1, \dots, e_n\}$ be given. Let R be a commutative ring and let U denote the exterior algebra over U generated in degree 1 by the elements of E. The algebra U is graded as

$$U = \bigoplus_{i=1}^{n} U_i, \quad \text{where} \quad U_1 = \oplus_{i=1}^{n} R e_i \quad \text{and} \quad U_i = \bigwedge{}^i U_1$$

For every index i the module U_i is free with basis the monomials of type $e_J := e_{j_1} \cdots e_{j_i}$ where $J = \{e_{j_1}, \dots, e_{j_i}\}$ ranges over the i-element subsets of E and $j_1 < \dots < j_i$.

For a given $J = \{e_{j_1}, \dots, e_{j_k}\} \subseteq E$ set

$$\partial e_J := \sum_{i=1}^{k} (-1)^{i-1} e_{(J \setminus e_{j_i})}$$

Definition 1.3.6 (See Definition 2.1 and Lemma 2.2 of [96])**.** The *Orlik-Solomon ideal* of a matroid M with set of circuits $\mathbf{C}$ is the ideal $I(M)$ of U generated by the monomials e_J with $\operatorname{rank}(J) > |J|$ and by the elements ∂e_C with $C \in \mathbf{C}$.

The *Orlik-Solomon algebra* $OS(M)$ *of* M is the quotient algebra $U/I(M)$.

Remark 1.3.7. The grading of U carries over to a grading of $OS(M)$. For $i = 0, \dots d$ a basis of $OS(M)_i$ is given by the monomials e_N where N ranges over the no-broken-circuit sets of size i.

Now we can state the main result of [75]: the cohomology ring of $\mathscr{M}(\mathscr{A})$ is determined by the associated matroid.

Theorem 1.3.8 (See Theorem 4.4 of [96])**.** *Let $\mathscr{A}$ be an arrangement of hyperplanes defined over $\mathbb{C}$ and M the associated matroid. Then the Orlik-Solomon algebra $OS(M)$ (with $R = \mathbb{Z}$) is naturally isomorphic to the cohomology ring (over the integers) of $\mathscr{M}(\mathscr{A})$.*

To prove this theorem one constructs an explicit isomorphism of $OS(M)$ with the de Rham complex for $\mathscr{M}(\mathscr{A})$. In turn, the construction relies on the following remarkable fact.

Lemma 1.3.9 (Corollary 2.13 of [96])**.** *If M is the matroid of the arrangement $\mathscr{A}$ and e is the element of the ground set of M corresponding to $H \in \mathscr{A}$, then there are short exact sequences*

$$0 \to OS(M \setminus e) \to OS(M) \to OS(M/e) \to 0$$

and hence

$$0 \to H^*(\mathscr{M}(\mathscr{A}')) \to H^*(\mathscr{M}(\mathscr{A})) \to H^*(\mathscr{M}(\mathscr{A}'')) \to 0$$

Randell's lattice-isotopy theorem

Consider smooth functions $\alpha_1, \ldots, \alpha_n : [0,1] \to \mathbb{K}^d$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ As above, for all $t \in [0,1]$ the vectors $\alpha_1(t), \ldots, \alpha_n(t)$ define an arrangement $\mathscr{A}_t$ and a corresponding matroid M_t.

Definition 1.3.10. The family $(\mathscr{A}_t)_{t\in[0,1]}$ is a *lattice isotopy* if for all $t_1, t_2 \in [0,1]$ the matroids M_{t_1} and M_{t_2} are isomorphic.

Two matroids are isomorphic if there is a bijection between their ground sets which preserves the set of circuits (or, equivalently, the sets of bases, or vectors, or...). Since the matroids appearing in this definition are simple, this amounts to requiring isomorphy of the lattices of flats, hence the name.

Theorem 1.3.11 (Randell's Lattice Isotopy Theorem, see [80])**.** *If $(\mathscr{A}_t)_{t\in[0,1]}$ is a lattice isotopy, then $\mathscr{M}(\mathscr{A}_0)$ and $\mathscr{M}(\mathscr{A}_1)$ are diffeomorphic.*

$K(\pi, 1)$ arrangements

One of the earliest interests in the topology of arrangements has been to determine when some spaces of the form $\mathscr{M}(\mathscr{A})$ are $K(\pi, 1)$ spaces (or *aspherical*, meaning that the homotopy groups vanish in degree bigger than 1). For a sketch of the motivation, see Section 2.3.

The so-called *$K(\pi, 1)$-problem* asks whether asphericity of $\mathscr{M}(\mathscr{A})$ is determined by the matroid of $\mathscr{A}$.

We mention two notable results which support a positive answer. Hattori [61] proved asphericity of *general position* arrangements (i.e., a genericity condition on the lattice of flats $\mathbf{L}(\mathscr{A})$). Combining work of Falk and Randell [55] and Terao [92], we obtain asphericity also in the case where $\mathbf{L}(\mathscr{A})$ is *supersolvable*, giving rise to *fiber-type arrangements*.

To date, this longstanding problem is still open in its generality.

Rybnikov's examples

The optimism which may have arisen in view of the results of Orlik-Solomon and of Randell turns out to be misleading: in general, the matroid of a complex hyperplane arrangement does not determine the homotopy type of its complement. This has been shown first by G. Rybnikov [85], who exhibited two arrangements $\mathscr{A}_1$ and $\mathscr{A}_2$ with isomorphic matroids but for which the fundamental groups $\pi_1(\mathscr{M}(\mathscr{A}_1))$ and $\pi_1(\mathscr{M}(\mathscr{A}_2))$ are different.

Topological representation of matroids

Recently, Swartz proved a "topological representation theorem for matroids", that is to say, he set up a bijection between isomorphism classes of simple matroids and the homotopy classes of *arrangements of homotopy spheres.* The result of Swartz has been improved successively by Anderson [2] and Engström [53]. To give a complete account of this new development would exceed the scope of this introductory section, and probably fail its goal, which is to convey the flavor of the results. We thus prefer to give a sketch of the main ideas which, albeit partial, we hope will highlight the general ideas and principles. For a complete and rigorous treatment we refer to the cited original sources.

Definition 1.3.12 (Definition 5.1 of [91]). A *homotopy d-sphere* is a CW-complex of dimension d which is homotopy equivalent to the d dimensional sphere S^d. The empty set is a homotopy (-1)-sphere.

A d-arrangement of homotopy spheres consists of a d-dimensional homotopy sphere S and a finite set of subcomplexes $\mathscr{A} = \{S_1, ..., S_n\}$ of S each of which is a homotopy $(d-1)$-sphere. Moreover,

(a) Every intersection of homotopy spheres in $\mathscr{A}$ is a homotopy sphere.

(b) If X is an intersection in $\mathscr{A}$ which is a k-dimensional homotopy sphere and $X \not\subseteq S_j$, then $X \cap S_j$ is a $(k-1)$-dimensional homotopy sphere.

Swartz goes on to restrict further this notion. The arrangement $\mathscr{A}$ is *fully partitioned* if the $(d-1)$-skeleton of the sphere S is contained in the subcomplex $\bigcup \mathscr{A}$; and it is *regular* with respect to a distinguished $S_j \in \mathscr{A}$ if all other S_i 'intersect nicely along S_j' (see Definition 5.9 of [91] for the precise definition).

Theorem 1.3.13 (Theorems 5.3 and 6.1 of [91]). *The poset of intersections of a (fully partitioned) d-arrangement of homotopy spheres $\mathscr{A}$ is a geometric lattice.*

Conversely, given a geometric lattice $\mathbf{L}$ *and a distinguished atom a of* $\mathbf{L}$*, there is a fully partitioned d-arrangement of homotopy spheres $\mathscr{A}$ which poset of intersections is isomorphic to* $\mathbf{L}$ *and which is regular with respect to the homotopy sphere corresponding to a under the isomorphism.*

If desired, the construction can be carried out so as to admit a fixed-point free involution on S.

As a comment to this result one should point out that the 'homotopy spheres' involved here are a rather general object - for instance, they must not be manifolds at all. Also, the proof of Swartz does not yield an explicit construction of the arrangement and depends on a large number of (non canonical) choices.

Inspired by Swartz's result, Laura Anderson [2] gave a version of the Representation Theorem with a completely explicit construction which depends only on the choice of a maximal chain in the lattice of flats. Moreover, for matroids that are *orientable*, there is a nice structural relationship between the topological representation of the matroid and the representations of its orientations given in Folkman and Lawrence's Topological Representation Theorem for oriented matroids (see next chapter). This nice fact is 'paid for' by the fact that Anderson's construction allows the homotopy spheres to be of much larger dimension that their ranks in the lattice of flats.

Alexander Engström [53] used the techniques of homotopy colimits of diagrams of spaces to give a topological representation of matroids as arrangements of homotopy d-fold joins X^{*d} inside a homotopy $d+1$ fold join $X^{*(d+1)}$, where X is any space. In the case where X is the 0-sphere S^0, this gives a topological representation as an arrangement of homotopy spheres "of the correct dimension". Moreover, every free group action on X induces a free action of the same group on the representation and each subspace intersection.

1.4 Grassmannians

We close this introductory chapter with a mention of the result that motivated Definition 1.0.2 in [59].

Let G^n_{n-k} be the Grassmann manifold of $(n-k)$-dimensional subspaces of $\mathbb{C}^n$ (the *complex Grassmannian*). To every subspace $V \in G^n_{n-k}$ we associate the vectors $v_1, \dots, v_n$ defined as the projections of the standard basis to the quotient $\mathbb{C}^n/V$. We can then associate to V the matroid M_V defined by the matrix with columns $v_1, \dots, v_n$. This is the matroid with set of vectors $\mathbf{V} = \{\mathrm{supp}(x) : x \in V\}$ (the analogous construction with the orthogonal complement to V, as noted before, will yield the dual matroid).

The *complex torus* $(\mathbb{C}^*)^n$ acts on the standard basis of $\mathbb{C}^n$ and thus on G^n_{n-k}. For $V \in G^n_{n-k}$ let $\overline{V}$ denote the closure of the orbit of V under this action.

Definition 1.4.1 (Definition and Proposition 2.1 of [59])**.** Given $V \in G^n_{n-k}$ and $J \subseteq \{1, \dots, n\}$ let M_J denote the matrix with column set $(v_j)_{j \in J}$. The *moment map* associated with the above torus action is the map

$$\mu : G^n_{n-k} \to \mathbb{R}^n$$

with i-th coordinate

$$\mu_i(V) := \frac{\sum_{\substack{|J|=k \\ i \in J}} |\det M_J|^2}{\sum_{|J|=k} |\det M_J|^2}$$

The image of the moment map satisfies

$$\mu(G^n_{n-k}) = \Delta^n_{n-k}.$$

Theorem 1.4.2 ([59])**.** *For $V \in G^n_{n-k}$, let $\mathbf{P}_V$ be the polyhedron of the matroid associated to V. Then*

$$\mathbf{P}_V = \mathrm{closure}(\mu(\overline{V})).$$

1.5 More tools

We close this chapter on combinatorial geometries by reviewing some technical material that will be needed in the exposition of the following chapters.

Minors

Definition 1.5.1 (Section 3.1 of [77])**.** Let M be a matroid on the ground set E with set of bases $\mathbf{B}$, and let $A \subseteq E$. Choose $\{a_1, \dots, a_l\}$ a maximal independent set in A. We define

(1) the *contraction* M/A as the matroid given by the set of bases

$$\mathbf{B}(M/A) := \{B \subset E \mid B \cup \{a_1, \dots, a_l\} \in \mathbf{B}\}$$

(2) the *deletion* $M \setminus A$ as the matroid with set of bases

$$\mathbf{B}(M \setminus A) := \max\{B \setminus A \mid B \in \mathbf{B}\},$$

where max denotes inclusion-maximality.

For any $A \subseteq E$, we let $M(A)$ denote $M\backslash(E\backslash A)$. The matroids M/A, $M\setminus A$, $M(A)$ are called *minors* of M. In fact, in the representable case they encode data related to the minors of the original matrix.

Lemma 1.5.2 (Section 3.1 of [77])**.** *The contraction and deletion of a matroid M on the ground set E can also be defined by means of their set of circuits:*

$$\mathbf{C}(M \setminus A) = \{C \in \mathbf{C}(M) \mid C \cap A = \emptyset\},$$

$$\mathbf{C}(M/A) = \min\{C \setminus A \mid C \in \mathbf{C}(M),\ C \nsubseteq A\}.$$

Moreover, the operations of contraction and deletion are dual to each other in the sense that

$$(M/A)^* = M^* \setminus A.$$

The basis graph

We turn to the study of the 1-skeleton of the polytope $\mathbf{P}$ through which we first defined a matroid. As we have already seen, its vertices are naturally associated to bases of the matroid. The condition of parallelism on the edges can be then interpreted as an 'exchange' of an element happening 'along an edge' between the bases associated to the vertices incident to it. Let us phrase this in precise terms.

Lemma 1.5.3 (See [77], Corollary 1.2.6)**.** *If B is a basis of a matroid M on the ground set E and $e \in E \setminus B$ then there is a unique circuit $X \subseteq B \cup \{e\}$, called the* basic circuit *of e with respect to B and denoted by $C(B, e)$. In particular, for any pair of bases of the form $B_1 = A \cup e_1, B_2 = A \cup e_2$ there is a unique circuit supported on $B_1 \cup B_2$.*

Definition 1.5.4. (See [69]) The *basis graph* of a matroid M with set of bases $\mathbf{B}$ is the simple graph with vertex set

$$V(G) := \mathbf{B}$$

and edge set

$$E(G) := \{\{B_1, B_2\} \mid B_1 = A \cup e_1, B_2 = A \cup e_2 \text{ for some } e_1 \neq e_2 \in B_2 \setminus A\}.$$

Thus the edge between two vertices $B_1 = A \cup e_1$, $B_2 = A \cup e_2$ can be associated with the circuit $C(B_1, e_2) = C(B_2, e_1)$.

Maurer gave a thorough treatment of these graphs, giving for instance a complete characterization of which graphs are basis graphs of a matroid. For this paper we will only need the following Theorem 1.5.5.

A sequence of edges $e_1, \ldots, e_k$ in a graph G is a *path from the vertex A to the vertex B* if for all $j \in \{1, \ldots, k-1\}$, e_j and e_{j+1} share a vertex and if A (resp. B) is the vertex of e_1 (resp. e_k) that is not shared with e_2 (resp. e_{k-1}). We say that an *elementary move* on the given path is the substitution of any subpath $e_j e_{j+1}$ with another path consisting of at most two edges of G, and such that the replacement yields again a path. The *trivial path* is the path corresponding to an empty sequence of edges.

Theorem 1.5.5 ([69])**.** *Let G be the basis graph of a matroid M and choose a vertex A of G. Then every closed path in G from A to A can be reduced to the trivial path by a sequence of elementary moves and of inverses thereof.*

Part I

$\mathbb{R}$

2 Preliminaries: Euclidean space

The combinatorics and the topology of a geometric or algebraic object in a real vector space is vastly enriched by its Euclidean structure. In this chapter we give a glimpse at how this additional structure is modeled combinatorially in the theory of *oriented matroids*. With the exception of Theorem 2.2.10, which appeared recently in [42], the material covered here is standard and the interested reader is referred to the textbook of Björner, Las Vergnas, Sturmfels, White and Ziegler [20] for further detail and complete proofs.

Our goal here is to quickly review the material needed later, and to do so in a way that lends itself to (and inspires) the generalization of the forthcoming chapters on Complex Matroids. For the same reason our notation and phrasing of otherwise standard facts could appear slightly unorthodox to some reader already familiar with the topic.

We start with some elementary geometric motivation of the main definitions and then proceed to sketch some basics of the theory. Then, in preparation for the next chapters, we will show how the combinatorics of oriented matroids indeed determines the homotopy type of a certain class of complex hyperplane arrangements.

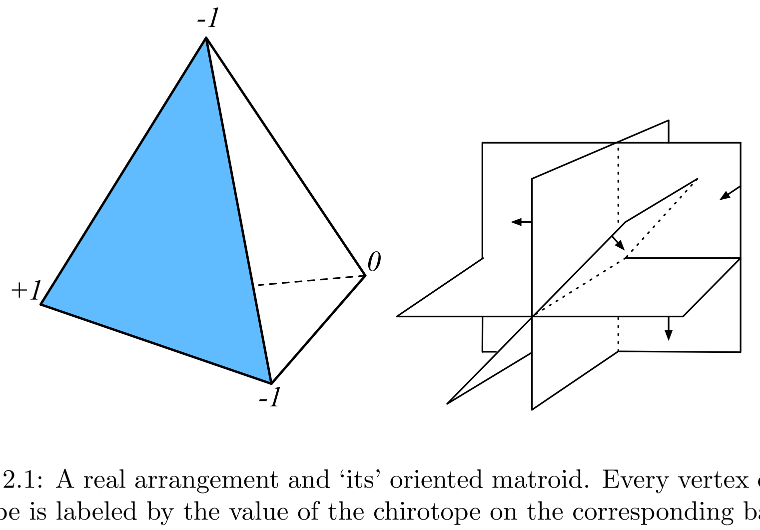


Figure 2.1: A real arrangement and 'its' oriented matroid. Every vertex of the polytope is labeled by the value of the chirotope on the corresponding basis.

2.1 Signs

We start with an introductory section, where we would like to draw the reader's attention to some elementary geometric facts that will be the main source of inspiration and motivation for the formal definitions of the next section.

Convexity

The *convex hull* of a finite subset $\{a_1, \ldots, a_k\}$ of $\mathbb{R}^n$ is the set

$$\mathrm{conv}(A) := \{\sum_{i=1}^{k} \lambda_i a_i \mid 0 \leq \lambda_i \leq 1, \sum_{i=1}^{k} \lambda_i = 1\}.$$

Theorem 2.1.1 (Carathéodory's Theorem, rephrasing Theorem 2.2.4 of [93])**.** *Let $A := \{a_1, \ldots, a_k\} \subset \mathbb{R}^n$ be such that* $\mathrm{conv}(A)$ *is d-dimensional. Then for every $v \in \mathrm{conv}(A)$ there is a subset $S \subseteq A$ with $|S| \leq d+1$ such that $v \in \mathrm{conv}(S)$.*

Recall that any linear dependence

$$\lambda_1 v_1 + \ldots + \lambda_n v_n = 0$$

among the column vectors of a given matrix M defines a *vector* of the associated matroid as the set $V := \{i : \lambda_i \neq 0\}$. Via Carathéodory's theorem we see that, if M has real coefficients and if the λ_i are real numbers, then there is a linear dependence of minimal support

$$\mu_1 v_1 + \ldots + \mu_n v_n = 0$$

where $\lambda_i = 0$ implies $\mu_i = 0$ and every nonzero μ_i has the same sign as the corresponding λ_i (the set $\{i : \mu_i \neq 0\}$ is then a *circuit* of the associated matroid). This can be used to refine the combinatorial information associated to "circuit elimination" in matroids described in Definition 1.1.2.3 as follows.

Consider the linear dependencies associated to two circuits $C_1, C_2 \in \mathbf{C}(M)$, and without loss of generality (after relabeling and scaling if needed) suppose that v_1 has coefficient 1 in the dependency associated to C_1 and -1 in the dependency associated to C_2. Now elimination of v_i amounts to adding

$$\begin{array}{rr} & v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n = 0 \\ + & -v_1 + \lambda'_2 v_2 + \ldots + \lambda'_n v_n = 0 \\ \hline & 0 + (\lambda_2 + \lambda'_2) v_2 + \ldots \quad\;\; = 0 \end{array}$$

and applying our former considerations to the (perhaps not minimal) bottomline linear dependency. We know that it contains a dependency of minimum support where the sign of the coefficient of v_i, if nonzero, agrees with the sign of $(\lambda_i + \lambda'_i)$. Now if the signs of λ_i and λ'_i agree or are zero, these determine the sign of their sum uniquely. If their sign disagree, then it is not enough to know the sign of the summands to know the sign of the sum. The signed circuit elimination axiom 2.2.4.3.($\mathcal{C}2$) is a formal combinatorial statement of this elementary fact.

Orthogonality

We have seen that matroid duality can be understood as a kind of "combinatorial orthogonality". The fact that orthogonality of vectors is characterized by the vanishing of their scalar product can be used to give a condition on signs of orthogonal vectors. Indeed for $v, w \in \mathbb{R}^n$ we have

$$v \cdot w = \sum_{i=1}^{n} v_i w_i = 0$$

only if the signs of the $v_i w_i$ are either all zero or else among them both signs ± 1 appear. In what follows we will phrase this fact by saying that "0 must be in the convex hull of the signs of the $v_i w_i$" (Compare Definition 2.2.3 and 2.2.4.1.(χ3)). This might seem at first some awkward way to put things - but we choose it, as many other mildly unorthodox aspects of our oresentation, in view of the forthcoming generalization to complex matroids.

2.2 Oriented matroids

Motivates by the previous geometric examples we now formulate the precise definitions that lay at the foundation of oriemted matroid theory.

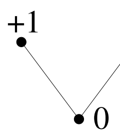
Signs

Definition 2.2.1. Given a finite ground set E, a *sign vector* (or *signed set*) is any

$$X \in (S^0 \cup \{0\})^E$$

where $S^0 = \{+1, -1\}$ is the unit sphere in $\mathbb{R}$. The e-th component of X will be denoted by $X(e)$ or, if no confusion can arise, by X_e. The signed set with value 0 for all components will be denoted by $\widehat{0}$.

We order $S^0 \cup \{0\}$ according to the following Hasse diagram.



The *sign* $\operatorname{sign}(x)$ of $x \in \mathbb{R}$ is defined to be 0 if $x = 0$ and $\frac{x}{|x|}$ otherwise. The sign $\operatorname{sign}(v)$ of $v \in \mathbb{R}^E$ is defined to be the sign vector with e-th component $\operatorname{sign}(v_e)$.

Convexity and orthogonality

We define the *convex hull* of a subset P of $S^0 \cup \{0\}$ to be the set of all signs of positive linear combinations of the elements of P. Thus

- $\operatorname{conv}(\emptyset) = \emptyset$
- $\operatorname{conv}(\{0\}) = \{0\}$
- $\operatorname{conv}(\{0, \epsilon\}) = \epsilon$ for $\epsilon \in S^0$

- $\text{conv}(P) = (S^0 \cup \{0\})$ if $S^0 \subseteq P$.

The following can be considered a combinatorial model of "addition of linear dependencies" (the reader should keep in mind that a linear dependency is, among other things, defined up to a global scalar multiple).

Definition 2.2.2. If $X, Y \in (S^0 \cup \{0\})^E$, the *composition* $X \circ Y \in (S^0 \cup \{0\})^E$ is defined as follows: for every $e \in E$,

$$X \circ Y(e) := \begin{cases} X(e) & \text{if } X(e) \neq 0 \\ Y(e) & \text{otherwise.} \end{cases}$$

What follows is a definition of orthogonality of sign vectors that is inspired by orthogonality of vectors in $\mathbb{R}^n$, and leads to a definition of orthogonality of oriented matroids that nicely models orthogonality of real vector spaces (see Theorem 2.2.7).

Definition 2.2.3. Two sign vectors $X, Y \in \{+, 0, -\}^E$ are defined to be *orthogonal* if $0 \in \text{conv}(\{X(e)Y(e) \mid e \in E\})$.

Axioms

Definition 2.2.4.

1. A function $E^d \to S^0 \cup \{0\}$ is called a *rank d chirotope* of an oriented matroid M if

(χ 1) χ is nonzero

(χ 2) χ is alternating

(χ 3) For any two subsets $x_1, \dots, x_{d+1}$ and $y_1, \dots, y_{d-1}$ of E, 0 is contained in the convex hull of the numbers

$$(-1)^k \chi(x_1, x_2, \dots, \widehat{x_k}, \dots, x_{d+1}) \chi(x_k, y_1, \dots, y_{d-1})$$

(Combinatorial Grassmann-Plücker relations).

2. A family $\mathcal{V} \subseteq (S^0 \cup \{0\})^E$ of signed sets is the set of *signed vectors* of an oriented matroid $\mathcal{M}$ if

($\mathcal{V}$0) $\widehat{0} \in \mathcal{V}^*$,

($\mathcal{V}$1) $\mathcal{V} = -\mathcal{V}$ *(Symmetry)*,

($\mathcal{V}$2) if $X, Y \in \mathcal{V}^*$ then $X \circ Y \in \mathcal{V}^*$ *(Composition)*,

($\mathcal{V}$3) for every $X, Y \in \mathcal{V}$ and $e \in E$ with $X(e) = -Y(e)$ there is some $Z \in \mathcal{V}$ with

- $Z(f) = \max\{X(f), Y(f)\}$ for all f for which this maximum exists, and
- $Z(e) = 0$

(Vector Elimination).

3. A family $\mathcal{C} \subseteq (S^0 \cup \{0\})^E \setminus \{\widehat{0}\}$ of signed sets is the set of *signed circuits* of an oriented matroid $\mathcal{M}$ if

($\mathcal{C}0$) $\mathcal{C} = -\mathcal{C}$ *(Symmetry)*,

($\mathcal{C}1$) if $X, Y \in \mathcal{C}$ and $\operatorname{supp}(X) \subseteq \operatorname{supp}(Y)$ then $X = \pm Y$ *(Incomparability)*,

($\mathcal{C}2$) for every $X, Y \in \mathcal{C}$ such that $X \neq -Y$ and $e, f \in E$ with $X(e) = -Y(e)$ and $X(f) \neq -Y(f)$, there is some $Z \in \mathcal{C}$ with $f \in \operatorname{supp}(Z) \subseteq \operatorname{supp}(X) \cup \operatorname{supp}(Y) \setminus e$ and $Z(f) = \max\{X(f), Y(f)\}$ *(Circuit Elimination)*.

As for matroids, there are cryptomorphisms allowing us to speak of "the oriented matroid with chirotopes χ and $-\chi$, vector set $\mathcal{V}$, and circuit set $\mathcal{C}$". The *underlying matroid* of this oriented matroid has basis set $\operatorname{supp}(\chi)$, vector set $\{\operatorname{supp}(X) : X \in \mathcal{V}\}$, and circuit set $\{\operatorname{supp}(X) : X \in \mathcal{C}\}$. We define the *rank* of an oriented matroid to be the rank of its chirotope or, equivalently, the rank of the underlying matroid.

As is well known:

Proposition 2.2.5. *Let M be a $d \times n$ matrix of rank d over $\mathbb{R}$. Let $v_1, \ldots, v_n$ denote the columns of M. Then there is a rank d oriented matroid with*

- *chirotope the function $[n]^d \to S^0 \cup \{0\}$ sending each $(i_1, \ldots, i_d)$ to the determinant of the matrix $(v_{i_1} \cdots v_{i_d})$*
- *signed vector set $\{\operatorname{sign}(x) \mid x \in \ker(M)\}$, and*
- *signed circuit set the set of all minimal nonzero signed vectors.*

Definition 2.2.6. The oriented matroids arising from matrices over $\mathbb{R}$ are called *realizable*.

Duality

Theorem 2.2.7. *If $\mathcal{M}$ is an oriented matroid with ordered ground set E, chirotope $\chi : E^r \to S^0 \cup \{0\}$, circuit set $\mathcal{C}$, and vector set $\mathcal{V}$, then there is an oriented matroid $\mathcal{M}^*$ with*

(0) ground set E,

(1) chirotope $\chi^* : E^{|E|-r} \to \{0, +, -\}$ given by

$$\chi^*(x_1, \ldots, x_{n-r}) = \chi(y_1, \ldots, y_r)\sigma(x_1, \ldots, x_{n-r}, y_1, \ldots, y_r),$$

where $\{y_1, \ldots, y_r\} = E \setminus \{x_1, \ldots, x_{n-r}\}$ and σ denotes the sign of the indicates permutation of E,

(2) covector set $\mathcal{V}^* = \mathcal{V}^\perp$, and

*(3) cocircuit set $\mathcal{C}^* = \min(\mathcal{V}^\perp - \{\widehat{0}\})$,
where* min *denotes support minimality.*

The underlying matroid of $\mathcal{M}^*$ is the dual of the underlying matroid of $\mathcal{M}$. If $\mathcal{M}$ is realized by a matrix with row space W, then $\mathcal{M}^*$ is realized by a matrix with row space $W^\perp$.

This $\mathcal{M}^*$ is called the *dual* to $\mathcal{M}$. The vectors of $\mathcal{M}^*$ are the *covectors* of $\mathcal{M}$, and the circuits of $\mathcal{M}^*$ are called the *cocircuits* of $\mathcal{M}$.

Proposition 2.2.8 (Axioms for dual pairs [24]). *Let $\mathcal{C}, \mathcal{C}^* \subseteq \{S^0 \cup \{0\}\}^E$. The sets $\mathcal{C}$ and $\mathcal{C}^*$ are the signed circuits resp. signed cocircuits of an oriented matroid if and only if:*

($\mathcal{S}1$) $\mathcal{C} = -\mathcal{C}$

($\mathcal{S}1^*$) $\mathcal{C}^* = -\mathcal{C}^*$

($\mathcal{S}2$) if $X, Y \in \mathcal{C}$ and $\operatorname{supp}(X) = \operatorname{supp}(Y)$ *then* $X = \pm Y$

($\mathcal{S}2^*$) if $X, Y \in \mathcal{C}^*$ and $\operatorname{supp}(X) = \operatorname{supp}(Y)$ *then* $X = \pm Y$

($\mathcal{S}3$) $\{\operatorname{supp}(X) \mid X \in \mathcal{C}\}$ *and* $\{\operatorname{supp}(X) \mid X \in \mathcal{C}^*\}$ *are the set of circuits resp. cocircuits of a matroid*

($\mathcal{S}4$) $X \perp Y$ for all $X \in \mathcal{C}$ and $Y \in \mathcal{C}^*$.

Modular elimination

While the axiomatization of signed circuits given in Definition 2.2.4 is the standard one, we present here another axiomatization via elimination which builds on Theorem 1.2.9 and strenghtens the following known fact.

Proposition 2.2.9 (Modular Elimination Axiom [20]). *In the definition of signed circuits (definition 2.2.4.3), if the set* $\{\operatorname{supp}(C) \mid C \in \mathcal{C}\}$ *is known to be the set of circuits of a matroid M, the Circuit Elimination Axiom can be replaced by the* Modular Elimination Axiom:

($\mathcal{C}2'$) for every $X, Y \in \mathcal{C}$ and $e, f \in E$ such that

- $\operatorname{supp}(X), \operatorname{supp}(Y)$ *is a modular pair of circuits of M,*
- $X \neq -Y$,
- $X(e) = -Y(e)$, *and*
- $X(f) \neq -Y(f)$,

there is some $Z \in \mathcal{C}$ with $f \in (\operatorname{supp}(Z) \subseteq \operatorname{supp}(X) \cup \operatorname{supp}(Y)) \setminus e$ *and, for all g,* $Z(g) \in \{0, X(g), Y(g)\}$.

From this proposition and Theorem 1.2.9, one readily obtains the following alternative aziomatization. For oriented matroids this can be viewed as 'a nice *curiosum*', but in fact it will turn out to be the only form of elimination possible for complex matroids.

Theorem 2.2.10. *In the definition of oriented matroids via signed circuits (Definition 2.2.4.3), it is enough to require $\mathcal{C}2$ to hold for all pairs $X, Y \in \mathcal{C}$ such that the supports* $\operatorname{supp}(X), \operatorname{supp}(Y)$ *are a modular pair in* $\operatorname{supp}(\mathcal{C})$ *(see Definition 1.2.8).*

Minors

Definition 2.2.11. Let $\mathcal{M}$ be a rank d oriented matroid on the ground set E with chirotope χ. Let $A \subseteq E$.

(1) Choose a maximal independent subset $\{a_1, \dots, a_l\}$ of A. Define the *contraction* M/A as the oriented matroid given by the chirotope

$$(\chi/A)(x_1, \dots, x_{d-l}) := \chi(x_1, \dots, x_{d-l}, a_1, \dots, a_l).$$

(2) Choose $\{a_1, \dots, a_{d-r}\} \subseteq A$ such that $(E \setminus A) \cup \{a_1, \dots, a_{d-r}\}$ spans E. Define the *deletion* $M \setminus A$ as the oriented matroid with chirotope

$$(\chi \setminus A)(x_1, \dots x_r) := \begin{cases} \chi(x_1, \dots, x_r) & \text{if } r = d, \\ \chi(x_1, \dots, x_r, a_1, \dots, a_{d-r}) & \text{if } r < d. \end{cases}$$

It is easily seen that these definitions are independent of the choices involved. (The chirotopes are independent of the choices involved up to global change of sign.)

For $X \in (S^0 \cup \{0\})^E$ and $A \subseteq E$ let $X_{\setminus A} \in (S^0 \cup \{0\})^{E \setminus A}$ be the restriction of X to $E \setminus A$.

Lemma 2.2.12. *Let $\mathcal{M}$ be an oriented matroid on the ground set E, and let $A \subseteq E$.*

(1) The deletion $M \setminus A$ *is the oriented matroid with set of signed circuits*

$$\mathcal{C}(\mathcal{M} \setminus A) = \{X_{\setminus A} \mid X \in \mathcal{C}, \operatorname{supp}(X) \cap A = \emptyset\}.$$

(2) The contraction $\mathcal{M}/A$ *is the oriented matroid given by the set of signed circuits*

$$\mathcal{C}(M/A) = \min\{X_{\setminus A} \mid X \in \mathcal{C}\},$$

where min *denotes support minimality.*

It's not hard to see that if χ is a chirotope of an oriented matroid $\mathcal{M}$ then $\chi \setminus A$ is a chirotope of $\mathcal{M} \setminus A$ and χ/A is a chirotope of $\mathcal{M}/A$.

As is the case with matroids, deletion and contraction are dual operations:

$$(\mathcal{M}/A)^* = (\mathcal{M}^*) \setminus A.$$

Topological representation

A *pseudosphere* S in the d-sphere S^d is a PL-homeomorphic image of the sphere S^{d-1} into S^d. In particular, removing S from S^d leaves two disconnetted open d-balls which are called the *sides* of S and labeled S^+, S^-.

Definition 2.2.13 (See Definition 5.1.3 of [20])**.** A finite (multi)set $\mathscr{A} = (S_e)_{e \in E}$ of pseudospheres in S^d is an *arrangement of pseudospheres* if

($\mathcal{A}1$) For all $I \subseteq E$, the subspace $S_I := \bigcap_{e \in I} S_e$ is a sphere.

($\mathcal{A}2$) If, for some $I \subseteq E$ and $e \in E$, $S_I \not\subseteq S_e$, then $S_I \cap S_e$ is a pseudosphere in S_I with sides $S_I \cap S_e^+$, $S_I \cap S_e^-$.

An arrangement of pseudospheres is called *signed* if it comes with a choice of a “positive side” of every S_e.

The arrangement is called *essential* if $\dim S_E = \emptyset$.

The cells of the (regular) CW-complex in which an arrangement $\mathscr{A}$ of pseudospheres sudivides the sphere S^d are called *faces*. The set of faces of $\mathscr{A}$ is denoted by $\mathcal{F}(\mathscr{A})$. Every face F is uniquely identified by the sign vector X_F such that $X_F(e) = 0$ if $F \subseteq S_e$, and $X_F(e) = \pm 1$ according to whether $F \subseteq S_E^{\pm}$.

The following is the celebrated Topological Representation Theorem for oriented matroids, first proved by Folkman and Lawrence in 1978 [56].

Theorem 2.2.14 (See 5.2.1 of [20])**.** *For every signed arrangement $\mathscr{A}$ of pseudospheres in S^d the set of signed vectors $\{X_F | F \in \mathcal{F}(\mathscr{A})\}$ satisfies the axioms for the set of vectors of a rank $d+1$ oriented matroid (which is loop-free).*

Conversely, for every loop-free oriented matroid $\mathcal{M}$ of rank $d+1$ on the ground set E there is a signed arrangement of pseudospheres $\mathscr{A}$ in S^d such that $\{X_F | F \in \mathcal{F}(\mathscr{A})\} = \mathcal{V}(\mathcal{M}^*)$

Observation 2.2.15. If all spheres in $\mathscr{A}$ are *linear* (i.e., they are great circles of S^d), then $\mathscr{A}$ can be realized by intersecting a (real) arrangement of hyperplanes with the unit sphere in $\mathbb{R}^n$.

This justifies the appearance of the *dual* of $\mathcal{M}$ in the definition above: the matroid of linear dependencies of the normal vectors of this hyperplane arrangement (defined in Section 1.3) is the *dual* to the underlying matroid of the oriented matroid whose vectors arise from the faces of the sphere in the Topological Representation Theorem above.

Topes

Motivated by the previous considerations about faces of oriented matroids, we define a partial order on $\mathcal{V}$ by extending componentwise the partial order on signs.

Definition 2.2.16. Let $\mathcal{M}$ be an oriented matroid with ground set E. On its set of vectors $\mathcal{V}$ define a partial ordering by

$$X \leq Y \Leftrightarrow \forall e \in E : X(e) \leq Y(e).$$

The set $\mathcal{V}(\mathcal{M}^*) = \mathcal{V}^*$ endowed with this ordering gives a poset which is isomorphic to the poset of (closed) cells of the decomposition of the Folkman-Lawrence sphere ordered by containment. This is then called the *face poset* of the oriented matroid $\mathcal{M}$ and is denoted by $\mathcal{F}(\mathcal{M})$. It has a unique minimal element but in general it possesses many maximal elements, that are called *topes* of the oriented matroid. the set of topes of an oriented matroid $\mathcal{M}$ will be denoted by $\mathcal{T}(\mathcal{M})$ (or just $\mathcal{T}$).

For $T \in \mathcal{T}$ and $F \in \mathcal{V}^*$ we define $T_F \in \mathcal{T}$ by $(T_F)_e = T_e$ if $F_e = 0$ and $(T_F)_e = F_e$ else (see Remark 3.3.6 for a geometric interpretation of this operation).

The set $\mathcal{T}$ can be given interesting partial orders. These were introduced by Edelman [49] in the context of arrangements of hyperplanes and independently by Edmonds and Mandel [51] for abstract oriented matroids.

Definition 2.2.17 (See also Definition 4.2.9 of [20]). Let an oriented matroid $\mathcal{M}$ be given and consider its set of topes $\mathcal{T}$. For $T, T' \in \mathcal{T}$ let

$$S(T,T') := \{e \in E \mid T_e = -T'_e\}.$$

To every tope $B \in \mathcal{T}$ we can associate a partial order $\prec_B$ on $\mathcal{T}$ defined by

$$T_1 \prec_B T_2 \Leftrightarrow S(B,T_1) \subset S(B,T_2).$$

The set $\mathcal{T}$ endowed with the order relation $\mathcal{T}_B$ is called *tope poset of* $\mathcal{M}$ *based at* B and will be denoted by $\mathcal{T}_B(\mathcal{M})$ or simply by $\mathcal{T}_B$. This poset is ranked by $r(T) = |S(B,T)|$.

Flipping and sweeping

Definition 2.2.18 (Compare Definition 7.3.4 of [20]). Let $\mathscr{A} := (S_e)_{e\in E}$ be an arrangement of pseudospheres on S^d. Pick a vertex w of the induced stratification of S^d and consider a pseudosphere S_f with $w \notin S_f$. Let $T_w := \{e \in E \mid S_e \ni w\} \cup \{f\}$ and set $\mathscr{U}_w := (S_e)_{E\setminus T_w}$.

We say that w is *near* S_f if all the vertices of the arrangement T_w are inside the two regions of $\mathscr{U}_w$ that contain w and $-w$.

Given an arrangement of pseudospheres, if a vertex w is near some pseudosphere S_f, one can perturb locally the picture by 'pushing S_f across w' and, symmetrically, across $-w$, so to obtain another valid arrangement of pseudospheres which oriented matroid differs from the preceding only in faces inside the two regions of T_w that contain w and $-w$. This operation was called a *flipping* of the oriented matroid at the vertex w by Fukuda and Tamura, who first described this operation [58]. For a formally precise description of flippings see also [20, p. 299 and ff.].

Every arrangement of linear hyperplanes in $\mathbb{R}^d$ induces on the unit sphere S^{d-1} an arrangement of spheres. An oriented matroid that can be realized in this way is called *realizable*. It is NP-hard to decide whether an oriented matroid is realizable [83].

Observation 2.2.19. Flippings preserve the underlying matroid (i.e.,the intersection lattice of the arrangement). However, a flipping of a realizable oriented matroid need not be realizable!

To be able to encode the data of an affine arrangement one uses *affine oriented matroids*. The idea is to add an hyperplane 'at infinity' to the oriented matroid represented by the cone of the given affine arrangement (for the precise definition, see [20, Section 4.5]). For the affine counterpart of the representation theorem we need one more definition.

Definition 2.2.20. A k-pseudoflat in $\mathbb{R}^d$ is any image of $\mathbb{R}^{d-k}$ under a (tame) selfhomeomorphism of $\mathbb{R}^d$. A pseudohyperplane clearly has two well-defined *sides*. An arrangement of pseudohyperplanes is a set of such objects satisfying the condition that every intersection of pseudohyperplanes is again a pseudoflat.

Then every affine oriented matroid is represented by an (affine) arrangement of pseudohyperplanes, and the notion of flipping is similar to the previous: the only difference is that there is no vertex "$-w$".

Notation 2.2.21. Let $\mathscr{A}$ be an affine arrangement of pseudohyperplanes, $\widetilde{H} \in \mathscr{A}$, and w a vertex of $\mathscr{A}$ near $\widetilde{H}$. The arrangement representing the oriented matroid obtained from the previous by flipping $\widetilde{H}$ across w will be denoted $\texttt{Flip}(\mathscr{A}, \widetilde{H}, w)$.

Consider an arrangement of affine pseudohyperplanes $\mathscr{A}$ and pick a pseudohyperplane H such that all points of $\mathscr{A}$ are on the same side of H. A *sweeping* (or 'topological sweeping') of H through $\mathscr{A}$ is a sequence of flippings, one for every point of $\mathscr{A}$, that fixes everything except H. At the end of a sweeping, the points of $\mathscr{A}$ are all on the opposite side of H with respect to the beginning.

It is a well-known fact that such a sweeping need not exist in general for all $\mathscr{A}$ and H. At every step, the flip through a point p of $\mathscr{A}$ is performed by extending $\mathscr{A}$ with a pseudohyperplane through p parallel to H, and then perturbing the resulting arrangement around p [20, Section 7.3]. While the 'perturbation' part is always feasible, the 'extension' part requires careful consideration.

The oriented matroid program $(\mathscr{A}, H)$ is called *Euclidean* if an extension of $\mathscr{A}$ by a pseudohyperplane parallel to H containing p exists for every point p [20, Definition 10.5.2]. The following characterization was first proved in Komei Fukuda's PhD. thesis. We refer to [20, Chapter 10] and the bibliography cited therein for a structured and complete exposition of the subject.

Definition 2.2.22 (See Section 10.5, Theorem 10.5.5 of [20])**.** Let an affine arrangement of pseudohyperplanes $\mathscr{A}$ be given, and let $H \in \mathscr{A}$ be such that all points of $\mathscr{A} \setminus \{H\}$ are on the same side of H. Every 1-dimensional face F of $\mathscr{A}$ that is not contained in H is supported on a pseudoline $\ell_F := \bigcap\{H \in \mathscr{A} : F \subseteq H\}$, and ℓ_F meets H in exactly one point p. We can then think of the 1-cell F as being directed away from p (along ℓ_F). Thus, we turn the union of the 0- and 1- dimensional faces of $\mathscr{A}$ not contained in H into an oriented graph we call G_H.

The oriented matroid program $(\mathscr{A}, H)$ is Euclidean if and only if G_H is acyclic.

Corollary 2.2.23. *If an oriented matroid program $(\mathscr{A}, H)$ is realizable (i.e., $\mathscr{A}$ is an arrangement of hyperplanes), then G_H is acyclic, and thus allows for a sweeping of H through $\mathscr{A}$.*

2.3 Complexified arrangements

In the language of Section 1.3, let $\mathscr{A}$ be an arrangement in $\mathbb{C}^d$ where the linear forms α_i are defined over the reals. Such an arrangement is called *complexified.*

The important structural property of a complexified arrangement $\mathscr{A}$ in $\mathbb{C}^d$ is that it induces a real arrangement $\mathscr{A}_{\mathbb{R}}$ in $\mathbb{R}^n$ with the same lattice of flats. In real space one can then take advantage of the combinatorics of oriented matroids.

Reflection groups

It is fair to say that the study of reflection groups has been a driving motivation for the study of the topology of the complement of complex hyperplane arrangements. The story began when Arnol'd computed the cohomology ring of the pure braid group on n strands by studying the configuration space of n distinct points in $\mathbb{C}$. This space M_n is nothing else than $\mathbb{C}^n \setminus \bigcup \mathscr{A}$, where $\mathscr{A}$ is the arrangement defined by the linear forms

$$\alpha_{i,j}(x) := x_j - x_i$$

where $1 \leq i < j \leq n$.

Arnol'd notes that M_n is an aspherical space, and thus is a $K(\pi, 1)$ for the pure braid group on n strands. He then computes the cohomology of M_n obtaining what we nowadays call the "Orlik-Solomon" algebra of this arrangement. The last lines of [5] are almost prophetic - Arnol'd considers the complement of a general complex arrangement of hyperplanes and states that its integer cohomology "Probably,... is torsion-free and is generated by the one-dimensional classes [...]".

Brieskorn went on to consider the same question for "generalized braid groups" (finite-type Artin groups). The first step was to take the arrangement $\mathscr{A}$ given by the reflecting hyperplanes of the associated Coxeter group G and to consider its complement $\mathscr{M}$ in the complexification of the (real) ambient space. Knowing that the fundamental group of $\mathscr{M}/G$ (resp. $\mathscr{M}$) is the corresponding (pure) Artin group, the question was whether such spaces are always aspherical. This was conjectured by Brieskorn in [26], where a proof is given for some cases. The proof of this conjecture has been given by Deligne in [38] in purely geometric terms. His striking result is the following.

Theorem 2.3.1 (Deligne [38])**.** *The complement to a complexified arrangement $\mathscr{A}$ is aspherical if all regions of $\mathscr{A}_{\mathbb{R}}$ are cones over simplices.*

This condition on the combinatorics of chambers is clearly satisfied by the mirrors of any Coxeter group - hence Brieskorn's conjecture was proved. But the theorem has a far wider reach: in particular, it gives a 'combinatorial' sufficient condition for asphericity of complexified arrangements. Here by 'combinatorial' we mean the information encoded in the arrangement's oriented matroid. Indeed Deligne's condition can be phrased as 'the poset of topes is a lattice for every choice of the base tope'. This becomes apparent in view of the following fact.

Fact 2.3.2 (See [48])**.** *Let $\mathscr{A}$ be the arrangement in $\mathbb{R}^d$ given by the mirrors of a Coxeter group G. Then the tope poset of the associated oriented matroid is isomorphic to the weak order on G.*

As a closing word for this section we point out that the lattice structure of the tope poset is *the* combinatorial structure which makes Deligne's argument work. This idea will be taken up again in the next part about unitary space.

Salvetti's complex

In this section we review a fundamental result by Mario Salvetti [86], which shows that if $\mathscr{A}$ is a complexified arrangement, then the homotopy type of $\mathscr{M}(\mathscr{A})$ is determined by the oriented matroid of $\mathscr{A}_{\mathbb{R}}$.

Definition 2.3.3. Given an oriented matroid $\mathcal{M}$, we define a poset $\mathcal{S}(\mathcal{M})$ (denoted simply by $\mathcal{S}$ if no confusion can arise). The elements of $\mathcal{S}(\mathcal{M})$ are all pairs $\langle F, T\rangle$ where $F \in \mathcal{F}(\mathcal{M})$, $T \in \mathcal{T}(\mathcal{M})$ and $F < T$ in $\mathcal{F}(\mathcal{M})$. The order relation in $\mathcal{S}$ will be denoted $<_s$ and defined by setting

$$\langle F, T\rangle <_s \langle F', T'\rangle \text{ if } F > F' \text{ in } \mathcal{F}(\mathcal{M}) \text{ and } T = T'_F.$$

The relevance of Definition 2.3.3 comes from the following fundamental result by Salvetti (which holds also in a general form for affine arrangements).

Theorem 2.3.4 (Theorem 1 of [86])**.** *Let $\mathscr{A}$ be a complexified arrangement of hyperplanes and $\mathcal{M}$ the oriented matroid associated to $\mathscr{A}_{\mathbb{R}}$. Then $\mathcal{S}(\mathcal{M})$ is the poset of cells of a regular CW-complex, called* Salvetti complex*, that can be embedded as a deformation retract in $\mathscr{M}(\mathscr{A})$.*

Notation 2.3.5. The order complex[1] $\Delta(\mathcal{S}(\mathcal{M}))$ is then the barycentric subdivision of the Salvetti complex, and is sometimes referred to as just 'the Salvetti complex'. When confusion cannot arise, we write $\mathcal{S}$ without distinction for both complexes.

Remark 2.3.6. We see that, although $\mathcal{S}$ can be defined for any oriented matroid, the main topological interest of the construction is in the context of arrangements of hyperplanes, i.e., of representable oriented matroids. However, work is being dedicated to the study of the homotopy type of the 'general' Salvetti complex [44].

[1]Recall that the order complex of a poset is the simplicial complex if its totally ordered subsets.

3 Shelling-type orderings of regular CW-complexes and acyclic matchings of the Salvetti complex

Motivated by the work of Salvetti and Settepanella ([87, Remark 4.5]) we introduce certain total orderings of the faces of any shellable regular CW-complex (called shelling-type orderings) that can be used to explicitly construct maximum acyclic matchings of the poset of cells of the given complex. Building on an application of this method to the classical zonotope shellings (i.e., those arising from linear extensions of the tope poset) we describe a class of maximum acyclic matchings for the Salvetti complex of a linear complexified arrangement. To do this, we introduce and study a new purely combinatorial stratification of the Salvetti complex. For the obtained acyclic matchings we give an explicit description of the critical cells that depends only on the chosen linear extension of the poset of regions. It is always possible to choose the linear extension so that the critical cells can be explicitly constructed from the chambers of the arrangement via the bijection to no-broken-circuit sets defined by Jewell and Orlik [63]. Our method generalizes naturally to abstract oriented matroids.

Introduction

The idea of *shelling* was initially introduced by Bruggesser and Mani [27] as a (geometrically defined) technique to deconstruct polytopes in a 'controlled way' allowing an accurate bookkeeping of certain combinatorial data. The required total ordering of the polytope's facets was obtained from the order in which a general position line in meets the affine hulls of the facets. Much work has been spent on a purely combinatorial characterization of this process, and on a corresponding generalization of the method beyond polytopes. In fact, shellability can be defined for general (possibly nonpure) regular cell complexes [21, 22]. A line of research initiated by Björner [18] studies combinatorial properties of posets that ensure shellability of the associated order complexes. A considerable amount of work was dedicated to this subject (see e.g. [16, 18, 21, 22]). Particular attention was dedicated to the posets of cells of regular CW-complexes: Björner characterized them combinatorially (see [16, Definition 2.1 and Proposition 3.1]), and proved that shelling orders of the facets of the associated CW-complex correspond to recursive coatom orderings of the posets ([16, Proposition 4.2], see also [22, Theorem 13.2]).

Recently, Forman [57] proposed a combinatorial version of Morse theory, called *Discrete Morse Theory*. The idea is that, given any regular CW-complex, one can define a combinatorial analog of the Morse vector fields (i.e., acyclic matchings on the poset of cells; see Definition 3.1.4 and [30, Proposition 3.3]) such that the original complex is homotopy equivalent to a complex having as many cells of dimension d as there are 'critical points' (i.e., non-matched cells) of rank $d + 1$. Moreover, the attaching maps can be reconstructed from the knowledge of the 'gradient paths' (i.e., alternating paths in the poset). Since at the topological core of both shellability and discrete Morse theory lies the idea of collapsing cells (along matched edges or along the shelling order), it is natural to study the relation between these concepts: this study was undertaken by various authors, e.g. in [8, 30, 64]. A comprehensive and careful exposition of the nowadays established combinatorial framework of discrete Morse theory can be found in the book of Kozlov [65].

The motivation for our considerations was given by a joint work of Mario Salvetti with Simona Settepanella [87], where discrete Morse theory is used to explicitly obtain a minimal CW-complex that models the homotopy type of the complement of a complexified arrangement of hyperplanes, thus providing a constructive proof of the minimality result for general arrangements that was obtained independently by Randell [81] and Dimca and Papadima [45]. Another recent study of minimal complexes for complexified arrangements is due to Yoshinaga [95]. For the basic definitions about arrangements of hyperplanes we refer to [76].

The starting point of [87] is the Salvetti complex, and the main tool used to construct a maximum acyclic matching of its poset of cells is a certain total order on the faces of the arrangement that is called *polar ordering* by the authors. The name refers to the fact that this total order is obtained by considering polar coordinates with respect to a generic flag and then ordering the faces according to their smallest point in the lexicographical order of the polar coordinates (for the precise definition see [87, Definition 4.4]). It is explicitly asked for a completely combinatorial formulation of this method [87, Remark 4.5].

In an attempt to answer this question, we keep the idea of constructing acyclic matchings by considering the arrangement from a 'generic' point of view, but we try to stay in the context of oriented matroids. These are widely studied combinatorial objects that encode the structure of real arrangements of pseudospheres, and in particular of linear hyperplanes (for an introductory reference see [20, Chapter 1]). Thus, we actually loose the generality of [87], where the results hold also for affine arrangements. However, our method has the advantage that it does not need the choice of a generic flag in the ambient space, and that it holds for general abstract oriented matroids.

One of the ways one can think to look 'generically' at an oriented matroid is to consider a shelling of its *zonotope*. This is a polytope that is classically associated to every oriented matroid and that, if the oriented matroid corresponds to a real arrangement, is combinatorially isomorphic to the polyhedral subdivision of the unit sphere given by the hyperplanes (for a precise account of this subject, see [20, Section 2.2 and Chapter 4]). It is well-known that to every linear extension of the *tope poset* of the oriented matroid corresponds a (class of) shelling(s) of the associated zonotope: in fact, one can construct recursive coatom orderings of the zonotope's poset of faces.

We first show a way to construct maximum acyclic matchings of (CW-)

posets that admit a recursive coatom ordering. We do this using *shelling-type orderings*: a class of total orderings of the involved poset that are associated to recursive coatom orderings. Then we apply this construction to the special case of a zonotope.

It turns out that linear extensions of tope posets describe also a nice decomposition of the Salvetti complex that, to the best of our knowledge, has not been considered up to now. The above obtained zonotope shellings give acyclic matchings of every 'piece' of this decomposition that can be 'pasted together' to give an acyclic matching of the poset of cells of the whole Salvetti complex. To every critical cell correspond canonically a (unique) chamber and a flat of the underlying matroid which codimension equals the dimension of the critical cell. Both are uniquely determined by the chosen linear extension of the tope poset. Maximality of the matching follows from the fact that the critical cells are in bijection with no-broken circuits, and thus with generators of the homology (by e.g. [63, 95]).

This correspondence can be made more precise and explicit: we show that, for an adequate choice of the ordering of the hyperplanes and of the linear extension of the base poset, the bijection between chambers and no-broken-circuits given by Jewell and Orlik in [63] associates to every chamber a basis of the flat that carries the corresponding critical cell.

The his chapter is organized as follows. After introducing the main characters, in Section 3.1 we prove that every recursive coatom ordering of a CW-poset induces a shelling-type total ordering of its faces (Lemma 3.1.10). From this total ordering, in Proposition 3.1.14 we construct an acyclic matching of the given poset that turns out to be 'optimal' (for a comparison with known related results of Chari [30] and Babson and Hersh [8] see Remark 3.1.8). Then, Section 3.2 introduces oriented matroids, explains the construction of the zonotope shelling associated to a linear extension of the tope poset and compares (in Remark 3.2.5) the associated shelling-type ordering with the polar orderings of [87]: this is our (kind of) answer to [87, Remark 4.5]. In Section 3.3 we study the stratification of the Salvetti complex induced by a linear extension of the tope poset (in the context of arrangements also called 'poset of regions' and first considered in [49]). First, we prove a general property of tope posets (Theorem 3.3.14) that, given a linear extension, allows to associate a unique flat X_C to every tope C. It turns out that the stratum associated to a tope C corresponds naturally to the oriented matroid obtained by contraction of the flat X_C. On the one hand, this allows to construct acyclic matchings for every stratum and to verify acyclicity and maximality of the 'patchworked' matching (Proposition 3.3.20). On the other hand, in Section 3.4 we show that for some orderings of the hyperplanes (Definition 3.4.4) there is a linear extension of the tope poset (Definition 4.2.5) for which the flat X_C is spanned by the no-broken circuit set that corresponds to C under the bijection described in [63] (Proposition 3.4.17).

3.1 Shellings and acyclic matchings

On partially ordered sets.

Some notation about partially ordered sets was already introduced in Section 1.2. For this chapter we will need some more refined notations and definitions. We review them here, our reference being, as usual, [90, Chapter 3].

Recall that a *poset* is a set (say P) endowed with a partial order relation (say $<$), and will be written as a pair $(P,<)$ or, if no misunderstanding about the partial order will be possible, just denoted by P. Moreover, the posets we will consider will be *locally finite*, meaning that for each $p \in P$ there are only finitely many q with $p < q$ or $p > q$. An element $p \in P$ is said to *cover* $q \in P$ whenever $p > q$ and there is no $x \in P$ with $p > x > q$. If p covers q with respect to the order relation $>$ (or $\rhd$, $\succ$,...), then we will write $p \gtrdot q$ (respectively $\rhd\!\!\!\cdot$, $\succ\!\!\!\cdot$). The set of all elements of P that are covered by p will be called, by slight abuse of notations, the set of *coatoms* of p, and denoted by $\mathrm{coat}(p)$. In fact, for every $q \in P$, the set $\mathrm{coat}(q)$ is the set of coatoms of the poset $P_{\leq q} := \{p \in P \mid p \leq q\}$. This poset is called the *principal lower ideal* generated by q, a *lower ideal* being in general any subposet of P that can be written as an intersection of principal lower ideals; (principal) *upper* ideals are defined accordingly. Any subset of the form $P_{\leq q} \cap P_{\geq p}$ is called an *interval* of P. We will write $P_{<q} := P_{\leq q} \setminus \{q\}$. A totally ordered subset $\omega \subset P$ will be called *chain*, and its *length* is defined by $\ell(\omega) := |\omega| - 1$. The length of P, $\ell(P)$, is then defined as the maximum length of a chain contained in P. If every maximal chain of P has the same length, then P is called *graded* and possesses a unique *rank function* $r : P \to \mathbb{N}$, $r(p) := \ell(P_{\leq p})$.

Recall also that a poset P is said to be a *lattice* if every two $p, q \in P$ have a unique least upper bound (called *join* and denoted $p \vee q$) and a unique greatest lower bound (called *meet* and denoted by $p \wedge q$).

Remark 3.1.1. An upper (lower) ideal in a lattice is principal if and only if it is closed under meet (join).

Sometimes we will have to consider different order relations on the same set. If needed, the concerned order relation will be specified in a subscript. Thus, for example, $\max_{\succ} A$ denotes the maximal element of A with respect to the order $\succ$. A *linear extension* of a partial order $<$ is a total order $\lhd$ such that $p \lhd q$ whenever $p < q$.

A poset P is called *bounded* if it possesses a maximal and a minimal element. Let $\widehat{P}$ denote the poset P with a maximal and a minimal element added, if P has none. The maximal and minimal elements of P are customarily denoted by $\widehat{1}$ and, respectively, $\widehat{0}$. In a poset with $\hat{0}$ a principal lower ideal is also called a *lower interval.*

Given a (possibly nonpure) CW-complex K, we define its *poset of faces* $\mathcal{F}(K)$ as the set of (closed) cells of K ordered by containment, with a minimal element $\widehat{0}$ added (the '-1 - dimensional cell'). Note that, for every cell k, every maximal chain in $\mathcal{F}(K)_{\leq k}$ has the same length. The *height* of k is $h(k) = \ell(\mathcal{F}(K)_{\leq k})$, the length of the corresponding lower interval. Geometrically, we have $\dim(k) = h(k) + 1$ for every cell k.

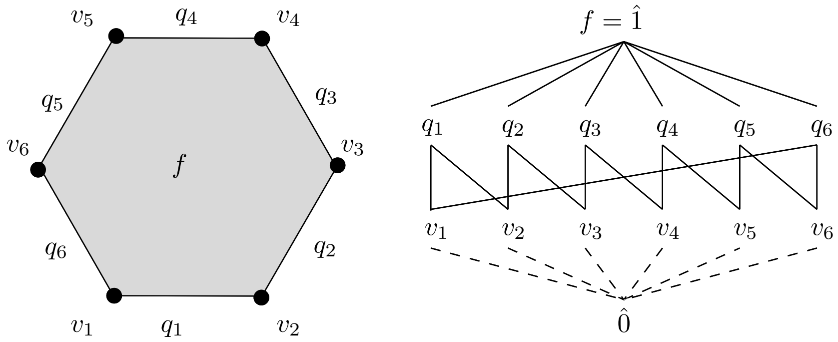


Figure 3.1: The regular CW-complex K_1 given by a filled hexagon, and its poset of faces $\widehat{\mathcal{F}}(K_1)$.

Shellability and Recursive Coatom Orderings.

A regular CW complex K is said to be *shellable* if its maximal cells can be given an order along which the complex can be 'rebuilt' in a very controlled way. For the precise definition we refer to [22, Definition 13.1], where shellability was first extended from simplicial complexes to regular CW-complexes. The complexes that we will consider are given by means of their poset of cells. Therefore we take a point of view that is more tailored to our context: we will define recursive coatom orderings of posets, and then see how they correspond to shellings of regular CW-complexes.

Definition 3.1.2 (Definition 5.10 of [21])**.** A bounded poset $(P, <)$ is said to admit a *recursive coatom ordering* $\prec$ if $\ell(P) = 1$, or if $\ell(P) > 1$ and there is a total ordering $\prec = \prec_{\hat{1}}$ on the set $\mathrm{coat}(\hat{1})$ of coatoms of P such that

(1) for all $p \in \mathrm{coat}(\widehat{1})$, the interval $[\widehat{0}, p]$ admits a recursive coatom ordering $\prec_p$ in which the coatoms of the intervals $[\widehat{0}, q]$ for $q \prec_{\widehat{1}} p$ come first.

(2) for all $p \prec_{\widehat{1}} q$, if $p, q > y$, then there is $p' \prec_{\widehat{1}} q$ and $z \in \mathrm{coat}(q)$ such that $p' > z \geq y$.

This definition is one of the criteria introduced by Björner to check shellability of the order complex of a poset. It turned out that, in the context of regular CW-complexes, this property is equivalent to shellability. We state these facts in the next theorem.

Theorem 3.1.3 (See [18], [22])**.** *If a poset P admits a recursive coatom ordering, then $\Delta(P)$ is shellable. If P is the poset of faces of a regular CW-complex K, then a total ordering of the maximal faces of K is a shelling order if and only if it is a recursive coatom ordering of $\widehat{P}$.*

Matchings and Discrete Morse Theory.

We introduce here some basic concepts of Discrete Morse Theory, omitting their proofs. The interested reader will find reference to the publications where the

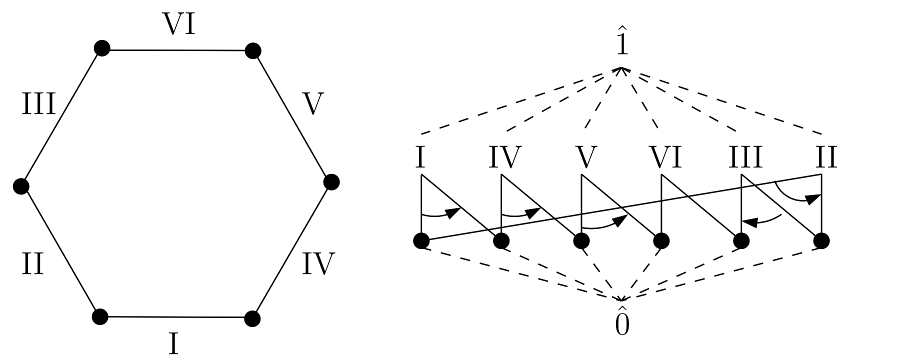


Figure 3.2: A shelling order of the maximal faces of the boundary complex K_2 of a hexagon, and a corresponding Recursive Coatom Ordering of the poset $\widehat{\mathcal{F}}(K_2)$. The arrows give the R.C.O. of the corresponding lower intervals. Below the VI face, the ordering does not matter.

statements first appeared. For a comprehensive exposition of the subject in its entirety we refer to the book of Kozlov [65].

Definition 3.1.4 (Compare Proposition 3.3 of [30])**.** Let $(P, <)$ be any poset. The set of *edges* of P is $\mathfrak{E} := \{(p, q) \in P \times P \mid p \gtrdot q\}$. A subset $\mathfrak{M} \subset \mathfrak{E}$ is called a *matching* of P if every element of P appears in at most one *matched pair*, i.e., a pair $(p, q) \in \mathfrak{M}$. A *cycle* in a matching $\mathfrak{M}$ is a subset $\{(p_1, q_1), \ldots, (p_k, q_k)\} \subseteq \mathfrak{M}$ such that

$$q_1 \lessdot p_2,\ q_2 \lessdot p_3,\ \ldots,\ q_k \lessdot p_1.$$

The matching $\mathfrak{M}$ is called *acyclic* if it contains no cycle.

Much of the terminology is borrowed from the theory of graphs, the idea being that $\mathfrak{M}$ is actually a matching of the *Hasse diagram* of P, i.e., the graph defined by the set of edges $\mathfrak{E}$ on the vertex set P (informally speaking, this is the graph one usually draws when graphically representing a poset). A matching $\mathfrak{M}$ will be called *maximal* if there is no matching $\mathfrak{M}' \supsetneq \mathfrak{M}$. If, in addition, $\mathfrak{M}$ has maximal cardinality among all matchings of P, then it is called a *maximum matching.* A *perfect matching* is a matching such that every element of P is contained in a matched pair. In general, $p \in P$ is called *critical* for $\mathfrak{M}$ if it is not contained in any matched pair.

The following result is very useful in dealing with acyclic matchings.

Lemma 3.1.5 (Theorem 11.2 of [65])**.** *A matching M of a poset P is acyclic if and only if there is a linear extension $\lhd$ of P such that $p \lhd q$ whenever $(p, q) \in M$.*

From a topological point of view, the interest of acyclic matchings of posets is explained in the following (weak) version of the main theorem of Discrete Morse Theory.

Theorem 3.1.6 (Theorem 11.13 of [65]. See also [30, 57])**.** *Let K be a regular CW-complex K and $\mathfrak{M}$ an acyclic matching of $\mathcal{F}(K) \setminus \{\widehat{0}\}$. Let c_i denote the number of critical elements of rank i. Then K is homotopy equivalent to a CW-complex that has c_i cells in dimension i.*

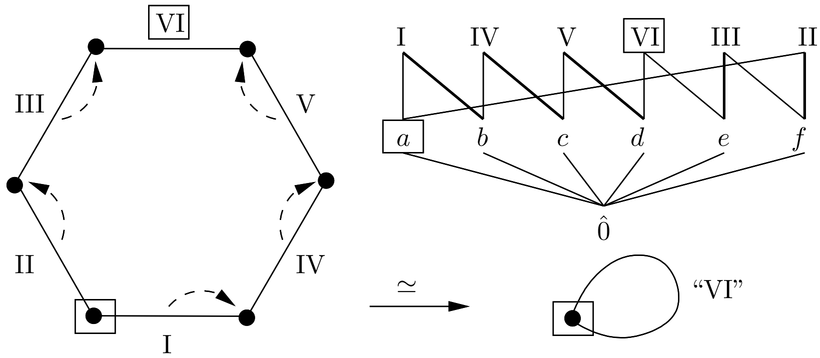


Figure 3.3: To illustrate Theorem 3.1.6 we take again the empty hexagon K_2 of Figure 3.2 and its face poset. The bold edges give an acyclic matching which critical cells are the ones in the boxes: one in dimension 1 and one in dimension 0. Indeed, the complex is homotopy equivalent to S^1. Although we will not get into it here, note that the homotopy equivalence here can be obtained as the concatenation of the collapses indicated by the dashed arrows - that are strongly related with the matching.

Remark 3.1.7. If we consider the whole $\mathcal{F}(K)$ we can say that if a perfect acyclic matching of $\mathcal{F}(K)$ exists then K is contractible. Moreover, if there is an acyclic matching of $\mathcal{F}(K)$ that has critical elements only in one rank level, say the i-th, then K is homotopy equivalent to a wedge of i-spheres.

From recursive coatom orderings to acyclic maximum matchings.

We now describe a construction of certain acyclic maximum matchings of the poset of cells of every shellable regular CW-complex. The core of the argument is Lemma 3.1.10, where a convenient linear ordering of all cells is produced.

Remark 3.1.8. It has to be pointed out that our approach via recursive coatom orderings differs from those taken in [8] and [30]. Babson and Hersh [8] consider a certain kind of shellability (i.e. lexicographic) of a particular class of simplicial complexes (order complexes of posets) and, in this case, they construct Morse functions "with a relatively small number of critical cells" ([8, Introduction]). Our argument works for any shelling order of any regular CW-complex K and gives always a 'best possible' matching. In this sense, when K is the order complex of a poset, and the the shelling order is the lexicographic one, our result improves [8, Theorem 2.2]. After the first version of this paper, we learned that also Chari [30] proved the existence of 'best possible' matchings for regular CW-complexes with a *generalized shelling* (for the precise meaning and the definitions see [30, Page 103 and Corollary 4.3]). Our approach is different, and more constructive. We use the algorithmic language of recursive coatom orderings, and exploit the structure given by the shelling-type linear orderings in the construction of the matching. This structure allows a more accurate understanding of the matchings, and we decided to include it as a stepping stone toward the study of the boundary relations in the minimal complexes produced in Proposition 3.3.20, a task that we plan to undertake in future work.

We would like to point out that our shelling-type orderings appear to be a kind of generalized shellings where the bounded faces are exactly the homology facets of the considered CW-complex.

As a first step, we define the class of posets that will be the object of our study. It is clear that these posets include the posets of cells of (possibly nonpure) regular CW-complexes.

Definition 3.1.9. A poset P will be called *locally ranked* if all its principal lower ideals are ranked. It then possesses a well-defined *height function* h that assigns to every element the rank of the lower principal ideal it generates. Let $h(P) := \max\{h(p) \mid p \in P\}$. For technical reasons, we will denote by P_i the set of all $p \in P$ with $h(p) = h(P) - i$.

The set of maximal elements of a given locally ranked poset P will be denoted by M_P or simply M if no misunderstanding can occur. If an ordering $\prec$ of M_P is specified, then we can associate to every $p \in P$ a unique element

$$m_p := \max_{\prec}\{m \in M_P \mid m \geq p\}$$

(informally, the last among the maximal elements that lie above p).

We proceed to prove the key technical lemma toward Proposition 3.1.14.

Lemma 3.1.10. *Let $(P, <)$ be a locally ranked poset, and let a recursive coatom ordering $\prec$ be defined on $\widehat{P}$. Then it is possible to define a family of total orders $\{(P_i, \sqsubset_i)\}_{i=0,\dots,h(P)}$ with the following properties:*
given $p \in P_i$, and writing $Q_p := \bigcup_{p' \sqsubset_i p} \operatorname{coat}(p')$,
(1) the order induced by $\sqsubset_{i+1}$ on $D_p := \operatorname{coat}(p) \setminus Q_p$ can be extended to a recursive coatom ordering $\prec_p$ of $\operatorname{coat}(p)$ in which the elements of Q_p come first.
(2) for all $p' \sqsubset_i p$ in P_i, if $p', p > z$, then there is $p'' \sqsubset_i p$ and $w \in \operatorname{coat}(p)$ such that $p'' > w \geq z$.

Proof. The orderings $\sqsubset_i$ will be defined recursively for increasing i. First, since $P_0 \subset M$, it makes sense to let $\sqsubset_0$ coincide with the given recursive coatom ordering $\prec$. By hypothesis, for every $p \in P_0$ there is a recursive coatom ordering $\prec_p$ of $P_{\leq p}$ in which the elements of Q_p come first. Therefore we can define $\sqsubset_1$ by declaring

$$x \sqsubset_1 y \Leftrightarrow \begin{cases} x \prec_p y & \text{if there is } p \in P_0 \text{ with } x, y \in D_p, \\ p \sqsubset_0 q & x \in D_p, y \in D_q, \\ m_x \prec_{\hat{1}} y & \text{if } y \in M. \end{cases}$$

This ordering is well-defined because by construction $D_P \cap D_q = \emptyset$ if $p \neq q$. Moreover, it clearly satisfies the requirement.

Now let $i > 1$ and suppose that the orderings $\sqsubset_j$ are defined for $j \leq i$.
Definitions: For $p \in P_i$ let $q_p := \min_{\sqsubset_{i-1}}\{q \in P_{i-1} \mid q > p\}$ (and note that this implies $p \in D_q$). Moreover, let

$$N_p := \operatorname{coat}(p) \setminus \{y \in \operatorname{coat}(p') \mid p' \in \operatorname{coat}(q_p),\, p' \sqsubset_j p\}$$

and note that by definition $P_{i+1} = \coprod_{p \in P_i} N_p$. We define also

$$A_p := \{y \in \mathrm{coat}(p) \mid y < q' \text{ for } q' \sqsubset_{i-1} q_p\} \text{ and } B_p := Q_p \cap P_{<q_p},$$

so that $N_p = \mathrm{coat}(p) \setminus B_p$ (see Figure 3.4).

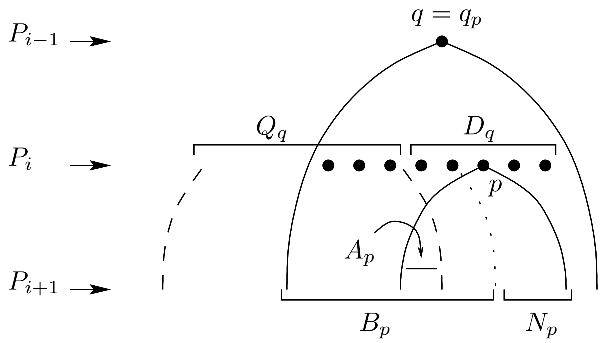


Figure 3.4:

Remark: For every $p \in P_i$ we have $A_p \subseteq B_p$. In fact, given $p \in P_i$ and $x \in A_p$, by assumption on $\sqsubset_{i-1}$ there is $w \in \mathrm{coat}(q_p)$ such that $w > x$ and $w \sqsubset_i p$.

Because $\sqsubset_i$ induces a recursive coatom ordering on $\mathrm{coat}(q_p)$, we know that there is a recursive coatom ordering $\prec_p$ of $\mathrm{coat}(p)$ such that the elements of B_p come first.

For $x, y \in P_{j+1}$ we define:

$$x \sqsubset_{i+1} y \Leftrightarrow \begin{cases} x \prec_p y & \text{if there is } p \text{ such that } x, y \in N_p, \\ p \sqsubset_i p' & \text{if } x \in N_p,\ y \in N_{p'}, \\ m_x \prec y & \text{if } y \in M \end{cases}$$

At this point it is worth to point out that, given $p \in P_i$, $Q_p = \bigcup_{p' \sqsubset_i p} N_p$ and $D_p = N_p$.

We have now to check the conditions. (1) is clear: given $p \in P_i$ and $D_p = N_p$, $\prec_p$ is a recursive coatom ordering of $\mathrm{coat}(p)$ such that the elements of B_q, and thus every $x \in \mathrm{coat}(p) \setminus Q_p$, come first. For (2) take $x, x' \in P_{i+1}$ such that $x' \sqsubset_{i+1} x$ and $z < x', x$. If $x' \in N_{p'}$ and $x \in N_p$ for $p \neq p'$, then we have $p' \sqsubset_i p$, and by property (2) of $\sqsubset_i$ there is $p'' \sqsubset_i p$ and $y \in \mathrm{coat}(p)$ such that $z \leq y \leq p''$. Applying Definition 3.1.2.(2) to $\prec_p$ we obtain an $x'' \prec_p x$ and a $w \in \mathrm{coat}(x)$ such that $z \leq w < x''$. The proof is concluded by the remark that $x'' \prec_p x$ implies $x'' \sqsubset_{i+1} x$ because $x \in N_p$. $\square$

Definition 3.1.11 (Shelling-type orderings)**.** Let P be a locally ranked poset. We introduce functions $\pi_i : P_i \to P_{i+1}$ defined by

$$\pi_i(q) := \max_{\sqsubset_{i+1}} \{p \in P_{i+1} \mid q > p\},$$

where the $\sqsubset_i$ are the orderings associated to some shelling via Lemma 3.1.10.

Then we define a linear extension $\lhd$ of P by:

$$p \lhd q \Leftrightarrow \begin{cases} p \sqsubset_i q & \text{if there is } i \text{ such that } p, q \in P_i, \\ p \sqsubseteq_i \pi_i \pi_{i+1} \cdots \pi_{j-1}(q) & \text{if } p \in P_i,\ q \in P_j \text{ and } i > j. \end{cases}$$

The easy check that this is a well-defined linear order is left to the reader. Every linear extension $\lhd$ of P that is constructed in this way from a recursive coatom ordering will be called a *shelling-type ordering* of P.

We can now construct an acyclic matching for any shelling-type ordering of a locally ranked poset.

Lemma 3.1.12. *Every shelling-type ordering $\lhd$ of a locally ranked poset P induces an acyclic matching $\mathfrak{M}$ on P.*

Proof. Definition of the matching $\mathfrak{M}$:
We start with the one-element matching $\mathfrak{M} := \{(p_1^1, \pi_1(p_1^1))\}$. For every $j = 2, \ldots, k_1$ we add $(p_j^1, \pi_1(p_j^1))$ to $\mathfrak{M}$ if $\pi_1(p_j^1)$ is not already matched (or, equivalently, if $\pi_1(p_l^1) \neq \pi_1(p_j^1)$ for all $l < j$).
For $i = 1, \ldots h(P)$ we further expand $\mathfrak{M}$ as follows: for $j = 1, \ldots, k_i$, if p_j^i is not already matched and $\pi_i(p_j^i) \neq \pi_i(p_l^i)$ for all $l < j$, then add $(p_j^i, \pi_i(p_j^i))$ to $\mathfrak{M}$.
Since, by construction, $p \rhd \pi_i(p)$ whenever $(p, \pi_i(p)) \in \mathfrak{M}$, this matching is acyclic by Lemma 3.1.5. $\square$

So far we stayed in the full generality of locally ranked posets. If we restrict ourselves to the case of posets of cells of CW-complexes, we can have even more control on the critical elements. The stepping stone for this is the following easy lemma, that we prove for completeness.

Lemma 3.1.13. *Let K be a regular CW-decomposition of a sphere. Then in every shelling order of K the only homology facet is the last one.*

Proof. The argument is by contraposition. Indeed, if the claim would not hold, then there would be a counterexample, say a regular CW-complex K, a homeomorphism $\phi : K \to S^d$, and a shelling order on the facets of K such that the last facet, call it F, is not a homology facet. This means that the union K' of all the facets other than F is a shellable complex with still a homology facet - in particular, it is not contractible. But on the other hand, this complex has to be homeomorphic to $S^d \setminus \phi(F \setminus K')$, which is contractible because $F \setminus K'$ is. A contradiction follows. $\square$

Proposition 3.1.14. *Every shelling of a regular CW-complex K induces an acyclic matching of the poset of faces of K. Moreover, the critical cells of this matching correspond to the homology facets of the given shelling.*

Proof. Consider a critical element $p \in P_i$ such that p is not maximal in P. Several situations can occur:
(i) There is $q \gtrdot p$ such that $(q, \pi_{i-1}(q)) \in \mathfrak{M}$. Then $p \sqsubset_i \pi_{i-1}(q)$ and, since p was not matched, there must be $\tilde{p} \sqsubset_i p$ such that $\pi_i(p) = \pi_i(\tilde{p})$. In particular, every element of $x \in \operatorname{coat}(p)$ is coatom of some $p' \sqsubset_i p$ by 3.1.10.(1). We may assume without loss of generality that $p' \in \operatorname{coat}(q)$, because else by property (2) of Lemma 3.1.10 we can find $p'' \in \operatorname{coat}(q)$ such that $x < p''$. This all means that, in the shelling of $P_{<q}$ that is induced by $\sqsubset_i$, the whole boundary of p is already taken when the turn of p comes. But since p is not the last element of this shelling (which is $\pi_{i-1}(q)$), using Lemma 3.1.13 we get a contradiction with the fact that $P_{<q}$ is a shellable sphere. This case can therefore not enter.$\diamondsuit$

(ii) There is $q \gtrdot p$ that is not matched. If for this q we have $p \sqsubset_i \pi_{i-1}(q)$, then the same reasoning of item (i) applies to get a contradiction. On the other hand, if $\pi_{i-1}(q) = p$ then our algorithm should have taken the edge $(q, \pi_{i-1}(q) = p)$ into $\mathfrak{M}$ when examining the elements of P_{i-1}: indeed, p was not already taken as $\pi_{i-1}(q')$ for any $q' \sqsubset_{i-1} q$ (and actually it will remain free until the end!). So, this second situation can also not happen.$\diamondsuit$
(iii) Else: every $q \gtrdot p$ is matched 'from above', i.e., by an edge $(w, \pi_{i-2}(w) = q)$. In this case, let $q_1, \ldots, q_k$ be any enumeration of the elements that cover p. We know that no edge (q_j, p) is matched, but we have supposed also that for every $j = 1, \ldots, k$ there is w_j such that $(w_j, q_j) \in \mathfrak{M}$. Since P is a CW-poset, we know (e.g. by [16, Proposition 2.2]) that every interval of length 2 has four elements - so that to every $j \in \{1, \ldots, k\}$ we can associate $\phi(j) \in \{1, \ldots, k\}$ such that the interval $[p, w_j]$ has elements $\{p, q_j, q_{\phi(j)}, w_j\}$. In this interval by assumption the edge (w_j, q_j) is matched, and therefore for sure $(w_j, q_{\phi(j)}) \notin \mathfrak{M}$. This implies in particular $w_j \neq w_{\phi(j)}$ for every j. But then the alternating path $q_j, w_j, q_{\phi(j)}, w_{\phi(j)}, q_{\phi^2(j)}, \ldots$ must be a cycle, because ϕ can take only finite many values (we supposed the CW-complexes to be locally finite). Thus, also this case cannot enter.$\diamondsuit$

It follows that every critical element is a maximal element of P, i.e., by a facet of K. But a maximal element $m \in P_i$ is not matched exactly when $\max_{\sqsubset_{i+1}} \operatorname{coat}(m)$ is matched by some $p \sqsubset_i m$ (and hence, by item (i) above, when all its coatoms are). In topological words, m is critical exactly if, when its turn in the shelling comes, its whole boundary was already taken. This means exactly that m is a homology facet of the given shelling. $\square$

Example 3.1.15. The acyclic matching depicted in Figure 3.3 is induced from the shelling order and the recurdive coatom ordering of Figure 3.2 by the following shelling-type ordering:

$$I \lhd a \lhd b \lhd II \lhd f \lhd III \lhd e \lhd IV \lhd c \lhd V \lhd d \lhd VI \lhd \hat{0}.$$

Remark 3.1.16. Proposition 3.1.14 gives a perfect acyclic matching of $\widehat{P}$ whenever P is the poset of faces of a regular CW-complex that is homotopy equivalent to a sphere. Indeed, in that case the only critical cell of P can be matched by the added element $\widehat{1} = \widehat{P} \setminus P$.

3.2 Shelling-type orderings of oriented matroids

In this section we apply Proposition 3.1.14 to a special situation, as an attempt to answer [87, Remark 4.5] and as a stepping stone to the results of Section 3.3. If we consider the fan defined by a set of real linear hyperplanes, we see that the boundary of the associated polar polytope is a shellable (CW-) sphere. The combinatorics of real arrangements of hyperplanes is customarily encoded by *oriented matroids.* These combinatorial objects are more general than real linear hyperplane arrangements; however, to every oriented matroid corresponds a shellable CW-sphere that, in case the oriented matroid describes an arrangement, is combinatorially isomorphic to the associated polar polytope.

It is a nice fact that, for any oriented matroid $\mathcal{M}$, $\mathcal{F}(\mathcal{M})^{op}$ (suitably augmented by an additional $\hat{0}$-element, if needed) is the poset of faces of a convex polytope that is called the *zonotope* of $\mathcal{M}$. The 1-skeleton of its dual polytope is isomorphic, as a graph, to the Hasse diagram of $\mathcal{T}_B(\mathcal{M})$ for every $B \in \mathcal{T}$. In this sense, specifying a linear extension of $\mathcal{T}_B$ amounts to somehow 'specify a direction' in the ambient space of the zonotope. Indeed, such a linear ordering is all what one needs to get a shelling of the zonotope.

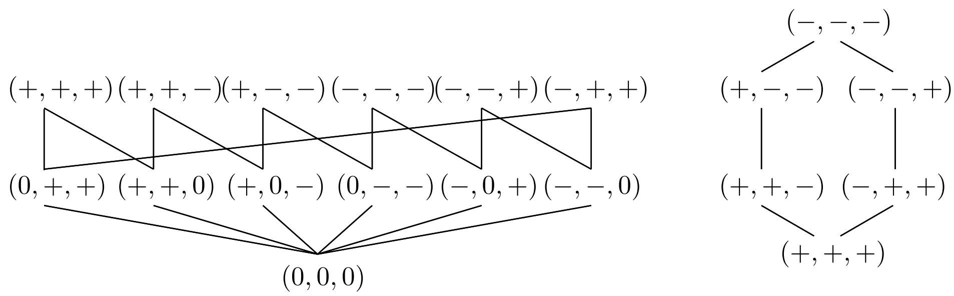


Figure 3.5: On the left is the face poset $\mathcal{F}(\mathcal{M})$ of an oriented matroid on 3 elements. Its (augmented) dual $\mathcal{F}(\mathcal{M})^{op}$ appeared already in Figure 3.1, so that the zonotope of this oriented matroid is the hexagon K_1. The dual polytope of K_1 is again an hexagon, so that the tope poset $\mathcal{T}_{(+,+,+)}(\mathcal{M})$ for this oriented matroid is the poset depicted on the right.

Theorem 3.2.1 (Proposition 4.3.2 of [20])**.** *Let $\mathcal{M}$ be a simple oriented matroid and B be a tope of $\mathcal{M}$. Every linear extension of the tope poset $\mathcal{T}_B(\mathcal{M})$ induces a recursive coatom ordering of $\mathcal{F}(\mathcal{M})$.*

Notation 3.2.2. We will use the symbol $\dashv$ to indicate total orderings that are linear extensions of the ordering of a tope poset.

Thus, an application of Proposition 3.1.14 gives immediately the following existence result.

Theorem 3.2.3. *Let $\mathcal{M}$ be a simple oriented matroid and B be a tope of $\mathcal{M}$. Every linear extension $\dashv$ of the tope poset $\mathcal{T}_B(\mathcal{M})$ defines an acyclic matching $\mathfrak{M}$ of the face poset $\mathcal{F}(\mathcal{M})$ such that the only critical element is $-B$, the tope opposite to B.*

Example 3.2.4. One possible linear extension of the tope poset of Figure 3.5 is given by

$$(+,+,+) \dashv (+,+,-) \dashv (+,-,-) \dashv (-,+,+) \dashv (-,+,+) \dashv (-,-,+) \dashv (-,-,-).$$

Comparing Figure 3.2 we see that this linear extension corresponds indeed to the shelling order $I, II, \ldots, VI$ of K_2 via the correspondence of the posets of faces, and thus induces on the poset $\mathcal{F}(\mathcal{M}) = \mathcal{F}(K_2)$ the acyclic matching indicated in Figure 3.3.

Remark 3.2.5 (On polar orderings)**.** As we will explain in detail in the next Section, to every real linear arrangement of hyperplanes is associated an oriented matroid whose covectors correspond to the induced stratification of $\mathbb{R}^n$. These 'special' oriented matroids can be therefore also given a *polar ordering* in the sense of Salvetti and Settepanella [87]. This makes a comparison of the two orderings possible. The outcome is that shelling-type orderings are different from the polar orderings of [87] on linear arrangements (although they can be used for the same scope, as we will see in the next section): indeed, a polar ordering is *never* a linear extension of the face poset (as can be easily seen comparing Theorem 4 of [87]). Moreover, the order induced on the chambers by a polar ordering is never a linear extension of a poset of regions: otherwise, there would be no other choice for the base chamber B as to take the chamber containing the basepoint of the polar ordering. But then we see that there is a maximal chain in $\mathcal{T}_B$ (determined by the general position line V_1 of [87]) whose elements form by definition an initial segment in the order of chambers induced by the polar ordering. This is clearly incompatible with being a linear extension of $\mathcal{T}_B$.

Nevertheless, at a first glance the ordering induced on the chambers by the polar orders seems to be a shelling order for the zonotope. We leave this as an open question.

Remark 3.2.6. The proofs of [20, Proposition 4.3.1 and 4.3.2] are constructive. Hence, by taking a closer look at the arguments used there one can give an explicit description of the shelling-type orderings (and thus of the matchings) that result from our construction. To do this, we need some notation. For every element e of the oriented matroid let $R(e) := \min_{\dashv}\{B_F \mid |F| = e\}$, and let $F(e)$ be the unique face with $F(e) \lessdot R(e)$ and $|F(e)| = e$. Then, for every $R \in \mathcal{T}_B$ choose a maximal chain ω_R in the interval $[B, -R] \subset \mathcal{T}_R(\mathcal{M})$. For $i = 0, \ldots, n$ let $\omega_R(i)$ denote the i-th element of the chain (counted from the bottom).

For every maximal element R of $\mathcal{F}$ we can express D_R, Q_R and $\pi_0(R)$ (see Lemma 3.1.10) as follows:

$$
\begin{aligned}
D_R &= \{F \in \operatorname{coat}(R) \mid |F| \in S(-R)\} \quad (= \{F \in \operatorname{coat}(R) \mid R_F = T_F\}),\\
Q_R &= \{F \in \operatorname{coat}(R) \mid |F| \in S(R)\} \quad (= \{F \in \operatorname{coat}(R) \mid R_F \neq T_F\}),\\
\pi_0(R) &= \omega_R(n).
\end{aligned}
$$

We conclude that the induced ordering $\sqsubset_1$ on $\mathcal{F}$ can be expressed by

$$
F_1 \sqsubset_1 F_2 \Leftrightarrow \begin{cases} T_{F_1} \dashv T_{F_2} \text{ or} \\ T_{F_1} = T_{F_2} =: R \text{ and } |F_1| \prec_R |F_2|, \end{cases}
$$

where $\prec_R$ is the order in which the elements appear as $S(\omega_R(i), \omega_R(i+1))$ for increasing i.

Moreover, according to the construction of [20, 4.3.1 and 4.3.2], the recursive ordering of $\operatorname{coat}(F)$ is given as above by any maximal chain in $\mathcal{T}_F(\mathcal{M}/|F|)$ that contains F', where F' is the face where ω_R crosses $|F|$. In particular, the elements of D_F are ordered according to a maximal chain in the interval $[F', -F] \subset \mathcal{T}_F(\mathcal{M}/|F|)$, and so on.

3.3 Acyclic maximum matchings for the Salvetti complex

The main motivation of Salvetti and Settepanella for considering polar orderings in [87] was to use these total orderings in the construction of what they call the *polar gradient*. The polar gradient of [87] is essentially an acyclic maximum matching of the poset of cells of the Salvetti complex - a regular CW complex that was introduced by Mario Salvetti in order to model the homotopy type of the complement of a complexified arrangement of hyperplanes (see Definition 3.3.1 and [86]).

In this section we want to construct acyclic matchings for the Salvetti complex of linear arrangements using shelling-type orderings. In fact, the outcome is that linear extensions of tope posets give a very nice stratification of the Salvetti complex (see Lemma 3.3.19) and allow us to paste together different choices of acyclic matchings of the strata.

Let us begin by the definition of the poset of cells of the Salvetti complex for a general oriented matroid. In a second step we will introduce the terminology (and the geometric intuition) of arrangements of hyperplanes, which we will adopt for the following sections.

Definition 3.3.1. Recall the Salvetti complex associated to an oriented matroid $\mathcal{M}$. Note that the poset $\mathcal{F}(\mathcal{M})$ has a unique minimal element that we denote by P. For any given tope T let $\mathcal{S}_T := \mathcal{S}(\mathcal{M})_{\leq \langle P,T \rangle}$. It is clear that $\mathcal{S}_T$ is isomorphic to $\mathcal{F}(\mathcal{M})^{op}$ as a poset. If no confusion can arise we will write just $\mathcal{S}$, $\mathcal{F}$, $\mathcal{T}$ for $\mathcal{S}(\mathcal{M})$, $\mathcal{F}(\mathcal{M})$, $\mathcal{T}(\mathcal{M})$.

Now fix a "base tope" $B \in \mathcal{T}$. If a linear extension $\dashv$ of $\mathcal{T}_B$ is given, define, for every $R \in \mathcal{T}$,

$$\mathcal{S}(R) := \bigcup_{T \dashv R} \mathcal{S}_T \quad \text{and} \quad N(R) := \mathcal{S}(R) \setminus \mathcal{S}(R'),$$

where R' is the tope that precedes R in $\dashv$.

Example 3.3.2. The poset of Figure 3.8 is $\mathcal{S}(\mathcal{M})$ for the (realizable) oriented matroid $\mathcal{M}$ of Figure 3.5, where for better readability we denoted the covectors by the corresponding strata in $\mathbb{R}^2$ (see Figure 3.6).

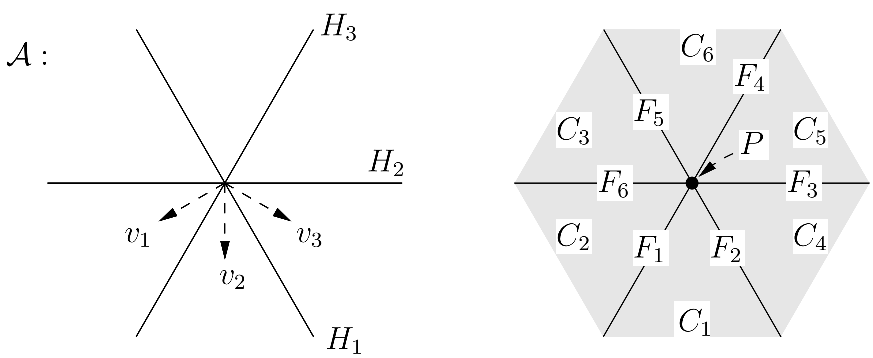


Figure 3.6: Our main example, the arrangement of three lines in the plane. On the left the 'plain' arrangement, with our choice of normal vectors to build the oriented matroid $\mathcal{M}_{\mathscr{A}}$. On the right, the cells of the induced stratification of $\mathbb{R}^2$.

If $\mathcal{M}$ is a realizable oriented matroid corresponding to the arrangement $\mathscr{A}$, then the poset $\mathcal{F}(\mathcal{M})$ is the poset of the closed strata determined by $\mathscr{A}$ in $\mathbb{R}^d$, ordered by inclusion of the topological closures.

Example 3.3.3. For $\mathscr{A}$ as in Figure 3.6, $\mathcal{M}_{\mathscr{A}}$ is the oriented matroid $\mathcal{M}$ of Figure 3.5. In particular, we can compare the poset $\mathcal{F}(\mathcal{M})$ of Figure 3.5 with the stratification of $\mathbb{R}^2$ on the right hand side of Figure 3.6. For instance, the covector $(+, 0, -)$ represents the stratum of all vectors of $\mathbb{R}^2$ which scalar product with v_1 is positive, with v_2 equals 0 and with v_3 is negative (i.e., the points 'in front of' H_1, 'on' H_2 and 'behind' H_3 with respect to the base chamber $B = (+, +, +)$).

The topes are the maximal strata - i.e., the closure of the connected components of the complement $\mathbb{R}^d \setminus \bigcup \mathscr{A}$ of $\mathcal{M}_{\mathscr{A}}$ - and are customarily called *chambers* (or *regions*) of $\mathscr{A}$ (given a set $A := \{a_1, a_2, \ldots, a_n\}$, we will write $\bigcap A$ for the set $a_1 \cap a_2 \cap \ldots \cap a_n$ and $\bigcup A$ for $a_1 \cup a_2 \cup \ldots \cup a_n$). Accordingly, $\mathcal{T}_B(\mathscr{A})$ is often referred to as the *poset of regions* of $\mathscr{A}$ (e.g., in his first appearance in the context of hyperplane arrangements, see [49]). For any two chambers C_1, C_2 of $\mathscr{A}$ (topes of $\mathcal{M}_{\mathscr{A}}$), the elements of $S(C_1, C_2)$ correspond to the hyperplanes that *separate*[1] C_1 from C_2, i.e., the hyperplanes that are met by any line segment connecting a point in the interior of C_2 with a point in the interior of C_1. Since the arrangements corresponding to oriented matroids are linear, every chamber is a convex cone. The hyperplanes supporting the facets of the cone determined by the chamber C are called *walls* of C. The set of walls of C is denoted by $\mathcal{W}_C$.

Remark 3.3.4. For every wall $H \in \mathcal{W}_C$ there is a chamber $K \in \mathcal{T}(\mathscr{A})$ such that $S(C, K) = \{H\}$. In fact, this can be taken as the 'abstract' definition in the setting of arbitrary oriented matroids.

Notation 3.3.5. We will denote by $\mathcal{L}(\mathscr{A}) = \mathcal{L}(\mathcal{M})$ (or just by $\mathcal{L}$) the *lattice of flats* of the underlying matroid; this is indeed a geometric lattice and we will think of it as of the poset of intersections of the hyperplanes ordered by reverse inclusion (see the top of Figure 3.7 for a picture of $\mathcal{L}(\mathscr{A})$ when $\mathscr{A}$ is the arrangement of three lines through the origin of the plane). For every face $F \in \mathcal{F}(\mathcal{M})$ we write $|F|$ for what corresponds to the "affine span" of F, i.e., the flat given by the elements of $\mathrm{supp}(F)$. Given any flat $Y \in \mathcal{L}$, we denote by $\mathscr{A}_Y$ the arrangement given by the hyperplanes that contain Y and set $\mathscr{A}_Y =: \mathrm{supp}(Y)$. We write $\mathscr{A}^Y$ for the arrangement $\{H \cap Y \mid H \notin \mathscr{A}_Y\}$ that is determined on Y by the hyperplanes that intersect Y nontrivially. The oriented matroid associated to $\mathscr{A}^Y$ is the *contraction* $\mathcal{M}(\mathscr{A})/Y$ of the oriented matroid associated to $\mathscr{A}$ (see [20, Section 3.3]). The natural map $\mathcal{T}(\mathscr{A}) \to \mathcal{T}(\mathscr{A}_Y)$ will be denoted by π_Y. We will use it to explain the geometric content of the operation described in Definition 2.2.16.

Remark 3.3.6. Let $\mathcal{M}$ be a realizable oriented matroid and $\mathscr{A}$ the corresponding arrangement. Let C be one of its topes (chambers) and F be some covector

[1]The use of the word 'separation' arose in the litarature while considering the chambers to be the *open* sets that are obtained subtracting $\mathscr{A}$ from $\mathbb{R}^d$, so that any two chambers are really disjoint and 'separated' by the hyperplanes in the set $S(C_1, C_2)$. For consistency we let here the chambers be, as any other face, closed. The combinatorics of course works as well, and we will save some cumbersome distinctions in the last section.

(face) of $\mathcal{M}(\mathscr{A})$. Then the tope T_F corresponds to the unique chamber that is contained in $\pi_{|F|}(T)$ and contains F.

Important Remark 3.3.7. In all what follows, unless explicitly stated,

$$\boxed{\mathscr{A} \textit{ will denote a finite arrangement of } n \textit{ linear hyperplanes in } \mathbb{R}^d.}$$

Moreover, we fix from now an (arbitrarily chosen) base chamber $B \in \mathcal{T}(\mathscr{A})$ and a (also arbitrary) linear extension $\dashv$ of $\mathcal{T}_B(\mathscr{A})$.

Let us also point out that everything we will say can be easily translated into the language of (and thus: holds for) abstract oriented matroids. As the 'grammar' and the 'vocabulary' for this translation we refer to [20].

Notation 3.3.8. Given $H \in \mathscr{A}$, let $\mathscr{A}' := \mathscr{A} \setminus \{H\}$. Given $C \in \mathcal{T}(\mathscr{A})$, we will write C' for the unique chamber of $\mathscr{A}'$ that contains C. This natural inclusion of chambers induces an order preserving map

$$\psi : \mathcal{T}_{B'}(\mathscr{A}') \to \mathcal{T}_B(\mathscr{A}); \qquad C' \mapsto \min_{\dashv}\{C \in \mathcal{T}_B(\mathscr{A}) \mid C \subseteq C'\}.$$

Note that if $C' \in \mathcal{T}(\mathscr{A}')$ contains two chambers $C_1, C_2 \in \mathcal{T}(\mathscr{A})$ then, up to renumbering, $C_1 \prec_B C_2$. So this definition could have been phrased as well in terms of $\preccurlyeq$, the partial ordering of $\mathcal{T}_B(\mathscr{A})$, instead of $\dashv$.

This map is clearly injective, and thus for $C_1', C_2' \in \mathcal{T}(\mathscr{A}')$ the ordering $\preccurlyeq'$ of $\mathcal{T}_{B'}(\mathscr{A}')$ satisfies

$$C_1' \preccurlyeq' C_2' \Leftrightarrow \psi(C_1') \preccurlyeq \psi(C_2'). \tag{3.3.1}$$

Given any linear extension $\dashv$ of $\mathcal{T}_B(\mathscr{A})$ we let let $\dashv'$ denote the linear extension of $\mathcal{T}_{B'}(\mathscr{A}')$ that is in a sense the 'pullback' of $\dashv$ along ψ:

$$C_1' \dashv' C_2' :\Leftrightarrow \psi(C_1') \dashv \psi(C_2').$$

As we will see this construction is canonical.

Lemma 3.3.9. *Given two distinct hyperplanes $H_1, H_2 \in \mathscr{A}$, for both $i = 1, 2$ write $\mathscr{A}_i := \mathscr{A} \setminus \{H_i\}$ and let B_i be the unique chamber of $\mathscr{A}_i$ containing B. Let ψ_i denote the map $\mathcal{T}_{B_i}(\mathscr{A}_i) \to \mathcal{T}_B(\mathscr{A})$ defined in 3.3.8. Let then $\widehat{B}$ be the unique chamber of $\mathscr{A}_1 \cap \mathscr{A}_2$ that contains B_1 and B_2, and write $\widehat{\psi}_i$ for the corresponding map $\mathcal{T}_{\hat{B}} \to \mathcal{T}_{B_i}(\mathscr{A}_i)$. Then the diagram of poset maps*

$$\begin{tikzcd}
\mathcal{T}_{\widehat{B}}(\mathscr{A}_1 \cap \mathscr{A}_2) \arrow[r, "\widehat{\psi}_1"] \arrow[d, "\widehat{\psi}_2"'] & \mathcal{T}_{B_1}(\mathscr{A}_1) \arrow[d, "\psi_1"'] \\
\mathcal{T}_{B_2}(\mathscr{A}_2) \arrow[r, "\psi_2"] & \mathcal{T}_{\hat{B}}(\mathscr{A})
\end{tikzcd}$$

commutes.

Proof. For brevity, let $\widehat{\mathscr{A}} := \mathscr{A}_1 \cap \mathscr{A}_2$. Consider $\widehat{C} \in \mathcal{T}(\widehat{\mathscr{A}})$. By definition we have

$$\widehat{\psi}_i(\widehat{C}) = \min_{\preccurlyeq_i}\{C' \in \mathcal{T}_{B_i}(\mathscr{A}_i) \mid C' \subset \widehat{C}\},$$

where $\preccurlyeq_i$ is the ordering of $\mathcal{T}_{B_i}(\mathscr{A}_i)$. This, in view of equation 3.3.1, means

$$\min_{\preccurlyeq}\{C \in \mathcal{T}(\mathscr{A}) \mid C \subset \widehat{\psi}_i(\widehat{C})\} \preccurlyeq \min_{\preccurlyeq}\{C \in \mathcal{T}(\mathscr{A}) \mid C \subseteq C' \subseteq \widehat{C} \text{ for a } C' \in \mathcal{T}_{B_i}(\mathscr{A}_i)\}$$

or, equivalently,

$$\min_{\preccurlyeq}\{C \in \mathcal{T}(\mathscr{A}) \mid C \subset \widehat{\psi}_i(\widehat{C})\} \preccurlyeq \min_{\preccurlyeq}\{C \in \mathcal{T}(\mathscr{A}) \mid C \subseteq \widehat{C}\}.$$

Now, because we are taking away from $\mathscr{A}$ exactly two hyperplanes, the right side of the last expression takes the minimum over a poset that either has only one element, or is a two-element chain, or has four elements and rank two (depending on whether none, one or both of H_1 and H_2 cut $\widehat{C}$). Thus, in any case the right side above identifies a unique $C \in \mathcal{T}_B(\mathscr{A})$, and this is $\psi_i\widehat{\psi}_i(\widehat{C})$. Summarizing, we have

$$\psi_i\widehat{\psi}_i(\widehat{C}) = \min_{\preccurlyeq}\{C \in \mathcal{T}(\mathscr{A}) \mid C \subseteq \widehat{C}\}.$$

Since this expression does not depend on i, we are done. □

We will need the following corollary.

Corollary 3.3.10. *In the setting of Lemma 3.3.9, for $i = 1, 2$ let $\dashv_i$ be the linear extension induced from $\dashv$ on $\mathcal{T}_{B_i}(\mathscr{A}_i)$, and $\widehat{\dashv}_i$ the linear extension of $\mathcal{T}_{\widehat{B}}(\widehat{A})$ induced from $\dashv_i$. Then*

$$\textit{for all } \widehat{C}, \widehat{K} \in \mathcal{T}_{\widehat{B}}(\mathscr{A}_1 \cap \mathscr{A}_2), \qquad \widehat{C} \mathrel{\widehat{\dashv}_1} \widehat{K} \;\Leftrightarrow\; \widehat{C} \mathrel{\widehat{\dashv}_2} \widehat{K}$$

Proof. For both $i = 1, 2$, $\widehat{C} \mathrel{\widehat{\dashv}_i} \widehat{K}$ if and only if $\psi_i\widehat{\psi}_i(\widehat{C}) \dashv \psi_i\widehat{\psi}_i(\widehat{K})$. The claim follows with Lemma 3.3.9. □

Now we can define the the object we will study in the next few statements. Recall that we fixed a linear extension $\dashv$ of the tope poset of $\mathscr{A}$.

Definition 3.3.11. For every $C \in \mathcal{T}(\mathscr{A})$ we let

$$\mathcal{J}(C) := \{X \in \mathcal{L}(\mathscr{A}) \mid \operatorname{supp}(X) \cap S(C, K) \neq \emptyset \text{ for every } K \dashv C\},$$

which is easily seen to be an upper ideal in $\mathcal{L}(\mathscr{A})$.

Notation 3.3.12. Let $H \in \mathscr{A}$ be given and recall the notation 3.3.8. We write $\mathcal{J}'(C')$ for the order ideal of $\mathcal{L}(\mathscr{A}')$ associated to C' and $\dashv'$ in Definition 3.3.11. The inclusion $\mathscr{A}' \hookrightarrow \mathscr{A}$ induces an order preserving injection

$$\iota : \mathcal{L}(\mathscr{A}') \to \mathcal{L}(\mathscr{A}),\ X \mapsto \bigcap \operatorname{supp}(X).$$

We will identify $\mathcal{J}'(C')$ with its image under this map.

Lemma 3.3.13. *Let a chamber $C \in \mathcal{T}_B(\mathscr{A})_{<\hat{1}}$ be given, choose $H \in \mathscr{A} \setminus S(B, C)$ (such an hyperplane exists because $C \neq -B$) and let $\mathscr{A}' := \mathscr{A} \setminus \{H\}$.*

For every $Y \in \mathcal{J}(C)$ we have

$$\bigcap(\operatorname{supp}(Y) \setminus \{H\}) \in \mathcal{J}'(C').$$

Proof. As a first step, observe that

($\star$) *If $H \notin \operatorname{supp}(Y)$, then $Y \in \mathcal{J}(C) \Leftrightarrow \bigcap(\operatorname{supp}(Y) \setminus \{H\}) \in \mathcal{J}'(C')$,* because in this case $\bigcap(\mathrm{supp}(Y) \setminus \{H\}) = Y$, and the conditions for being in $\mathcal{J}'(C')$ and $\mathcal{J}(C)$ become equivalent. Therefore suppose from now $H \in \mathrm{supp}(Y)$.

We want to argue by induction on $|\mathscr{A}|$. If $|\mathscr{A}| = 1$ the claim is trivial. So suppose $|\mathscr{A}| > 1$ and that the claim holds for every smaller arrangement. We need to distinguish two cases:

<u>*Case 1:*</u> $Y = \hat{1} \in \mathcal{L}(\mathscr{A})$. In this situation

$$\bigcap(\mathrm{supp}(Y) \setminus \{H\}) = \hat{1} \in \mathcal{L}(\mathscr{A}').$$

Since both $\mathcal{J}(C)$ and $\mathcal{J}'(C')$ are nonempty upper ideals, we have $Y \in \mathcal{J}(C)$ and $\bigcap(\mathrm{supp}(Y) \setminus \{H\}) \in \mathcal{J}'(C')$ and the claim holds.

<u>*Case 2:*</u> $Y \neq \hat{1} \in \mathcal{L}(\mathscr{A})$. Thus we can find $\widetilde{H} \in \mathscr{A} \setminus \mathrm{supp}(Y)$. Since $H \in \mathrm{supp}(Y)$, in particular $\widetilde{H} \neq H$. We need a couple of definitions, in order to apply Lemma 3.3.9.

Let $\widetilde{\mathscr{A}} := \mathscr{A} \setminus \{\widetilde{H}\}$, $\widetilde{\dashv}$ the induced linear extension, $\widetilde{\mathcal{J}}(\widetilde{C})$ the corresponding upper ideal (where $\widetilde{C}$ is the unique chamber containing C) and define $\widetilde{Y} := \bigcap(\mathrm{supp}(Y) \setminus \{\widetilde{H}\})$. Moreover, let $\widetilde{A}' := \widetilde{\mathscr{A}} \setminus \{H\}$ $(= \widetilde{\mathscr{A}} \cap \mathscr{A}')$ and define $\widetilde{\dashv}'$, $\widetilde{C}$ and $\widetilde{\mathcal{J}}'(\widetilde{C}')$ (noting that by Corollary 3.3.10 it does not matter to specify whether $\widetilde{\dashv}'$ is induced by $\widetilde{\dashv}$ or $\dashv'$). We have the following implications:

(I) $Y \in \mathcal{J}(C) \Rightarrow \widetilde{Y} \in \widetilde{\mathcal{J}}(\widetilde{C})$, e.g. by $(\star)$.

(II) $\widetilde{Y} \in \widetilde{\mathcal{J}}(\widetilde{C}) \Rightarrow \bigcap(\mathrm{supp}(\widetilde{Y}) \setminus \{\widetilde{H}\}) \in \widetilde{\mathcal{J}}'(\widetilde{C}')$ by the inductive hypothesis, since $H \in S(C, -B) \subseteq S(\widetilde{C}, \widetilde{B})$ and $|\widetilde{\mathscr{A}}| < |\mathscr{A}|$ (here $\widetilde{\dashv}'$ is viewed as being induced from $\widetilde{\dashv}$).

(III) $\bigcap(\mathrm{supp}(\widetilde{Y}) \setminus \{\widetilde{H}\}) \in \widetilde{\mathcal{J}}'(\widetilde{C}') \Rightarrow \bigcap(\mathrm{supp}(Y) \setminus \{H\}) \in \mathcal{J}'(C')$ again by $(\star)$, where we used Corollary 3.3.10 in switching point of view and considering $\widetilde{\dashv}'$ to be induced from $\dashv'$.

The lemma follows by chaining up these implications. $\square$

Theorem 3.3.14. *For every $C \in \mathcal{T}_B(\mathscr{A})$, $\mathcal{J}(C) \subset \mathcal{L}(\mathscr{A})$ is a principal upper ideal.*

Proof. If $C = -B$, then clearly $\mathcal{J}(C) = \{\hat{1}\} \subset \mathcal{L}(\mathscr{A})$ and the claim holds. If C is not $-B$, in particular there is $H \in S(C, -B) = \mathscr{A} \setminus S(B, C)$, and $\mathscr{A}' := \mathscr{A} \setminus \{H\}$ satisfies the theorem by induction hypothesis.

By Lemma 3.3.13 the (order preserving) map

$$\lambda : \mathcal{L}(\mathscr{A}) \to \mathcal{L}(\mathscr{A}'), Y \mapsto \bigcap(\mathrm{supp}(Y) \setminus \{H\}$$

satisfies $\lambda(\mathcal{J}(C)) \subseteq \mathcal{J}'(C')$. Note that the inclusion ι of $\mathcal{J}'(C')$ into $\mathcal{J}(C)$ is well defined because whenever $K \dashv C$, then $K' \dashv C'$ and $S(C', K') \cap \mathrm{supp}(Y) \subset S(C, K) \cap \mathrm{supp}(Y)$: if the former is nonempty, then so is the latter.

If we look at the composition of λ with ι, we see that $\iota\lambda(Y) \leq Y$ in $\mathcal{L}(\mathscr{A})$ for every $Y \in \mathcal{J}(C)$. Now consider two elements $Y_1, Y_2 \in \mathcal{J}(C)$: by induction hypothesis $\lambda(Y_1) \wedge \lambda(Y_2)$ exists in $\mathcal{J}'(C')$. In $\mathcal{J}(C)$ we then have an element $\iota(\lambda(Y_1) \wedge \lambda(Y_2)) \leq Y_1 \wedge Y_2$. Since $\mathcal{J}(C)$ is an upper ideal in the lattice $\mathcal{L}(\mathscr{A})$, the proof is complete. $\square$

This theorem ensures the existence of the object that we are going to define. For a *construction* of this object one needs some more refined considerations that we will carry out in Section 3.4.

Definition 3.3.15. Choose, as usual, a base chamber $B \in \mathcal{T}(\mathscr{A})$, let a linear extension $\dashv$ of $\mathcal{T}_B(\mathscr{A})$ be given, and recall Definition 3.3.11.

For every $C \in \mathcal{T}(\mathscr{A})$ define

$$X_C := \min \mathcal{J}(C).$$

From the arguments stated above we can also obtain

Corollary 3.3.16. *With the assumptions and notations of Definition 3.3.15:*

$$\text{if we define } F_C := X_C \cap C, \text{ we have } |F_C| = X_C.$$

Proof. consider $C \in \mathcal{T}(\mathscr{A})$. We will show that $\dim(X_C \cap C) = \dim(X_C)$ whenever $C \neq -B$ (in the remaining case, there is nothing to show).

Since the claim is trivial when $|\mathscr{A}| = 1$, we will proceed by induction, assuming from now that $|\mathscr{A}| > 1$ and that the claim holds for every arrangement with at most $|\mathscr{A}| - 1$ hyperplanes.

Choose $H \in \mathcal{W}_C \cap S(C, -B)$ (this can be done whitout loss of generality) and note that then C is the intersection of C' with the (closed) halfspace H^+ bounded by H and containing B. Thus,

$$C = C' \cap H^+.$$

We will write $X_C = \min \mathcal{J}(C)$ and $X'_{C'} := \min \mathcal{J}'(C')$. By induction hypothesis:

$$\dim(X'_{C'} \cap C') = \dim(X'_{C'}).$$

Recall now the maps defined in the proof of Theorem 3.3.14. We have

$$\lambda(X_C) = X'_{C'}$$

by injectivity of ι.

Therefore, only two cases can happen: either $\bigcap \operatorname{supp}(X_C) = \bigcap \big(\operatorname{supp}(X_C) \setminus \{H\}\big)$, and thus $X_C = X'_{C'}$, or $\bigcap \operatorname{supp}(X_C) \neq \bigcap \big(\operatorname{supp}(X_C) \setminus \{H\}\big)$, which implies $X_C = X'_{C'} \cap H$.

If $X_C = X'_{C'}$, then in particular $X_C \subset H$ and thus

$$\dim(C \cap X_C) = \dim(C' \cap H^+ \cap X'_{C'}) = \dim(X'_{C'} \cap H^+) = \dim(X_C).$$

If on the contrary $X_C = X'_{C'} \cap H$, then

$$\dim(C \cap X_C) = \dim(C' \cap H^+ \cap X'_{C'} \cap H)$$

$$= \dim(C' \cap X'_{C'} \cap H) = \dim(X'_{C'} \cap H) = \dim(X_C)$$

□

Question 3.3.17. It seems likely that the previous arguments can be carried out also for arrangements of affine hyperplanes, at least if B is assumed to be an *unbounded* chamber. Since this is not directly relevant for this work, we leave this as a question.

We return to the 'linear' case. The following lemma states, for later reference, an easy reformulation of the definition of X_C.

Lemma 3.3.18. *By Definitions 3.3.11 and 3.3.15, the flat X_C is uniquely determined by the following properties:*

(1) $S(K,C) \cap \operatorname{supp}(X_C) \neq \emptyset$ *for all* $K \dashv C$*, and*

(2) For every $Y \in \mathcal{L}(\mathscr{A})$ *such that* $Y \not> X_C$ *there is a chamber* $K \dashv C$ *such that* $S(K,C) \cap \operatorname{supp}(Y) = \emptyset$.

Proof. Clear. □

The next lemma shows the point of the above definitions: the X_C actually describe in very compact way the strata $N(C)$ of Definition 3.3.1.

Lemma 3.3.19. *Let* $\mathcal{M}$ *denote the oriented matroid associated to a real, linear arrangement* $\mathscr{A}$*, choose a base region* $B \in \mathcal{T}(\mathscr{A})$ *and a linear extension* $\dashv$ *of* $\mathcal{T}_B(\mathscr{A})$*, and recall Definition 3.3.1. Then*

$$N(C) \simeq \mathcal{F}(\mathcal{M}/X_C).$$

Proof. The right-to-left inclusion is easy. Indeed, if $F \in \mathcal{F}(\mathcal{M}/X_C)$, then $S(C_F,K) \cap \operatorname{supp}(F) = S(C,K) \cap \operatorname{supp}(F)$ for all K. By Lemma 3.3.18.(1), for all $K \dashv C$ we have $S(C,K) \cap \operatorname{supp}(F) \neq \emptyset$, and thus $C_F \neq K_F$. For the other direction, suppose $\langle F; C\rangle \in N(C) \setminus \mathcal{F}(\mathcal{M}/X_C)$,

so that $F < F'$ in $\mathcal{F}^{op}$, hence $|F'| < X_C$. Then by Lemma 3.3.18.(2) there is $K \dashv C$ with $S(K,C) \cap \operatorname{supp}(F) = \emptyset$, and thus $K_F = C_F$: a contradiction. □

Now we can apply the preceding work to construct a family of maximum acyclic matchings of the Salvetti complex.

Proposition 3.3.20. *Let* $\mathscr{A}$ *be an arrangement of linear hyperplanes in real space and fix any* $B \in \mathcal{T}(\mathscr{A})$*. To every linear extension of* $\mathcal{T}_B(\mathscr{A})$ *corresponds a family of acyclic maximum matchings of the associated Salvetti complex* $\mathcal{S}(\mathcal{M}_{\mathscr{A}})$ *which critical cells are in natural bijection with the chambers of* $\mathscr{A}$*.*

Proof. Let $\dashv$ denote a linear extension of the ordering $\prec_B$ of $\mathcal{T}_B$ and recall Definition 3.3.1.

We will prove recursively that every poset $\mathcal{S}(C)$ possesses a maximum acyclic matching with as many critical cells as there are chambers $C' \dashv C$.

For $\mathcal{S}(B)$ this follows from Theorem 3.2.3; so let the claim hold for a chamber $C \vdash B$. We have to find an acyclic matching of the 'new' part $N(C)$.

For any chamber K let

$$N(C,K) := \mathcal{S}_C \setminus \mathcal{S}_K = \{\langle F, C\rangle \in \mathcal{S}_C \mid C_F \neq K_F\}.$$

Clearly $N(C) = \bigcap_{K \dashv C} N(C,K)$, and thus, with every $N(C,K)$, also $N(C)$ is an upper ideal in $\mathcal{S}(C)$. Since by Lemma 3.3.19 $N(C)$ is the face poset of an oriented matroid, with Theorem 3.2.3 we have an acyclic matching of $N(C)$. These matchings can be pasted together to give a matching of the whole $\mathcal{S}$. The acyclicity of the 'patchwork-matching' can be shown with Lemma 3.1.5 by

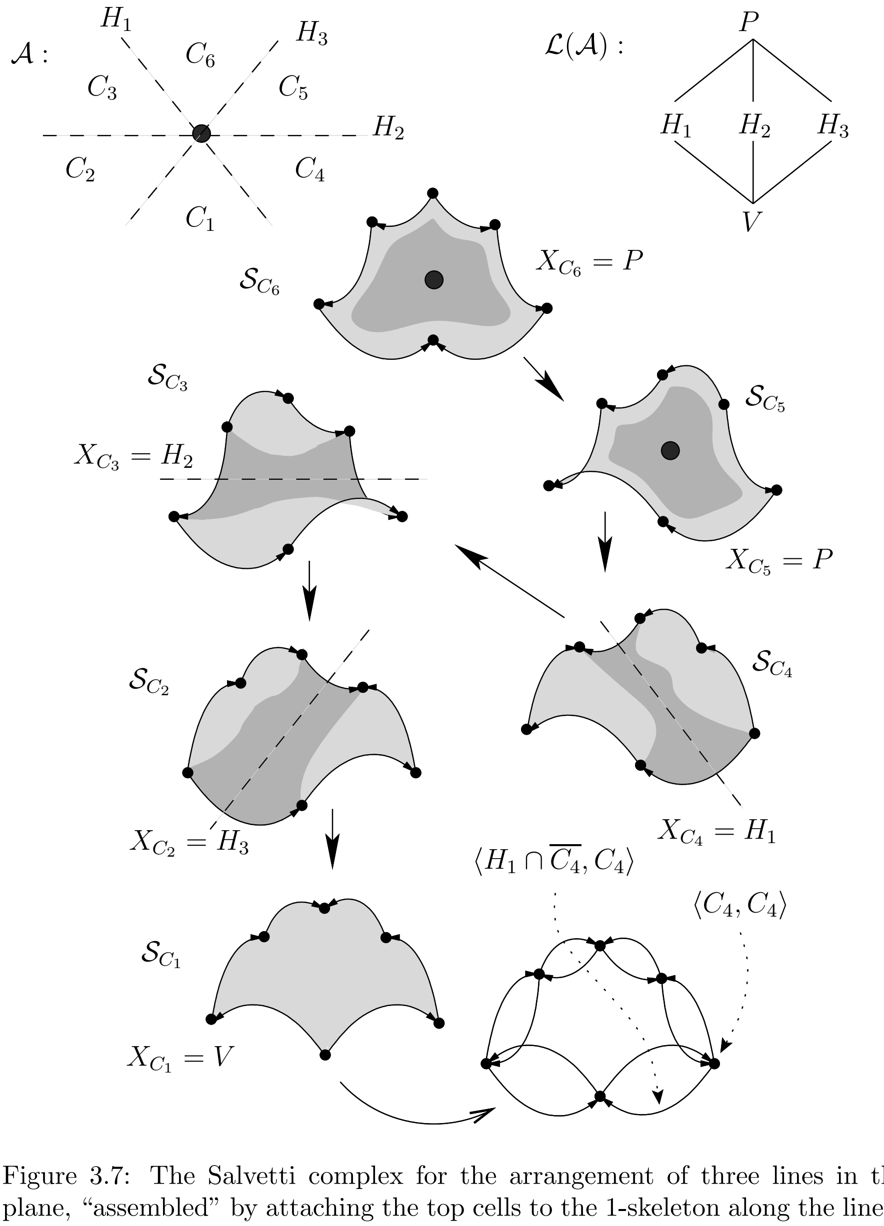


Figure 3.7: The Salvetti complex for the arrangement of three lines in the plane, "assembled" by attaching the top cells to the 1-skeleton along the linear extension of the tope poset that was described in Example 3.2.4 (see also Figure 3.5 and 3.6) The shaded regions represent the 'contributions to homotopy' that every top cell gives to the total complex.

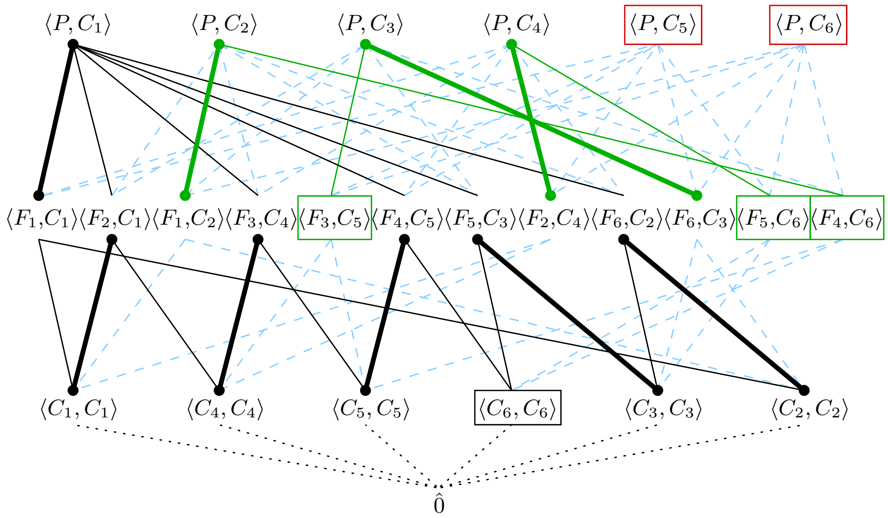


Figure 3.8: The poset of cells of the Salvetti complex for the arrangement of Figure 3.6, where the chambers were numbered according to our chosen linear extension of the tope poset (see Example 3.2.4). The dashed lines relate elements in different strata; elements of the same stratum are joined by solid lines. The stratum $N(C_1)$ is drawn in black, the strata $N(C_2), N(C_3), N(C_4)$ are drawn in green, while for $i = 5, 6$ we have $N(C_i) = \langle P, C_i \rangle$. The stratification corresponds to the one of Figure 3.7. Note that the induced shelling-type ordering of Example 3.1.15 translates into: $C_1 \lhd F_1 \lhd F_2 \lhd C_2 \lhd F_6 \lhd C_3 \lhd F_5 \lhd C_4 \lhd F_4 \lhd C_5 \lhd F_3 \lhd C_6 \lhd P$. On each stratum we depict the associated acyclic matching by thickening the edges of the matching. The resulting critical cells are enclosed into boxes.

considering the linear extension of $\mathcal{S}$ given by the concatenation of the linear extensions of the $N(C)$s so that an element of $N(C_1)$ comes after an element of $N(C_2)$ whenever $C_1 \dashv C_2$ (for a precise proof see the more general statement of [65, Theorem 11.10] on 'patchwork of acyclic matchings').

By Theorem 3.2.3 , the shelling induced on $N(C)$ has only one homology cell, and thus the corresponding acyclic matching has exactly one critical element. With the 'pigeon hole principle' we now see that the obtained 'global' acyclic matchings on $\mathcal{S}$ are in fact maximum acyclic matchings: indeed, the number of critical elements and the number of generators in homology both equal the cardinality of the family of the *no broken circuit sets* (see e.g. [63]).

□

Remark 3.3.21. The matchings of the previous proposition are obtained by pasting together acyclic matchings for the different $N(C)s$. In principle, any choices of acyclic maximum matchings of the $N(C)$s can be pasted together. But since it is easy to see that a shelling-type ordering of a locally ranked poset restricts to a shelling-type ordering of any of its lower ideals, we can construct the whole matching keeping the freedom of choice to a minimum: it is possible to give an explicit description of the critical elements of the matching induced on $\mathcal{S}$ by the choice of a base chamber B, of a linear extension $\dashv$ of $\mathcal{T}_B$, and of maximal chains ω_C in $[B, -C]$ for all $C \in \mathcal{T}$: the critical point added with $N(C)$ is $\langle F(C), C_{F(C)} \rangle$, with

$$F(C) := \max_{\sqsubset_{r(C)}} \{F' \in \mathcal{F} \mid |F'| = X_C\},$$

where $\sqsubset$ is the shelling-type ordering induced on $\mathcal{F}^{op}$ and $r(C)$ is the rank (i.e., the codimension) of X_C.

3.4 No broken circuits and critical elements

In this last section we want to relate our construction to no-broken-circuit sets. It is not easy to track back the origin of these widely studied combinatorial objects that can be defined for every geometric lattice; let us here mention just [19, 29] as 'early references'. We only recall that they give a basis for the Whitney homology of the associated geometric lattice (see [10, 19]) and, in the context of arrangements of hyperplanes, the no-broken-circuit sets of size k index a basis of the k-th degree of the Orlik-Solomon algebra (see e.g. [62, 75] and the textbook [76]), which is known to be isomorphic to the (integral) cohomology algebra of the arrangement's complement [75]. For a comprehensive and very readable account of these objects, and for more bibliography, see the survey of Yuzvinsky [96].

We will continue our 'geometric' treatment of the subject and, as above, leave to the interested reader the translation into the language (and the strength) of abstract oriented matroids.

Definition 3.4.1. (no-broken-circuit sets) Translating the classical definition for matroids, a *circuit* of $\mathscr{A}$ is a minimal set C of hyperplanes such that every $H \in C$ contains the intersection of the other elements of C. In particular, for every $H \in C$ the set $C \setminus \{H\}$ is minimal with the property that the intersection of its hyperplanes equals $\bigcap C$. If a linear ordering of the set of hyperplanes is given, a *broken circuit* is a subset $B \subset \mathscr{A}$ that can be written as $C \setminus \{H\}$, where H is the minimal element of C in the chosen total order.

A *no-broken-circuit set*, also called simply *nbc set*, is an independent subset of $\mathscr{A}$ that contains no broken circuit, or the empty set. It is clear that the nbc sets give a simplicial complex, denoted $\mathbf{nbc}(\mathscr{A})$, on the ground set $\mathscr{A}$. Note that we formally consider also the simplex of dimension -1 given by the empty set - thus, $\emptyset \in \mathbf{nbc}(\mathscr{A})$ for all $\mathscr{A}$.

Example 3.4.2. For the arrangement $\mathscr{A}$ of three lines in the plane, with the lattice depicted on the top right of Figure 3.7, we have only one circuit, namely $\{H_1, H_2, H_3\}$, and thus we get

$$\mathbf{nbc}(\mathscr{A}) = \big\{\emptyset, \{H_1\}, \{H_2\}, \{H_3\}, \{H_1, H_2\}, \{H_1, H_3\}\big\}.$$

A corresponding notion exists for arbitrary geometric lattices (i.e., for arbitrary matroids): the interested reader is referred to [19].

It is important to point out that, for technical reasons, our definitions differ from those of [63] in that our broken circuits fail to contain a *minimal* (instead of a maximal) element. The other definitions are then adapted to this change.

Before to state the main definitions, let us fix some notation that will accompany us through the remainder of this paper.

Notation 3.4.3. We keep the conventions of the Important Remark 3.3.7 but now, in addition, we suppose a linear ordering $\{H_1, \ldots, H_n\}$ to be given on the set of hyperplanes. For the moment no special requirements are made on this ordering.

We will write

$$\mathscr{A}_j := \{H_1, \ldots, H_j\} \text{ for } 1 \leq j \leq n, \qquad \mathscr{A}' := \mathscr{A}_{n-1}, \qquad \mathscr{A}'' := \mathscr{A}^{H_n},$$

where $\mathscr{A}^{H_n} = \{H \cap H_n \mid H \in \mathscr{A}'\}$, according to the Notation 3.3.5. Clearly every $\mathscr{A}_j$ inherits the ordering from $\mathscr{A}$. Moreover, there is a canonical ordering of $\mathscr{A}^{H_n}$ obtained by numbering every element $L \in \mathscr{A}''$ according to the 'smallest' hyperplane $H(L) \in \mathscr{A}$ in which it is contained. As above, every $C \in \mathcal{T}(\mathscr{A})$ is contained in exactly one chamber of $\mathscr{A}'$, that we will denote by C'. Thus, B' is the only chamber of $\mathscr{A}'$ that contains the base chamber B of $\mathscr{A}$.

For every $H \in \mathscr{A}$ let H^+ denote the closed halfspace that is bounded by H and contains B. Clearly $B = \bigcap_{H \in \mathscr{A}} H^+$ and $B' = \bigcap_{H \in \mathscr{A}'} H^+$. More generally, there is a canonical choice of a base region B_j for $\mathscr{A}_j$: we define $B_j := \bigcap_{i \leq j} H_i^+$. Turning our attention to $\mathscr{A}''$, for $L \in \mathscr{A}''$ it is natural to define $L^+ := H_n \cap H(L)^+$. Now, *if H_n is a wall of B* write $B'' := \bigcap_{L \in \mathscr{A}''} L^+$.

The last requirement on H_n is necessary to ensure that the intersection defining B'' has indeed maximal dimension inside H_n. It is clear that with this hypothesis

$$B'' = B' \cap H_n.$$

We will need this property to hold inductively: this is the motivation of the following definition.

Definition 3.4.4 (Cut property)**.** A total ordering $\{H_1, \ldots, H_n\}$ of $\mathscr{A}$ satisfies the *cut property* with respect to the base chamber B if, for every $j = 2, \ldots, n$, H_j intersects the interior of B_{j-1} (we will say: H_n *cuts* B_{j-1}).

We need to check that an ordering with this property exists. The next Lemma explains that those orderings correspond to known objects. Namely: maximal chains in the poset of regions.

Lemma 3.4.5. *An ordering $\{H_1, \ldots, H_n\}$ of the hyperplanes of an arrangement $\mathscr{A}$ satisfies the cut property if and only if there is a maximal chain*

$$B = C_0 \prec C_1 \prec \ldots \prec C_n = -B$$

in $\mathcal{T}_B(\mathscr{A})$ such that $S(C_{i-1}, C_i) = \{H_i\}$ for all $1 \leq i \leq n$.

Proof. $\square$

We see that every arrangement can be ordered so to satisfy the cut property (for example, the ordering of the hyperplanes in figure 3.6 satisfies the cut property). Indeed, Definition 3.4.4 turns out to describe the property we were seeking for.

Remark 3.4.6. If the ordering $\mathscr{A} = \{H_1, \ldots, H_n\}$ satisfies the cut property with respect to the chamber B, then for every $j = 1, \ldots, n$ there is a canonical choice of a base region in $(\mathscr{A}_j)''$:

$$B''_j := H_j \cap B_{j-1}.$$

Moreover, the induced ordering of $(\mathscr{A}_j)''$ satisfies the cut property with respect to B''_j.

Definition 3.4.7. Let $\mathscr{A} := \{H_1, \dots, H_n\}$ be ordered such that $H_n \in \mathcal{W}_B$. With the Notations of 3.4.3 we define:

$$\mathcal{T} := \mathcal{T}_B(\mathscr{A}), \qquad \mathcal{T}' := \mathcal{T}_{B'}(\mathscr{A}'), \qquad \mathcal{T}'' := \mathcal{T}_{B''}(\mathscr{A}'').$$

Moreover, let $\mathcal{B}'$ (or $\mathcal{B}'(\mathscr{A})$ if specification is needed) denote the set of all chambers of $\mathscr{A}'$ that are 'cut' by H_n. Every $C' \in \mathcal{B}'$ contains therefore two chambers $C^{\downarrow} \prec_B C^{\uparrow}$ of $\mathcal{T}$. Define

$$\mathcal{B}^{\uparrow} := \{C^{\uparrow} \mid C \in \mathcal{B}\}, \quad \mathcal{B}^{\downarrow} := \{C^{\downarrow} \mid C \in \mathcal{B}\},$$

$$\mathcal{U} := \mathcal{T}' \setminus \mathcal{B}', \qquad \mathcal{B}'' := \{H_n \cap C \mid C \in \mathcal{B}'(\mathscr{A})\}.$$

Remark 3.4.8. Clearly,

$$\mathcal{T} = \mathcal{U} \uplus \mathcal{B}^{\uparrow} \uplus \mathcal{B}^{\downarrow}, \qquad \mathcal{T}' = \mathcal{U} \uplus \mathcal{B}', \qquad \mathcal{T}'' = \mathcal{B}'',$$

with the evident order preserving bijections:

$$\beta' : \mathcal{B}' \to \mathcal{B}^{\downarrow}, \qquad \beta'' : \mathcal{B}^{\uparrow} \to \mathcal{B}''.$$

We want to describe a particular linear extension of $\mathcal{T}$ that allows us to explicitly index the critical elements of the associated acyclic matchings with the no broken circuit sets of the arrangement. We will make use of an indexing of the chambers of $\mathscr{A}$ by *nbc* sets that is inspired by a result of Jewell and Orlik [63].

Definition 3.4.9 (see Section 3.4 of [63])**.** Consider an ordering $\mathscr{A} = \{H_1, \dots, H_n\}$ that satisfies the cut property with respect to the chamber B and keep the notations introduced above. We define a map

$$\eta : \mathcal{T}_B(\mathscr{A}) \to \mathcal{P}(\mathscr{A})$$

recursively in the number of elements of $\mathscr{A}$ as follows:

• If $\mathscr{A} = \{H_1\}$, let $\eta_1(H_1^+) := \emptyset$ and $\eta_1(-H_1^+) := \{H_1\}$.

• Let $\mathscr{A} = \{H_1, \dots, H_n\}$ with $n > 1$ and suppose we are able to define such functions for every arrangement of cardinality at most $n-1$. In particular the functions η' and η'' associated to $\mathscr{A}'$, $\mathscr{A}''$ are defined. Then, for $C \in \mathcal{T}(\mathscr{A})$ we define

$$\eta(C) := \begin{cases} \eta'(C) & \text{if } C \in \mathcal{U} \cup \mathcal{B}^{\downarrow} \\ \big\{ \min\{H \in \mathscr{A} \mid H \cap H_n = L\} \,\big|\, L \in \eta''(\beta''(C)) \big\} & \text{if } C \in \mathcal{B}^{\uparrow} \end{cases}$$

where we slightly abused notation in implicitly identifying $\mathcal{T}'$ with $\mathcal{U} \cup \mathcal{B}^{\downarrow}$ using the bijection β' of Definition 3.4.8.

In particular, for $C \in \mathcal{B}'(\mathscr{A})$ we have $\eta(C^{\downarrow}) = \eta'(C)$ and a natural bijective correspondence between $\eta(C^{\uparrow})$ and $\eta''(C \cap H_n) \cup \{H_n\}$. The map η was introduced in [63] as a bijection between no-broken circuit sets and chambers of the arrangement, as we state in the following lemma.

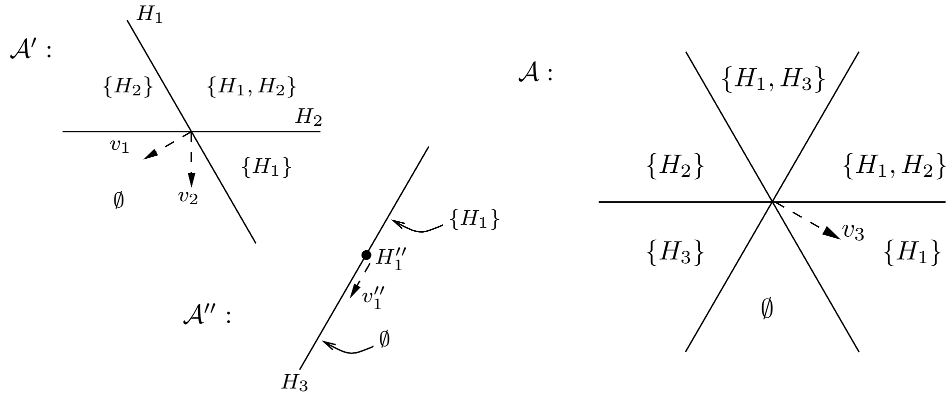


Figure 3.9: The last step in the inductive construction of η for the arrangement given on the left of Figure 3.6, where we see that $\mathcal{B}^{\uparrow} = \{C_3, C_5\}$, $\mathcal{B}^{\downarrow} = \{C_2, C_4\}$, $\mathcal{U} = \{C_1, C_6\}$. For every chamber C, the set $\eta(C)$ is written inside C to show the bijective correspondence.

Lemma 3.4.10 (see Lemma 3.14 of [63])**.** *The map η is a bijection $\mathcal{T}(\mathscr{A}) \to$ $\mathbf{nbc}(\mathscr{A})$ with $\eta(B) = \emptyset$.*

As a first step let us prove a technical property that derives from our particular choice of the ordering of the hyperplanes.

Lemma 3.4.11. *Let $\mathscr{A} = \{H_1, \dots, H_n\}$ be an arrangement of linear real hyperplanes and B a chamber of $\mathscr{A}$. Suppose that the ordering of the hyperplanes satisfies the cut property with respect to B. Then*

$$\bigcap \eta'(C) \cap H_n = \bigcap \eta''(C \cap H_n) \qquad \forall C \in \mathcal{B}'(\mathscr{A}).$$

Proof.

$$\bigcap \widehat{\eta}'(C) \cap H_n = \bigcap \widehat{\eta}''(C \cap H_n) \qquad \forall C \in \mathcal{B}'(\widehat{\mathscr{A}}).$$

Also, the induction hypothesis applies to the arrangement $\mathscr{A}''$ with respect to the induced order and the chamber $B'' = B \cap H_n$; thus, if we define $L :=$ $H_n \cap H_{n-1}$, *when there is no $j < n-1$ with $H_J \supset L$* we can write

$$\bigcap \nu'(C) \cap L = \bigcap \nu''(C \cap L) \qquad \forall C \in \mathcal{B}'(\mathscr{A}''),$$

where ν, ν', ν'' are the maps obtained by applying Definition 3.4.9 to $\mathscr{A}''$. Finally, let us denote by μ, μ', μ'' the maps associated to $\mathscr{A}' = \{H_1, \dots, H_{n-1}\}$. We know that the order induced on $\mathscr{A}'$ satisfies the cut property with respect to the unique chamber $B' \supset B$ and thus, by induction,

$$\bigcap \mu'(C) \cap H_{n-1} = \bigcap \mu''(C \cap H_{n-1}) \qquad \forall C \in \mathcal{B}'(\mathscr{A}').$$

We would like to point out the following (tautological) relations:

$$\mu = \eta', \quad \widehat{\eta}' = \mu', \quad \widehat{\eta}'' = \nu', \quad \nu = \eta''.$$

Now we proceed with the proof. Let $\mathscr{A}$ be as above, and choose $C \in \mathcal{B}'(\mathscr{A})$. It is easy to see that if $C \subset H_{n-1}^+$ or if H_{n-1} is not a wall of C, then the claim holds because it holds for $\widehat{\mathscr{A}}$.

So suppose that H_{n-1} is a wall of C and that $C \not\subset H_{n-1}^+$. Then we have

$$\eta'(C) = \mu(C) = \{H_{n-1}\} \cup \mu''(C \cap H_{n-1})$$

and

$$\eta''(C \cap H_n) = \begin{cases} \widehat{\eta}''(C \cap H_n) & \text{if there is } j < n-1 \text{ with } L \subset H_j, \\ \{L\} \cup \nu''((C \cap H_n) \cap H_{n-1}) & \text{else.} \end{cases}$$

Moreover, we can write

$$\begin{aligned} \bigcap \eta'(C) \cap H_n &= \bigcap \left[\{H_{n-1}\} \cup \mu''(C \cap H_{n-1})\right] \cap H_n \\ &= \bigcap \mu'(C) \cap H_{n-1} \cap H_n = \bigcap \widehat{\eta}'(C) \cap H_n \cap H_{n-1} \\ &= \bigcap \widehat{\eta}''(C \cap H_n) \cap H_{n-1}. \end{aligned}$$

Since we know that $H_{n-1} \in \eta'(C)$, this implies $\bigcap \eta'(C) \cap H_n = \bigcap \widehat{\eta}''(C \cap H_n)$. To conclude the proof we distinguish two cases:

Case 1. If there is $j < n-1$ with $L \subset H_j$, the claim follows immediately, because then $\widehat{\eta}''(C \cap H_n) = \eta''(C \cap H_n)$.

Case 2. If there is no such j, then the induction hypothesis applies to ν and gives

$$\bigcap \widehat{\eta}''(C \cap H_n) \cap H_{n-1} = \bigcap \nu'(C \cap H_n) \cap H_{n-1} = \bigcap \nu''(C \cap H_n \cap H_{n-1}) = \eta''(C),$$

where the last inequality holds because every element of $\nu''(C \cap H_n \cap H_{n-1})$ is contained in L.

Thus, in any case the claim holds. □

Now the idea is to consider a linear extension that behaves well under 'taking $\mathscr{A}'$ and $\mathscr{A}''$'.

Definition 3.4.12. For every $H \in \mathscr{A}$ let H^+ denote the open halfspace that is bounded by H and contains the base chamber B. To every $C \in \mathcal{T}$ we associate an array $\sigma(C) := (\sigma_1(C), \ldots, \sigma_n(C))$ by setting $\sigma_i(C) = 0$ if $C \subset H_i^+$, and $\sigma_i(C) = 1$ else.

We denote by $\dashv^\ell$ (or $\dashv^\ell_{\mathscr{A},B}$ when specification is needed) the total order on $\mathcal{T}$ induced by the lexicographic ordering of the corresponding arrays.

Example 3.4.13. The linear extension of example 3.2.4 translates into

$$(0,0,0) \dashv (0,0,1) \dashv (0,1,1) \dashv (1,0,0) \dashv (1,1,0) \dashv (1,1,1)$$

and is therefore $\dashv^\ell_{\mathscr{A}}$ for the arrangement $\mathscr{A}$ of Figure 3.6.

Remark 3.4.14. In the language of oriented matroids the above definition just fixes the acyclic orientation associated with the tope B and then associates to every tope its signed covector.

Lemma 3.4.15. *The ordering $\dashv^\ell_{\mathscr{A},B}$ is a linear extension of $\mathcal{T}_B(\mathscr{A})$, and satisfies:*

(1) the ordering of $\mathcal{T}'$ induced via the maps δ, β', γ is $\dashv^{\ell}_{\mathscr{A}',B'}$.

(2) the ordering of $\mathcal{T}''$ induced via the map β'' is $\dashv^{\ell}_{\mathscr{A}'',B''}$.

Proof. We have to show that if $C \prec_B C'$, then $C \dashv^{\ell} C'$. But the former means $S(B,C) \subset S(B,C')$: thus, $\sigma(C')$ is obtained from $\sigma(C)$ by switching from 0 to 1 the entries corresponding to the elements of $S(C,C')$, and $\dashv^{\ell}$ is therefore a linear extension. Item (1) is easy to see. For (2), recall that every hyperplane of $\mathscr{A}''$ corresponds to a codimension 2 subspace of $\mathscr{A}$ and gets the number of the smallest $i < n$ such that H_i contains the subspace. □

The next step will be to prove that the critical cells of the acyclic matching of Proposition 3.3.20 are completely determined by the associated chamber, provided that the chosen linear extension is the one associated via Definition 3.3.15 to an ordering of the hyperplanes that satisfies the cut property.

We will show that, for every base chamber B and every ordering of $\mathscr{A}$ satisfying the cut property with respect to B, $\eta(C)$ is a basis of the flat X_C if the chosen linear extension of $\mathcal{T}_B(\mathscr{A})$ is the one of Definition 4.2.5.

Theorem 3.4.16. *Let the ordering $\{H_1, \ldots, H_n\}$ of $\mathscr{A}$ satisfy the cut property with respect to the chamber B and consider the linear extension $\dashv^{\ell}$ of $\mathcal{T}_B$. We have*

$$X_C = \bigcap \eta(C).$$

Proof. Again, the claim is trivial if $\mathscr{A} = 1$. So let $n := |\mathscr{A}| > 1$ and suppose that the claim holds for every arrangement of at most $n-1$ hyperplanes (and thus, in particular, for $\mathscr{A}'$ and $\mathscr{A}''$).

Given $C \in \mathcal{T}(\mathscr{A})$, let

$$Y_C := \bigcap \eta(C).$$

We are going to prove that Y_C satisfies 3.3.18.(1) and 3.3.18.(2).

It is easily seen that this is true if $H_n \in S(B,C)$, because the above properties hold for $\mathscr{A}'$ and depend only on the position of the flat with respect to the union of the chambers K that come before C. In fact, the chosen linear extension is such that the union of all $K \dashv^{\ell} C$ equals (as a subset of $\mathbb{R}^d$) the union of the chambers that come before C' with respect to the ordering $\dashv^{\ell}_{\mathscr{A}',B'}$ (recall that C' is the unique chamber of $\mathscr{A}'$ containing C).

So let $C \in \mathcal{B}^{\uparrow}$ and recall that by definition we have

$$\eta(C) = \{H_n\} \cup \big\{ \min\{H \in \mathscr{A} \mid H \cap H_n = L\} \,\big|\, L \in \eta''(C \cap H_n) \big\}.$$

We now have to check the properties of Definition 3.3.18.
<u>*3.3.18.(1):*</u> $\operatorname{supp}(Y_C) \cap S(C,K) \neq \emptyset$ *for all* $K \dashv^{\ell} C$.
This assertion is clear if $H_n \in S(B,K)$, since then $H_n \in S(C,K) \cap \operatorname{supp}(Y_C)$. On the other hand, if $H_n \notin S(B,K)\mathcal{U}$, then we know that $S(C,K) \cap \operatorname{supp}(\bigcap \eta'(C')) \neq \emptyset$ by induction hypothesis. But Lemma 3.4.11 allows us to write

$$Y_C = \bigcap \eta(C) = \bigcap \eta''(C \cap H_n) = \bigcap \eta'(C) \cap H_n,$$

whence $\operatorname{supp}(Y_C) \supseteq \operatorname{supp}(\bigcap \eta'(C))$, and the claim follows.
<u>*3.3.18.(2):*</u> *For every flat $Z \not\geq Y_C$ in $\mathcal{L}(\mathscr{A})$ there is a chamber $K \dashv^{\ell} C$ such that* $\operatorname{supp}(Z) \cap S(C,K) = \emptyset$.

Clearly if $H_n \notin \text{supp}(Z)$, we are easily done by taking $K = (C')^{\downarrow}$ so that $S(C,K) = \{H_n\}$. We are left with the case where $H_n \in \text{supp}(Z)$. Then $Z \not\geq \bigcap \eta''(C'')$ in $\mathcal{L}(\mathscr{A}'')$ - recall Lemma 3.4.11 and that $C'' := C' \cap H_n$ - and by induction hypothesis we know that there is $K'' \dashv^{\ell}_{\mathscr{A}'',B''} C''$ with no hyperplane of $\mathscr{A}''$ containing Z and separating K'' from C''. Now let K be the chamber of $\mathscr{A}$ that is 'just above' (or: the preimage with respect to β''^{-1} of) K'' (so that $K \dashv^{\ell} C$ by Lemma 3.4.15). For every $H \in S(C,K)$, $H \cap H_n$ separates C'' from K'' in $\mathscr{A}''$. Thus, if there were $H \in \text{supp}(Z) \cap S(C,K)$, then there would be $L := H \cap H_n \in \text{supp}''(Z)$ separating C'' from K'' (where $\text{supp}''(Z)$ is naturally defined as $\{L \in \mathscr{A}'' \mid Z \subset L\}$) - a contradiction. □

We can now summarize our results leaving the greatest generality in the attempt to approach the greatest *naturality.* The proof is an easy combination of Proposition 3.3.20, Theorem 3.4.16, Remark 3.3.21 and Corollary 3.3.16.

Proposition 3.4.17. *Let $\mathscr{A}$ denote a real arrangement of linear hyperplanes and choose a chamber $B \in \mathcal{T}(\mathscr{A})$. Every ordering of $\mathscr{A}$ that satisfies the cut property with respect to B gives rise to a bijection η between chambers and nbc-sets as in Definition 3.4.9 and to an acyclic matching of the Salvetti complex which critical cells are precisely those of the form*

$$\langle \bigcap \eta(C) \cap C,\ C \rangle.$$

In particular, the resulting CW-complex has one cell of dimension $|\eta(C)|$ for every $C \in \mathcal{T}(\mathscr{A})$.

Example 3.4.18. By comparing Figure 3.6 with Figures 3.7, 3.8 and 3.9 one sees immediately the claimed correspondence:

$$\begin{array}{llll}
\eta(C_1) = \emptyset, & \bigcap \emptyset = \mathbb{R}^d = \hat{0} = X_{C_1}, & \mathbb{R}^d \cap C_1 = C_1, & \langle C_1, C_1 \rangle \text{ is critical;} \\
\eta(C_2) = \{H_3\}, & \bigcap \{H_3\} = H_3 = X_{C_2}, & H_3 \cap C_2 = F_1, & \langle F_1, C_2 \rangle \text{ is critical;} \\
\eta(C_3) = \{H_2\}, & \bigcap \{H_2\} = H_2 = X_{C_3}, & H_2 \cap C_3 = F_6, & \langle F_6, C_3 \rangle \text{ is critical;} \\
\eta(C_4) = \{H_1\}, & \bigcap \{H_1\} = H_1 = X_{C_4}, & H_1 \cap C_4 = F_2, & \langle F_2, C_4 \rangle \text{ is critical;} \\
\eta(C_5) = \{H_1, H_3\}, & H_1 \cap H_3 = P = X_{C_5}, & P \cap C_5 = P, & \langle P, C_5 \rangle \text{ is critical;} \\
\eta(C_6) = \{H_1, H_2\}, & H_1 \cap H_2 = P = X_{C_6}, & P \cap C_6 = P, & \langle P, C_6 \rangle \text{ is critical;}
\end{array}$$

and there are no further critical cells.

Remark 3.4.19. The importance of the chambers in the above characterization of the critical cells is mainly to give the order along which we decompose the Salvetti complex. It is now natural to ask if such ordering can be defined purely in terms of the no-broken-circuit sets. This would actually allow to describe the situation without referring to the geometry of $\mathbb{R}^d$. However, this task might be particularly subtle: for instance, compare the arrangement of Coxeter type A_2 and the coordinate arrangement in $\mathbb{R}^3$ (let us call it K_3). Up to symmetry, in both cases there is only one linear ordering induced on the families of no-broken-circuit sets:

$$A_2: \quad \emptyset, \quad \{3\}, \quad \{2\}, \quad \{1\}, \quad \{1,2\}, \quad \{1,3\},$$

$$K_3: \quad \emptyset, \quad \{3\}, \quad \{2\}, \quad \{2,3\}, \quad \{1\}, \quad \{1,3\}, \quad \{1,2\}, \quad \{1,2,3\}$$

(where we wrote j for H_j) and we see that $\{1,2\}$ and $\{1,3\}$ are switched in the two orderings. This seems to indicate that one should consider also some 'global' property of the lattice, other than just examining the no-broken-circuit sets.

4 Combinatorial polar orderings and recursively orderable arrangements

This chapter reproduces, with slight notational and editorial modifications, the paper [43], written jointly with Simona Settepanella.

Introduction

The approach taken by Salvetti and Settepanella in [87] to construct the discrete Morse vector field relies on the choice of a so-called *generic flag* and on the associated *polar ordering* of the faces of the real arrangement. Once this polar ordering is determined, the description of the vector field and of the obtained minimal complex is quite handy, e.g. yielding an explicit formula for the algebraic boundary maps.

But the issue about actually *constructing* such a polar ordering for a given arrangement remains. This motivates the first part of this chapter, where we use *flippings* in oriented matroids to give a fully combinatorial characterization of a class of total orderings of the faces of a complexified arrangement that can be used to carry out the construction of the discrete vector field described in [87]. Our *combinatorial polar orderings* still require a flag of general position subspaces as a starting point, but does not need this flag to satisfy the requirements that are requested from a *generic flag* in the sense of [87].

To make sure our construction works under these milder assumptions we need the strength of the theory of *oriented matroid programs* (see Definition 2.2.22).

Once the (combinatorial) polar ordering is constructed, one has to figure out the discrete vector field and follow its gradient paths to actually construct the minimal complex. Although the 'recipe' is fairly straightforward, this task soon becomes very challenging. For instance, this was accomplished in [87] for the family of real reflection arrangements of Coxeter type A_n. The key fact allowing one to carry out the construction in these cases is that the general flag can be set so that the associated polar orderings enjoy a special technical property (see Definition 4.3.1) that keeps the complexity of computations down to a reasonable level.

Thus it is natural to ask whether this property is shared by other arrangements. Since the obtained discrete vector fields are the same, it turns out that instead of restricting to 'actual' polar orderings, it is natural to work in our broader combinatorial setting, and say that an arrangement is *recursively orderable* if it admits a combinatorial polar ordering that satisfies this property.

In Section 4.3 a complete characterization of recursively orderable arrangements of lines is reached. Furthermore, we prove that every supersolvable arrangement is recursively orderable. On the other hand, an example shows that not every reflection arrangement is recursively orderable. As what concerns asphericity, already in dimension 3 there is a recursively orderable arrangement that is not $K(\pi, 1)$.

Polar ordering and polar gradient.

Salvetti and Settepanella introduced *polar orderings* of real hyperplane arrangements in [87] as the basic tool for the construction of minimal models for $\mathcal{M}(\mathscr{A})$. The construction starts by considering the polar coordinate system induced by any *generic flag* with respect to the given arrangement $\mathscr{A} \subset \mathbb{R}^d$, i.e., a flag $\{V_i\}_{i=0,\dots,d}$ of affine subspaces in general position, such that $\dim(V_i) = i$ for every $i = 0, \dots, d$ and such that 'the polar coordinates $(\rho, \theta_1, \dots, \theta_{d-1})$ of every point in a bounded face of $\mathscr{A}$ satisfy $\rho > 0$ and $0 < \theta_i < \pi/2$, for every $i = 1, \dots, d$' (see [87, Section 4.2] for the precise description). The existence of such a generic flag is not trivial ([87, Theorem 2]). Every face F is labeled by the coordinates of the point in its closure that has lexicographically least polar coordinates.

The *polar ordering* associated to a generic flag is the total order $\lhd$ on $\mathcal{F}$ that is obtained by ordering the faces lexicographically according to their labels. This extends the order in which V_{d-1} intersects the faces while rotating around V_{d-2}. If two faces share the same label - thus, the same minimal point p -, the ordering is determined by the general flag induced on the copy of V_{d-1} that is rotated 'just past p' and the ordering it generates by induction on the dimension (see [87, Definition 4.7]).

The main purpose of the polar ordering is to define a discrete Morse function on the Salvetti complex, which amounts to specifying an acyclic matching Φ on the poset of cells of $\mathcal{S}$ that is called the *polar gradient* (see Definition 3.1.4 for the terminology). The original definition of Φ is by induction in the dimension of the subspace V_k containing the faces [87, Definition 4.6]. For the sake of brevity let us here define Φ through an equivalent description that is actually the one we will use later (compare Definition 4.2.7)

Definition 4.0.20 (Compare Theorem 6 of [87])**.** For any two faces F_1, F_2 with $F_1 \prec F_2$, $\mathrm{codim}(F_1) = \mathrm{codim}(F_2) - 1$ and any chamber $C \prec F_1$, the pair

$$(\langle F_1, C\rangle, \langle F_2, C\rangle)$$

belongs to Φ if and only if the following conditions hold:

(a) $F_2 \lhd F_1$, and

(b) for all $G \in \mathcal{F}$ with $\mathrm{codim}(G) = \mathrm{codim}(F_1) - 1$ such that $C \prec G \prec F_1$, one has $G \lhd F_1$.

We conclude by pointing out that the above definition indeed has the required features.

Theorem 4.0.21 (See Theorem 6 of [87])**.** *The matching Φ is acyclic with the minimal possible number of critical elements.*

Moreover, the set of $k-$*dimensional critical cells is given by*

$$\mathrm{Crit}_k(\mathcal{S}) = \left\{ \langle F, C \rangle \;\middle|\; \begin{array}{l} \mathrm{codim}(F) = k,\, F \cap V_k \neq \emptyset, \\ G \lhd F \textit{ for all } G \textit{ with } C \prec G \precneqq F \end{array} \right\}$$

(equivalently, $F \cap V_k$ *is the maximum in polar ordering among all facets of* $C \cap V_k$*).*

The first step on the way to generalizing the construction of [87] is to give a combinatorial (i.e., 'coordinate-free') description of it. The idea is to let the hyperplane V_{k-1} 'sweep' across the arrangement $\mathscr{A} \cap V_k$ instead of rotating it around V_{k-1}.

As explained in the introduction, we want to put the polar ordering into the broader context of the orderings that can be obtained by letting an hyperplane sweep across an affine arrangement along a sequence of flippings. By Remark 2.2.19 we must then work with general oriented matroids, since realizability of every intermediate step is not guaranteed (and, indeed, rarely occurs). This raises the question of whether such a 'sweeping' is always possible throughout the construction. We will see that indeed all occurring oriented matroid programs are Euclidean.

4.1 Definitions and setup

Let $\mathscr{A}$ denote an affine real arrangement of hyperplanes in $\mathbb{R}^d$. A flag $(V_k)_{k=0,\ldots,d}$ of affine subspaces is called a *general flag* if every one of its subspaces is in general position with respect to $\mathscr{A}$ and if, for every $k = 0, \ldots d-1$, V_k does not intersect any bounded chamber of the arrangement $\mathscr{A} \cap V_{k+1}$. Note that this is a less restrictive hypothesis than the one required for being a *generic* flag in [87].

Moreover, we write

$$\mathscr{A}^k := \{H \cap V_k \mid H \in \mathscr{A}\}, \quad \mathcal{F}^k := \{F \in \mathcal{F} \mid F \cap V_k \neq \emptyset\} (= \mathcal{F}(\mathscr{A}^k)),$$
$$\mathcal{P}^k = \{p_1, p_2, \ldots\} := \max \mathcal{F}^k, \quad \mathcal{P} := \mathcal{P}^0 \cup \mathcal{P}^1 \cup \ldots \cup \mathcal{P}^d,$$

where of course the set $\mathcal{F}^k$ is partially ordered as the face poset of the arrangement $\mathscr{A}^k$.

If a total ordering $\rightsquigarrow^k$ of each $\mathcal{P}^k$ is given, we define a total ordering of $\mathcal{P}$ by setting, for any $p \in \mathcal{P}^i$ and $q \in \mathcal{P}^j$,

$$p \rightsquigarrow q \Leftrightarrow \begin{cases} p \rightsquigarrow^k q & \text{if } k = i = j \\ i < j & \text{if } i \neq j \end{cases}$$

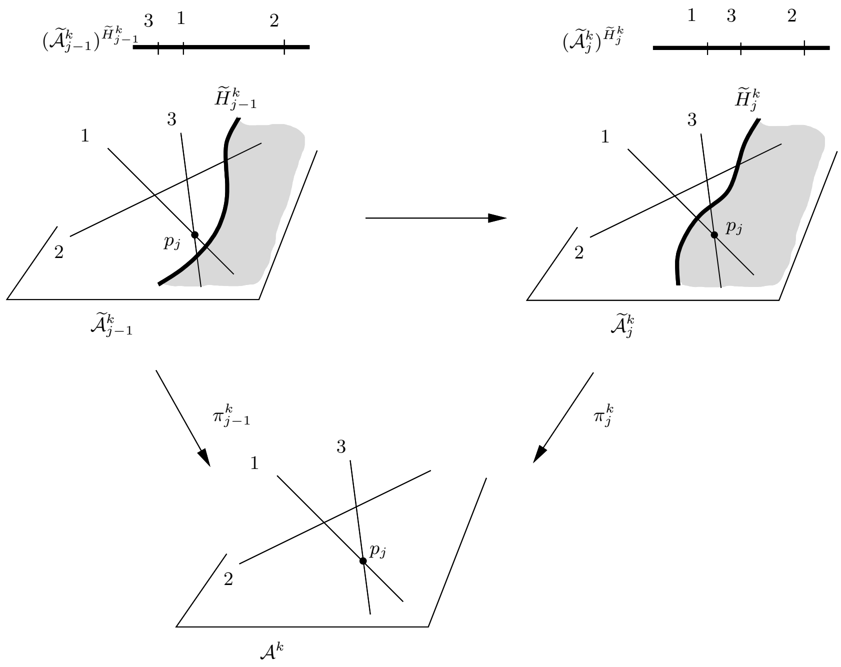
We want to let the hyperplane V_{k-1} sweep across $\mathscr{A}^k$. Let us introduce the necessary notation. For every $k = 1, \ldots, d$, let

$$\widetilde{H}_0^k := V_{k-1}, \quad \mathcal{F}_0^k := \mathcal{F}^{k-1}, \quad \widetilde{\mathcal{A}}_0^k := \mathscr{A}^k \cup \{\widetilde{H}_0^k\}.$$

For all $j > 0$, let $p_j \in \mathcal{P}^k$ be *near* $\widetilde{H}_{j-1}^k$ in the sense of Definition 2.2.18 and set

$$\widetilde{\mathcal{A}}_j^k := \mathtt{Flip}(\widetilde{\mathcal{A}}_{j-1}^k, \widetilde{H}_{j-1}^k, p_j), \quad \widetilde{H}_j^k : \quad \widetilde{\mathcal{A}}_j^k \setminus \mathscr{A} = \{\widetilde{H}_j^k\},$$
$$\mathcal{H}_j^k := (\widetilde{\mathcal{A}}_j^k)^{\widetilde{H}_j^k}, \quad \mathcal{F}_j^k := \mathcal{F}(\mathcal{H}_j^k), \quad \mathcal{P}_j^k := \max \mathcal{F}_j^k$$

where the definitions refer to the natural inclusions $\mathcal{F}_i^k \hookrightarrow \mathcal{F}^k \hookrightarrow \mathcal{F}$. Moreover, we will make use of the natural forgetful projection $\pi_j^k : \mathcal{F}(\widetilde{\mathcal{A}}_j^k) \to \mathcal{F}^k$ ('forgetting' $\widetilde{H}_j^k$).



Remark 4.1.1. Our construction will be inductive in the dimension. The definitions and arguments we make here about $\mathscr{A}$ will be applied to every $\mathcal{H}_j^k$, and so on. The involved oriented matroids can become quickly nonrealizable. Thus, it has to be stressed that our arguments hold in the generality of affine arrangements of pseudohyperplanes. The reason why we carry out this section by referring to $\mathscr{A}$ as an arrangement of hyperplanes is mainly to keep the terminology lighter and help the intuition. The reader will obtain proof of the corresponding statements for pseudoarrangements by just adding throughout the next section the prefix "pseudo" to the appropriate words.

We have to understand how the combinatorics of the arrangement induced on the "moving hyperplane" $\widetilde{H}_j^k$ changes, as j becomes bigger. By the definition of flippings, we know that nothing changes in $\widetilde{\mathcal{A}}_j^k$ outside

$$\mathcal{Y}(p_j) := (\pi_j^k)^{-1}(\mathcal{F}_{\preceq p_j}^k)$$

- *a fortiori*, nothing changes in $\mathcal{F}_{j-1}^k$ outside

$$\mathcal{X}(p_j) := \mathcal{F}_{j-1}^k \cap \mathcal{Y}(p_j).$$

Notation 4.1.2. Given two faces $F \prec G$, let us from now denote by $\mathrm{op}_G(F)$ the unique element of $\mathcal{F}$ such that $\mathrm{op}_G(F) \prec G$ and the face that represents $\mathrm{op}_G(F)$ is on the opposite side (with respect to F) of every pseudohyperplane that contains G but not F.

The next Lemma states an explicit (and order-preserving) bijection between the set of 'new faces' that are cut by the moving hyperplane after the flip at p_j and the following set of 'old faces':

$$\mathcal{C}(p_j) := \{X \in \mathcal{X}(p_j) \mid \mathrm{op}_{p_j}(X) \notin \mathcal{X}(p_j)\}.$$

Lemma 4.1.3. *With the notations explained above, let $\widetilde{\mathcal{A}}^k_{j-1}$ be given and let $p_j \in \mathcal{P}^k$ be near $\widetilde{H}^k_{j-1}$. Then, if $<_{j-1}$ denotes the ordering of $\mathcal{F}^k_{j-1}$, $\mathcal{F}^k_j$ is isomorphic to the poset given on the element set*

$$\left(\mathcal{F}^k_{j-1} \setminus \mathcal{C}(p_j)\right) \cup \{(p_j, X) \mid X \in \mathcal{C}(p_j)\}$$

by the order relation

$$F \leq_j F^* :\Leftrightarrow \begin{cases} F, F^* \in \mathcal{F}^k_{j-1} \setminus \mathcal{C}(p_j) & \text{and } F \leq_{j-1} F^*, \\ F = (p_j, X), F^* = (p_j, X^*) & \text{and } X \leq_{j-1} X^*, \\ F = (p_j, X), F^* \in \mathcal{F}^k_{j-1} \setminus \mathcal{C}(p_j) & \text{and } \mathrm{op}_{p_j}(X) \leq_{j-1} F^*, \end{cases}$$

the isomorphism being given by the correspondence $(p_j, X) \mapsto \mathrm{op}_{p_j}(X)$, and the identical mapping elsewhere.

Proof. Compare [20, Corollary 7.3.6]. □ □

Note that the faces represented by (p_j, X) for $X \in \mathcal{C}(p_j)$ are exactly the faces F whose minimal k-face is p_j.

Corollary 4.1.4. *If $p_i, p_{i+1} \in \mathcal{P}^k$ are both near $\widetilde{H}^k_{i-1}$, then the structure of $\widetilde{\mathcal{A}}^k_{i+1}$ does not depend on the order in which the two flippings are carried out.*

In particular, any $q \in \mathcal{P}^k$ near $\widetilde{H}^k_{i-1}$ and different from p_i is also near $\widetilde{H}^k_i$.

Proof. The fact that both are near $\widetilde{H}^k_{i-1}$ implies in particular $\mathcal{C}(p_i) \cap \mathcal{C}(p_j) = \emptyset$, and thus the modifications do not influence each other. □ □

Notation 4.1.5. Every $\mathcal{H}^k_j$ contains an isomorphic copy of $\mathcal{F}^{k-1}_0 \simeq \mathcal{F}^{k-2}$ because $\mathcal{F}(\mathcal{H}^k_0) = \mathcal{F}^{k-1}$. We may then add to $\mathcal{H}^k_j$ a pseudohyperplane $\widetilde{L}^{k,j}_0$ that intersect exactly the faces of $\mathcal{F}^{k-2}$ ('a copy of $\mathcal{F}(\mathcal{H}^{k-1}_0)$') and consider consecutive flippings $\widetilde{L}^{k,j}_i$ of it along the elements of $\mathcal{P}^k_j$.

Remark 4.1.6. It is not difficult to see that $\widetilde{L}^{k,j}_0$ indeed *can* be swept through $\mathcal{H}^k_j$. First of all, the oriented matroid program defined by $\mathcal{H}^k_0$and $\widetilde{L}^{k,0}_0$ is euclidean because the oriented matroid associated to $\mathcal{H}^k_0$is realizable (this arrangement is obtained by intersecting V_{k-1} with $\mathscr{A}$). To conclude that $\widetilde{L}^{k,j}_0$ can be swept through $\mathcal{H}^k_j$ for $j > 0$ it is enough to see that, for every $j \geq 0$, euclideanness of the program associated with $\mathcal{H}^k_j$ and $\widetilde{L}^{k,j}_0$ implies euclideanness of the program associated with $\mathcal{H}^k_{j+1}$ and $\widetilde{L}^{k,j+1}_0$.

This last fact is readily checked by considering in both cases the orientation of the graph associated to the programs. By Lemma 4.1.3 we know how $\mathcal{H}^k_j$ changes to $\mathcal{H}^k_{j+1}$ after the flip through p_j, and since $\widetilde{L}^{k,j}_0 = \widetilde{L}^{k,j+1}_0$, the orientation of the edges agrees everywhere except in $\mathcal{C}(p_j)$. Now by inspecion of the possible situations one concludes that the existence of a directed cycle in the graph associated to $\mathcal{H}^k_{j+1}$, $\widetilde{L}^{k,j+1}_0$, implies the existence of a directed cycle in the graph associated to $\mathcal{H}^k_j$, $\widetilde{L}^{k,j}_0$. Then, by 2.2.22 we are done.

Special orderings

Definition 4.1.7. Given an essential affine real (pseudo)arrangement $\mathscr{A}$ and a general position (pseudo)hyperplane $\widetilde{H}_0$, a total ordering $p_1, p_2, \dots$ of the points of $\mathscr{A}$ is a *special ordering* if there is a sequence of arrangements of pseudohyperplanes $\widetilde{\mathcal{A}}_0, \widetilde{\mathcal{A}}_1, \dots$ such that $\widetilde{\mathcal{A}}_0 = \mathscr{A} \cup \{\widetilde{H}_0\}$, and for all $j > 0$, $\widetilde{\mathcal{A}}_j$ is obtained from $\widetilde{\mathcal{A}}_{j-1}$ by flipping $\widetilde{H}_j$ across p_j.

We collect some fact for later reference.

Remark 4.1.8. It is clear that every $\widetilde{H}_j^k$ is in general position with respect to $\mathscr{A}$, because $\widetilde{H}_0^k$ was chosen so. Therefore, any two p, q that are near some $\widetilde{H}_j^k$ satisfy $\mathcal{C}(p) \cap \mathcal{C}(q) = \emptyset$ (just by definition of 'near', see [20]). This means amongst other that every element of $\mathcal{F}_{\preceq p} \cap \mathcal{F}_{\preceq q}$ is already in $\mathcal{H}_j^k$, thus either is in V_{k-1} or in some 'earlier' $\mathcal{C}(z)$, for $z \rightsquigarrow^k p_j \rightsquigarrow^k p, q$.

Lemma 4.1.9. *Let a special ordering $\rightsquigarrow$ of the points of an affine arrangement $\mathscr{A}$ with respect to a generic hyperplane $\widetilde{H}_0$ be given. Choose two consecutive points $p \rightsquigarrow q$ and let $\rightsquigarrow^*$ be the total ordering of obtained from $\rightsquigarrow$ by reversing the order of p and q. Then, the following are equivalent:*

(1) $\rightsquigarrow^*$ is a special ordering with respect to $\widetilde{H}_0$,

(2) In the induced flipping sequence just before the flipping through p, both p and q are near the moving pseudohyperplane.

(3) For all $F \in \mathcal{F}_{\preceq p} \cap \mathcal{F}_{\preceq q}$, the minimum vertex of F comes before p and q in $\rightsquigarrow$.

Proof. (1)$\Leftrightarrow$(2) is clear, and (2)$\Leftrightarrow$(3) follows from Remark 4.1.8 above. $\square$ $\square$

Let us return to the setup of Section 4.1 and fix $k \in \{1, \dots, d\}$ for this section. We want to understand whether (and how) it is possible to deduce a valid special ordering of the elements of $\mathcal{P}_j^k$ from a special ordering of the elements of $\mathcal{P}_{j-1}^k$.

Definition 4.1.10. Let a total ordering $\rightsquigarrow_{j-1}^k$ of $\mathcal{P}_{j-1}^k$ be given. For every line ℓ of $\mathcal{H}_{j-1}^k$ that contains some element of $\mathcal{X}(p_j) \cap \mathcal{P}_{j-1}^k$ let $y^+(\ell)$, $y^-(\ell)$ denote the points of $\mathcal{H}_{j-1}^k$ where ℓ intersects the (topological) boundary of $\mathcal{X}(p_j)$, ordered so that $y^+(\ell) \rightsquigarrow_{j-1}^k y^-(\ell)$.
Moreover, call $\overline{y}$ the maximum with respect to $\rightsquigarrow_{j-1}^k$ of all $y^+(\ell)$ (for varying ℓ).

Then define a total ordering of $\mathcal{P}_j^k$ by setting, for every $z_1, z_2 \in \mathcal{P}_j^k$:

$$z_1 \rightsquigarrow_j^k z_2 \Leftrightarrow \begin{cases} z_1, z_2 \in \mathcal{P}_j^k \cap \mathcal{P}_{j-1}^k & \text{and } z_1 \rightsquigarrow_{j-1}^k z_2 \\ z_1 \notin \mathcal{P}_{j-1}^k, z_2 \in \mathcal{P}_{j-1}^k & \text{and } \overline{y} \rightsquigarrow_{j-1}^k z_2 \\ z_i = (p_j, x_i) \text{ for } i = 1, 2 & \text{and } x_2^* \rightsquigarrow^{k-1} x_1^*, \end{cases}$$

where x_i^* denotes the unique element of $\mathcal{P}^{k-1}$ with the same support as x_i.

Our goal will be to prove the following statement.

Theorem 4.1.11. *For every $k \geq 0$ and every $j > 0$, if $\rightsquigarrow^k_{j-1}$ is a special ordering, so is $\rightsquigarrow^k_j$ too.*

Notation 4.1.12. To investigate the situation, we will focus on $\mathcal{X}(p_j) \subset \mathcal{H}^k_{j-1}$. Let us write $x_1, \ldots, x_s$ for the points of this complex. Also, let $\ell_1, \ldots, \ell_l$ be the (pseudo)lines of $\mathcal{H}^k_j$ that contain some x_i and write $y_1, y_2, \ldots$ for the intersection points of the ℓ's with the hyperplanes bounding $\mathcal{X}(p_j)$.

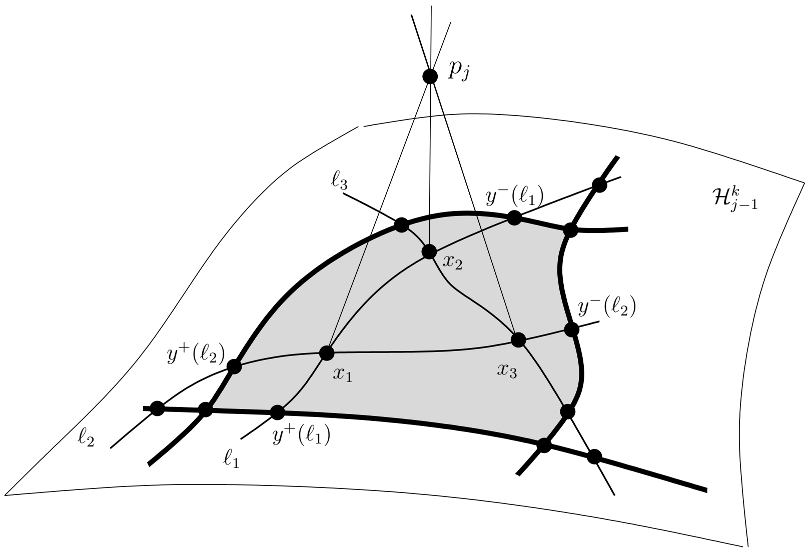


Figure 4.1: An illustration of our setup. The shaded region is $\mathcal{X}(p_j)$, and the subcomplex $\mathcal{C}(p_j)$ is spanned by x_1, x_2, x_3.

Remark 4.1.13. It is useful to consider the lines passing through a point $q \in \mathcal{P}^k$. For instance, one can see that if two points $p, q \in \mathcal{P}^k$ lie on a common line ℓ of $\mathscr{A}^k$ so that p is nearer than q to $\ell \cap V_{k-1}$, then there is no sequence of flippings of $\widetilde{H}^k_0$ in which q comes before p.

Lemma 4.1.14. *Let a special ordering of $\mathcal{P}^k_{j-1}$ be given. Let $\mathcal{X}(p_j) = \{x_1, \ldots, x_s\}$ be numbered so that $V_{k-1} \cap |x_r| \rightsquigarrow^{k-1} V_{k-1} \cap |x_t|$ if and only if $r < t$ (remember that $|x|$ denotes the support of x). Moreover, let $p_1, p_2, \ldots$ denote the elements of $\mathcal{P}^k_{j-1} \setminus \{x_1, \ldots, x_s\}$ ordered according to $\rightsquigarrow^k_{j-1}$ and let m be so that $p_m = \overline{y}$. Then the following is a special ordering of $\mathcal{P}^k_{j-1}$:*

$$p_1, p_2, \ldots, \overline{y}, x_1, x_2, \ldots, x_s, p_{m+1}, p_{m+2}, \ldots.$$

Proof. The proof is subdivided in three steps.

Claim 4.1.14.1. *Every y_i is contained in exactly one of the lines $\ell_1, \ldots, \ell_l$. Moreover, for all $1 \leq i < j \leq l$, there is r, $1 \leq r \leq s$, such that $x_r = \ell_i \cap \ell_j$.*

Proof of claim 4.1.14.1. Note that $\ell_i \cap \ell_j \neq \emptyset$ because both lines are flats of the central arrangement $\mathscr{A}_{p_j}$, and these intersections are points of the arrangement

$\widetilde{H}^k_{j-1} \cup \mathscr{A}_{p_j}$. Now both claims follow because the subcomplex $\mathcal{X}(p_j)$ contains, by definition of flipping, every point of the arrangement given by $\widetilde{H}^k_{j-1} \cup \mathscr{A}_{p_j}$ (see Definition 2.2.18 and ff.). $\square$

Now recall that, in *any* special ordering of $\mathcal{P}^k_{j-1}$, the 0-dimensional faces on every ℓ_i must be ordered 'along ℓ_i'. Thus, on every line ℓ_i the segment contained in $\mathcal{X}(p_j)$ is bounded by two points, say $y^+(\ell_i) \rightsquigarrow^k_{j-1} y^-(\ell_i)$.

Claim 4.1.14.2. *Consider a special ordering of $\mathcal{P}^k_{j-1}$. Then the ordering remains special after the following modifications:*

(1) Switching $y^+(\ell)$ and x whenever x comes right before $y^+(\ell)$.

(2) Switching $y^-(\ell)$ and x whenever x comes right after $y^-(\ell)$.

(3) Switching x and any $z \notin \mathcal{X}(q)$ whenever x and z are consecutive.

Proof of claim 4.1.14.2. In case (1) note that Claim 4.1.14.1 ensures that $\mathcal{C}(y^+(\ell))$ lies fully outside $\mathcal{X}(p_j)$ and so it is disjoint from any $\mathcal{C}(x)$. Now let x be, say, the r-th element of $\mathcal{P}^k_{j-1}$. Since x comes right before $y^+(\ell)$ we must have that $y^+(\ell)$ is already near $\widetilde{L}^{k,j-1}_{r-1}$: indeed, in that case x cannot be contained in ℓ and by definition also not in the boundary hyperplane that intersects ℓ in $y^+(\ell)$. Since the only change in passing from $\widetilde{L}^{k,j-1}_{r-1}$ to $\widetilde{L}^{k,j-1}_{r}$ happens at faces which supports contain x, we have $\mathcal{Y}(y^+(\ell)) \cap \widetilde{L}^{k,j-1}_{r-1} = \mathcal{Y}(y^+(\ell)) \cap \widetilde{L}^{k,j-1}_{r}$. By Corollary 4.1.4 we are done.

The case (2) is handled similarly, by reversing the order of the flippings, and case (3) is clear. $\square$ $\square$

At this point we know that the ordering

$$p_1, p_2, \ldots, p_m, [\cdots], p_{m+1}, p_{m+2}, \ldots,$$

where the square brackets contain the x_i's, is indeed a special ordering of $\mathcal{P}^k_{j-1}$. We have to prove that we can indeed arrange the elements in the square bracket as required.

First, if x_1 is not near $\widetilde{L}^{k,j-1}_m$, then there is a line $\ell \ni x_1$ and some other x_i that lies on ℓ between x_1 and $\ell \cap \widetilde{L}^{k,j-1}_m$. In particular, x_i lies between x_1 and $\ell \cap \widetilde{L}^{k,j-1}_0 = \ell \cap \mathcal{F}^{k-1}_0 = \ell \cap V_{k-2}$. The points $x_1, \ldots, x_s$ are given by the intersection of the pseudohyperplane $\mathcal{H}^k_{j-1}$ with lines $g_1, \ldots, g_s$ of $\mathscr{A}^k$, and ℓ is the intersection of $\mathcal{H}^k_{j-1}$ with the plane E generated by g_1 and g_i. For all r let $x^*_r := g_r \cap V_{k-1}$. Since $g_1 \cap g_i = p_j$, that lies outside the segments $\overline{x_1 x^*_1}$ and $\overline{x_i x^*_i}$, we get that in V_{k-1} the point x^*_i lies on the line $\ell^* := E \cap V_{k-1}$ between x^*_1 and $\ell^* \cap \widetilde{H}^{k-1}_0 = \ell^* \cap V_{k-2}$. With Remark 4.1.13, and by the way the numbering of the x_r was chosen, we reach a contradiction. We may now repete the argument with x_2, and all the following points until we reach x_s, concluding the proof. $\square$

We are now ready to prove the main result of this section.

Proof of claim 4.1.11. We can assume that $\rightsquigarrow^k_{j-1}$ is modified so to agree with the statement of Lemma 4.1.14. Let $U^{k,j}_m := \bigcup_{i \leq m} \widetilde{L}^{k,j}_i$ (meaning the set of all faces that are contained in some $\widetilde{L}^{k,j}_i$). Since the orderings $\rightsquigarrow^k_{j-1}$ and $\rightsquigarrow^k_j$ now agree up to $p_m = \overline{y}$ and clearly $U^{k,j}_m = U^{k,j-1}_m$ by Lemma 4.1.3, we are left with proving that it is possible to perform the flippings of the x_i just after $\overline{y}$, and in the reverse order as the corresponding flippings are performed in $\widetilde{H}^k_{j-1}$.

To this end, let us consider $\widetilde{L}^{k,j}_m$, i.e., the moving pseudohyperplane 'just after' the flipping through $p_m = \overline{y}$. Recall that $\widetilde{L}^{k,j}_m \simeq \widetilde{L}^{k,j-1}_m$, and in particular we can compare the points $z_1, \dots, z_l$ where the lines containing some x_i intersect the pseudohyperplane corresponding to $\widetilde{L}^{k,j}_m$. Let $F_1, \dots, F_l$ be the faces such that $z_i = F_i \cap \widetilde{L}^{k,j-1}_m$. Then we see that the 'same' points z_i are given by $(p_j, F_i) \cap \widetilde{L}^{k,j}_m$. So by the correspondence established in Lemma 4.1.3 we have that a point (p_j, x) is near $\widetilde{L}^{k,j}_m$ if and only if x is near (but "on the backside" of) $\widetilde{L}^{k,j}_{m+s}$. This shows that (p_j, x_s) is near $\widetilde{L}^{k,j}_m$. After performing this flipping we may repeat the argument to conclude that (p_j, x_{s-l}) is near $\widetilde{L}^{k,j}_{m+l}$ for every $l \leq s$, and the claim of the Theorem follows. □

4.2 Combinatorial polar orderings

After having looked inside each V_k, let us study the structure that arises by considering all strata.

Definition 4.2.1 (Compare Theorem 5. of [87])**.** Given total orderings $\rightsquigarrow^k$ of each $\mathcal{P}^k$, we define a total ordering $\lhd$ of $\mathcal{F}$. All faces of codimension d are elements of $\mathcal{P}^d$ and are ordered accordingly. Assuming the ordering is defined for all faces of codimension $k+1$ and bigger, then given two k-codimensional faces F and G we have:

(1) if $F, G \in \mathcal{P}^k$, $F \lhd G$ if $F \rightsquigarrow G$;

(2) if $F \in \mathcal{P}^k$ and $G \notin \mathcal{P}^k$, then $F \lhd G$;

(3) if $F, G \notin \mathcal{P}^k$, let F', (resp. G') be the $k+1$-codimensional facet in the boundary of F (resp. G), which is minimum with respect to $\lhd$. Then:

  (3.1) if $F' \lhd G'$, then $F \lhd G$.

  (3.2) if $F' = G'$, then $F \lhd G$ if and only if $F_0 \rightsquigarrow G_0$, where F_0 and G_0 are the unique elements of $\mathcal{P}^k$ that have the same linear span as F, respectively G.

(4) If $F \in \mathcal{P}^k$, then F is lower than any $k+1$-codimensional facet

(5) If $F \notin \mathcal{P}^k$, then F is bigger than its minimal boundary F' and lower than any $(k+1)$-codimensional facet which is bigger than F'.

Thus, if the orderings on the $\mathcal{P}_k$s are given by lexicografic order on the polar coordinates, we reproduce the polar order of [87].

Definition 4.2.2. Let an affine real arrangement $\mathscr{A}$ be given. A *combinatorial polar ordering* of $\mathcal{F}(\mathscr{A})$ is any total ordering $\lhd$ induced via Definition 4.2.1 by the choice of a general flag $(V_k)_{k=0\dots d}$ and of special orderings $\rightsquigarrow^k$ of the points of V_k with respect to V_{k-1}, for every $k = 1, \dots, d$.

Let us next give an alternative characterization of the combinatorial polar orderings that will turn out to be useful later on.

Definition 4.2.3. Given $F \in \mathcal{F}$, define the signature of F as $\sigma(F) = (k_F, j_F, m_F)$, where

$$
\begin{aligned}
k_F &:= \min\{k \mid V_k \cap F \neq \emptyset\} \\
j_F &:= \min\{j \mid F \in \mathcal{F}(\mathcal{H}_j^{k_F})\} \\
m_F &:= \min\{m \mid F \in \mathcal{F}(\widetilde{L}_m^{k_F, j_F})\},
\end{aligned}
$$

where we agree to put $j_F = 0$ when $k_F = 0$ and $m_F = 0$ if $k_F \leq 1$ because in those cases the above definition is void.

Lemma 4.2.4. *Let special orderings $\rightsquigarrow^k$ be given for every k, and let $\lhd$ be the total ordering of $\mathcal{F}$ induced by them. For $F_1, F_2 \in \mathcal{F}$, if $\sigma(F_1) < \sigma(F_2)$ in the lexicographic order, then $F_1 \lhd F_2$.*

Proof. If $k_{F_1} < k_{F_2}$, then by Definition 4.2.1.(4) $F_1 \lhd F_2$.

Suppose now $k_{F_1} = k_{F_2}$ but $j_{F_1} < j_{F_2}$. If $F_1, F_2 \in \mathcal{P}^k$, then we are already done by Definition 4.2.1.(1). Else, the condition means that the minimal codimensional-$k+1$ face of F_1 comes before the minimal codimensional-$(k+1)$ face of F_2, and by Remark 4.1.13 we are done.

The same line of reasoning applies to show that $k_{F_1} = k_{F_2}$, $j_{F_1} = j_{F_2}$ and $m_{F_1} < m_{F_2}$ implies $F_1 \lhd F_2$. $\square$ $\square$

Remark 4.2.5. It is now easy to see that one could go on and define for every face F a vector

$$(\sigma_1(F), \dots, \sigma_{k_F}(F))$$

with $\sigma_1(F) := j_F$ and $\sigma_i(F) := \min\{m \mid F \in \widetilde{L}_m^{k_F, \sigma_1(F), \dots, \sigma_{i-1}(F)}\}$ (where $\widetilde{L}_m^{k_F, a_1, a_2, \dots, a_j}$ is defined for $j > 1$ as the moving hyperplane of $\mathcal{H}_{a_j}^{k_F, a_1, \dots, a_{j-1}}$ after the m-th flipping). From this, a signature

$$\sigma(F) := (\underbrace{0, 0, \dots, 0}_{d-k_F \text{ times}}, \sigma_1(F), \dots, \sigma_{k_F}(F))$$

can be defined, so that for all $F_1, F_2 \in \mathcal{F}$, $F_1 \lhd F_2$ if and only if $\sigma(F_1) < \sigma(F_2)$ lexicographically. This yields an alternative equivalent formulation of the ordering defined in 4.2.1.

Remark 4.2.6. From the point of view of the computational complexity, the translation of Remark 4.2.5 shows that the whole work amounts indeed to determine special orderings of the V_k's. Effective algorithms for this kind of tasks were developed in the last few years by Edelsbrunner et al. [50].

"Polar" vector fields and switches

Recall that for $F \in \mathcal{F}$ we denote by F' the smallest facet of F with respect to the given ordering $\lhd$. We rephrase Definition 4.0.21 in our broader context.

Definition 4.2.7. Let an affine real arrangement $\mathscr{A}$ and a general flag $\{V_k\}_{k=0,\ldots,d}$ be given. For every total ordering $\lhd$ of $\mathcal{F}$ we define

$$\Phi(\lhd) := \left\{ \begin{array}{cc} [C \preceq F] < [C \preceq F'] \in \mathcal{S} : & \begin{array}{c} (i)\ F \notin \mathcal{P}, \\ (ii)\ G' \neq F \text{ for all } G \text{ with} \\ C \prec G \prec F. \end{array} \end{array} \right\}.$$

Remark 4.2.8. If $\lhd$ is the polar ordering defined in [87], then by Theorem 4.0.21 we know that $\Phi(\lhd)$ is a maximum acyclic matching on the poset of cells of the Salvetti complex, i.e., it defines a discrete Morse function on $\mathcal{S}$ with the minimum possible number of critical cells.

Our aim is to show that the total ordering can be slightly modified without affecting the resulting acyclic matching.

Definition 4.2.9 (Switch)**.** Let special orderings $\rightsquigarrow^k$ of the $\mathcal{P}^k$'s with respect to V_{k-1} be given and let $\lhd$ denote the induced total ordering of $\mathcal{F}$.
Two faces $F_1 F_2 \in \mathcal{P}^k$ are called *c-independent* if
(1) they are consecutive with respect to $\rightsquigarrow^k$, and
(2) $G \lhd F_1, F_2$ for every $G \in \mathcal{F}_{\preceq F_1} \cap \mathcal{F}_{\preceq F_2}$.
The ordering $\rightsquigarrow^*$ is obtained from $\rightsquigarrow$ by a *switch* if there are two c-independent faces $F_1 \rightsquigarrow F_2$ so that $F_2 \rightsquigarrow^* F_1$, while $F \rightsquigarrow G$ implies $F \rightsquigarrow^* G$ for every other F, G. We will write $\lhd^*$ for the corresponding combinatorial polar ordering.

The following fact is an easy consequence of Corollary 4.1.4.

Theorem 4.2.10. *If an ordering $\rightsquigarrow$ of the points of an affine arrangement is special with respect to a general position hyperplane $\widetilde{H}$, then so is $\rightsquigarrow^*$.*

Now we need to study how the induced total orderings $\lhd$ of $\mathcal{F}$ vary by switching two c-independent faces.

Lemma 4.2.11. *Let a special ordering $\rightsquigarrow$ of $\mathcal{P}$ be given, and $\lhd$ be the associated total ordering of $\mathcal{F}$. Moreover, let $\rightsquigarrow^*$ be obtained from $\rightsquigarrow$ by a switch and let $\lhd^*$ be defined accordingly. Then the minimum facet F' of any $F \in \mathcal{F}$ with respect to $\lhd$ is also the minimum facet with respect to $\lhd^*$.*

Proof. Let F_1, F_2 denote the two faces involved in the switch, and write $k_0 := k_{F_1} = k_{F_2}$. The claim is easily seen to be true if $k_F < k_0$ or if $k_F > k_0 + 1$.

Consider the case where $k_F = k_0$. Since the ordering $\rightsquigarrow^{k_0-1}$ does not change, if

$$\min\nolimits_{\rightsquigarrow}\{p \in \mathcal{P}^{k_0} \mid p \succeq F\} = \min\nolimits_{\rightsquigarrow^*}\{p \in \mathcal{P}^{k_0} \mid p \succeq F\} \tag{4.2.1}$$

then the claim is clearly true by Lemma 4.2.4.

Because F_1, F_2 are consecutive, condition (4.2.1) fails only if both $F_1, F_2 \succ F$. But then by Definition 4.2.9.(2) $F \lhd F_1, F_2$, implying that the minimum facet of F comes before F_1 and F_2, and thus remains unchanged by passing from $\lhd$ to $\lhd^*$.

Now let $k_F = k_0 + 1$. If $\operatorname{codim}(F) = k_0$, then F' (i.e., the minimal facet of F) is an element of $\mathcal{P}^{k_0+1}$, where the order remains unchanged; in any other case, $j_{F'} = j_F$. So after Lemma 4.2.4 we must prove that the claim holds for $F \in \operatorname{op}_{p_j} \mathcal{C}(p_j)$, for any $p_j \in \mathcal{P}^{k_0+1}$. Because the F_i are consecutive, the ordering on the set $\mathcal{P}^{k_0+1}_{j-1} \cap \mathcal{X}(p_j)$ does not change in passing from $\rightsquigarrow$ to $\rightsquigarrow^*$, unless p_j is the intersection point of the two lines of $\mathscr{A}^{k_0+1}$ that contain F_1 and F_2. But even in this last case, the corresponding points G_1, G_2 of $\mathcal{H}^k_j$ are again consecutive. Moreover, they are not joined by an edge in $\mathcal{H}^k_j$because F_1 and F_2 are not. By the construction of Lemma 4.1.14, all this implies that they are both near the moving pseudohyperplane $\widetilde{L}^{k_F,j}$ 'just before flipping across the first of them'. In turn, this means (by Remark 4.1.8) that the elements of $\mathcal{F}_{\preceq G_1} \cap \mathcal{F}_{\preceq G_2}$, and in particular F and F', come before G_1 and G_2 - i.e., the only elements of $\mathcal{P}^{k_F}_j$ that are switched. We can then apply the same reasoning as the case $k_0 = k_F$ to conclude the proof. $\square$ $\square$

In particular, just by looking at the definition of the matchings we obtain the following result.

Theorem 4.2.12. *Let a special ordering $\rightsquigarrow$ of $\mathcal{P}$ be given, and $\lhd$ be the associated total ordering of $\mathcal{F}$. Moreover, let $\rightsquigarrow^*$ be obtained from $\rightsquigarrow$ by a switch and let $\lhd^*$ be defined accordingly. Then*

$$\Phi(\lhd) = \Phi(\lhd^*).$$

The next step is to see that actually switches are rather powerful tools for transforming special orderings.

Theorem 4.2.13. *Let $\rightsquigarrow_1, \rightsquigarrow_2$ be any two special orderings of the point of an arrangement $\mathscr{A}$ with respect to a generic hyperplane $\widetilde{H}$. Then $\rightsquigarrow_2$ can be obtained from $\rightsquigarrow_1$ by a sequence of switches.*

Proof. Let $\mathcal{P}$ denote the set of points of $\mathscr{A}$. Write $\mathcal{P} = \{p_1, p_2, \ldots, p_m\}$ where $i < j$ if $p_i \rightsquigarrow_1 p_j$. Let σ be the permutation of $[m]$ so that $p_i \rightsquigarrow_2 p_j$ if $\sigma(i) < \sigma(j)$. We proceed by induction in the number $u(\sigma)$ of inversions in σ, the case $u(\sigma) = 0$ being trivial.

So suppose $u(\sigma) > 0$. Then there are numbers $i_1 < i_2$ such that $\sigma(i_1) = \sigma(i_2) + 1$. If τ is the transposition $(\sigma(i_2), \sigma(i_1))$, then the number of inversions of the permutation $\tau\sigma$ is strictly smaller than $u(\sigma)$.

Clearly the ordering of $\mathcal{P}$ associated to $\tau\sigma$ is obtained by changing the position of $v_1 := p'_{\sigma(i_1)}$ and $v_2 := p'_{\sigma(i_2)}$. Thus we will be done by showing that this is a valid 'switch' in $\rightsquigarrow_2$ according to Definition 4.2.9.

To this end, first remark that the elements are clearly consecutive in $\rightsquigarrow_2$. Next consider the fact that $v_2 \rightsquigarrow_1 v_1$ and $v_1 \rightsquigarrow_2 v_2$, where both $\rightsquigarrow_1$ and $\rightsquigarrow_2$ are valid special orderings. By Remark 4.1.13 there is no line containing both v_1 and v_2. Thus, in the sequence of flippings associated to $\rightsquigarrow_2$, just before flipping across v_1 the moving hyperplane is actually also near v_2. By Lemma 4.1.9 this ensures condition (2) of the definition of independence, and concludes the proof. $\square$ $\square$

If $\lhd$ is the polar ordering defined in [87], then by Theorem 4.0.21. we know that $\Phi(\lhd)$ is a maximum acyclic matching on the poset of cells of the Salvetti complex, i.e., it defines a discrete Morse function on $\mathcal{S}$ with the minimum possible number of critical cells. Moreover, the critical cells are given in terms of $\lhd$ by Theorem 4.0.21.

At this point, the main result of this section is evident.

Proposition 4.2.14. *Let a combinatorial polar ordering of the faces of an affine real arrangement $\mathscr{A}$ be given. Then the induced matching $\Phi(\lhd)$ is a discrete Morse vector field with the minimum possible number of critical cells.*

Remark 4.2.15. We already saw that the approach via flippings makes it unnecessary to request the stronger form of 'generality' for the flag $(V_k)_k$ that is needed in [87]. However, if this condition *is* satisfied, then the matching is the polar gradient of [87].

4.3 Recursively orderable arrangements

Having established that every special ordering of an arrangement with respect to a general flag gives rise to a combinatorial polar ordering - and thus to a minimal model for the complement of the arrangement's complexification, the problem of actually finding such an ordering remains.

However, some arrangements admit some particularly handy special orderings, that give rise to combinatorial polar ordering that appear particularly well-suited for explicit computations. The motivating example here is the braid arrangement, studied in [87]. In the following we state this nice property and look for other examples of arrangements that enjoy it.

The definition

Definition 4.3.1 (Recursive Ordering)**.** Let $\mathscr{A}$ be a real arrangement and $(V_k)_{k=0,\ldots,d}$ a general flag. The corresponding *recursive ordering* is the total ordering $\sqsubset$ of $\mathcal{P}$ given by setting $F \sqsubset G$ if one of the following occurs:

(i) $F \in \mathcal{P}^h$, $G \in \mathcal{P}^k$ for $h < k$.

(ii) there is k so that $F, G \in \mathcal{P}^k$ and, writing $F_0 := \min\{J \in \mathcal{P}^{k-1} \mid F \subset |J|\}$, $G_0 := \min\{J \in \mathcal{P}^{k-1} \mid G \subset |J|\}$,

  (a) either $F_0 \sqsubset G_0$,

  (b) or $F_0 = G_0$ and there exists a sequence of faces

$$F_0 \prec F_1 \succ J_1 \prec F_2 \succ J_2 \cdots \prec F$$

  such that $\operatorname{codim}(F_i) = \operatorname{codim}(J_i) + 1 = \operatorname{codim}(F)$, and every J_i, F_i intersect $|F_0| \cap V_k$, and $F_i \neq G$ for all i.

Definition 4.3.2. An arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is said to be *recursively orderable* if there is a general flag $(V_k)_{k=0,\ldots,d}$ so that the corresponding recursive ordering is special.

Example 4.3.3. The braid arrangement on n strands is recursively orderable for every n, as was shown (and exploited) in [87].

Remark 4.3.4. With the work done so far, we see that proving that an arrangement $\mathscr{A}$ is recursively orderable amounts essentially to finding a special ordering of $\mathcal{P}(\mathscr{A})$ such that in every V_k condition (ii).(a) of the above Definition 4.3.1 holds, since Conditions (i) and (ii).(b) are "standard features" in every special ordering.

Recursively orderable arrangements of lines

In this section $\mathscr{A}$ will be an affine arrangement of lines in $\mathbb{R}^2$. And we will suppose it to be *actually* affine, i.e. $\mathcal{P}^2$ consists of more than one element (otherwise the arrangement is central, and every central 2-arrangement is trivially recursively orderable). Here we do not need the detailed notation of the general case, so we will write $P := \mathcal{P}^2$ and abuse notation by writing $\mathscr{A} := \mathcal{P}^1$.

The generic flag here is a pair (b, ℓ), where b is a point in an unbounded chamber and $\ell \ni b$ is a line in general position with respect to $\mathscr{A}$ where all the points of $\mathscr{A}$ lie on the same side of ℓ, and the points $\mathscr{A} \cap \ell$ lie on the same halfline with respect to b. We shall sometimes confuse b with the chamber B it is contained in. In particular, we see that B cannot have two parallel walls.

Notation 4.3.5. Let an affine arrangement of lines $\mathscr{A}$ be given together with a general flag (b, ℓ). The line ℓ intersects a facet of B: let h_0 denote the element of $\mathscr{A}$ supporting it. Let $a_1, a_2, \ldots$ denote the points on h_0, numbered by increasing distance from b. Moreover, write $M_j := \{h_1^j, h_2^j, \ldots, h_{\max}^j\}$ for the set of all lines different from h_0 that contain a_j, ordered according to the sequence of points they generate on ℓ. For every $h \in \mathscr{A}$ let h^+ denote the (open) halfplane bounded by h and containing b, and set $h^- := \mathbb{R}^2 \setminus h^+$. Then we define, for every $j = 1, \ldots r$,

$$\Lambda_1 := \overline{h}_0^+ \cap (h_{\max}^1)^-,$$
$$\Lambda_j := (h_{\max}^{j-1})^+ \cap (h_{\max}^j)^- \text{ for } j > 1,$$

where overline denotes topological closure.

Definition 4.3.6. If for every $p \in P \cap \Lambda_j$ there is $h \in M_j$ with $a_j, p \in H$, then we will say that Λ_j is *complete* (with respect to (b, ℓ)). The arrangement $\mathscr{A}$ is *complete with respect to* (b, ℓ) if every Λ_j is complete and $P \subset \bigcup_{j=1,\ldots,r} \Lambda_j$

Lemma 4.3.7. *An affine line arrangement $\mathscr{A}$ is recursively orderable with respect to a general flag (b, ℓ) if and only if $\mathscr{A}$ is complete with respect to (b, ℓ).*

Sketch of proof. Fix an ℓ. If $\mathscr{A}$ is not complete at some j, then there is a point $x \in P$ so that $x \in \Lambda_j$ but there is no line containing a_j and x. Let $\tilde{h}$ denote the first line of M_j such that $x \in \tilde{h}^-$, and pick any line $h \in \mathscr{A}$ that contains x and is not parallel to $\tilde{h}$. Let $y := h \cap \tilde{h}$. By construction $h \in \bigcup_{i>j} M_i$, and since x is between y and $h \cap \ell$ on h, by Remark 4.1.13 there is no ordering that is special w.r.t. ℓ and in which y comes after x, as recursive orderability with respect to ℓ would require.

On the other hand, if $\mathscr{A}$ *is* complete at every a_j, then an explicit recursive combinatorial polar ordering can be described as follows. Write $\mathscr{A} =$

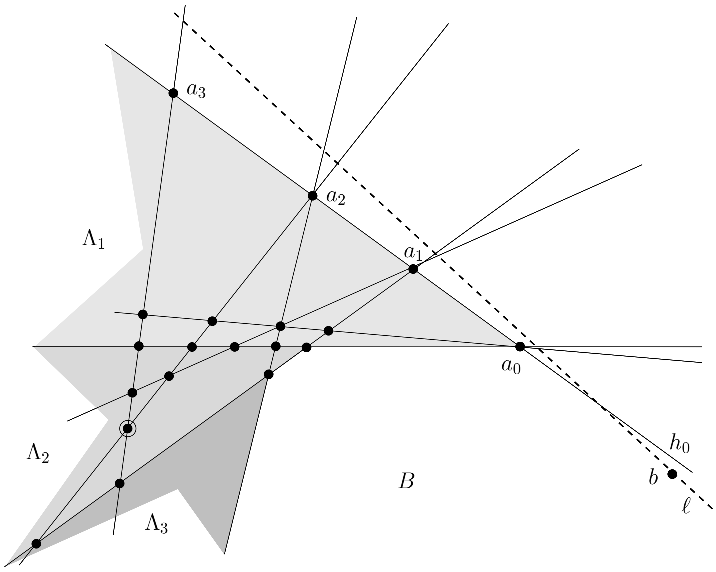


Figure 4.2: An affine line arrangement where Λ_1 is complete with respect to (b, ℓ) but Λ_2 is not. Thus, it is not recursively orderable

$\{h_0, h_1, \ldots\}$ according to the order in which the lines intersect ℓ. To begin with, being complete implies that there every point contained in h_0^- lies actually on h_0. It is now evident that the sequence $a_1, a_2, \ldots$ is a valid sequence of flippings, that leads to a pseudoline ℓ_1 with every point in $P \cap h_0$ on its "backside". Because there are no points in the interior of the cone $h_1^+ \cap h_2^-$, clearly one can now perform the flips across all points of h_2. Clearly one can go on this way until the moving pseudoline has flipped across every point in Λ_1.

We leave it to the reader to check that now one can perform all the flips of points in Λ_j for increasing j, each time following the order of lines induced by the intersection with ℓ. $\square$

We obtain a complete characterization of recursively orderable arrangements in the plane.

Theorem 4.3.8. *An affine arrangement of lines in the plane is recursively orderable if and only if there is a general flag (b, ℓ) so that $\mathscr{A}$ is complete with respect to (b, ℓ).*

Some general facts about recursively orderable arrangements can be deduced.

Remark 4.3.9. *Not all real reflection arrangements are recursively orderable.* For example consider the arrangement of type H_3. This is a central arrangement in $\mathbb{R}^3$, so it is recursively orderable if and only if there is a generic section of it that is recursively orderable. If we consider the projection of the associated dodecahedron on the plane of the section, we see that the points of this arrangement of lines correspond to vertices, to centers of edges or to centers of pentagonal faces. It is easy to see by case-by-case inspection that for every choice of a_0, of an adjacent chamber as B and of a suitable line for ℓ, Λ_1 is never complete with respect to (b, ℓ). Indeed, if a_0 corresponds to a pentagon p, the obstruction comes from a point corresponding to an edge e that is not adjacent to p but belongs to a pentagon adjacent to p (and vice-versa), while the obstruction for every 'vertex-type' choice of a_0 comes from another vertex that belongs to a common pentagon, but is not adjacent to a_0.

Remark 4.3.10. *Not all recursively orderable arrangements are* $K(\pi, 1)$. A counterexample can in fact be given already in dimension 3: consider the generic arrangement with defining form $xyz(x+y+z)$ in $\mathbb{R}^3$. By Hattori's theorem, this arrangement is not aspherical (see [76, Corollary 5.23]). However, it is central and any 2-dimensional section of it is easily seen to be recursively orderable.

Supersolvable arrangements are recursively orderable.

The class of "strictly linearly fibered" arrangements was introduced by Falk and Randell [55] in order to generalize the technique of Fadell and Neuwirth's proof [54] of asphericity of the braid arrangement (involving a chain of fibrations). Later on, Terao [92] recognized that strictly linearly fibered arrangements are exactly those which intersection lattice is supersolvable [89]. Since then these are known as *supersolvable arrangements*, and deserved intense consideration.

The goal of this section is to prove that every supersolvable real arrangement is recursively orderable. Let us begin by the definition.

Definition 4.3.11. A central arrangement $\mathscr{A}$ of complex hyperplanes in $\mathbb{C}^d$ is called supersolvable if there is a filtration $\mathscr{A} = \mathscr{A}_d \supset \mathscr{A}_{d-1} \supset \cdots \supset \mathscr{A}_2 \supset \mathscr{A}_1$ such that

(1) $\operatorname{rank}(\mathscr{A}_i) = i$ for all $i = 1, \ldots, d$

(2) for every two $H, H' \in \mathscr{A}_i$ there exits some $H'' \in \mathscr{A}_{i-1}$ such that $H \cap H' \subset H''$.

Before getting to the actual theorem, let us point out the key geometric fact.

Remark 4.3.12. Let $\mathscr{A}$ be as in Definition 4.3.11 and consider the arrangement $\mathscr{A}_{d-1}$ in $\mathbb{R}^d$. It is clearly not essential, and the top element of $\mathcal{L}(\mathscr{A}_{d-1})$ is a 1-dimensional line that we may suppose to coincide with the x_1-axis. The arrangement $\mathscr{A}_{d-1}$ determines an essential arrangement on any hyperplane H that meets the x_1-axis at some $x_1 = t$. For all t, the intersection of $\mathscr{A}_{d-1}$ with the hyperplane H determines an essential, supersolvable arrangement $\mathscr{A}'_{d-1} \subset \mathbb{R}^d$ with $\mathscr{A}'_r = \mathscr{A}_r$ as sets, for all $r \leq d-1$. Thus, given a flag of general position subspaces for $\mathscr{A}'_{d-1}$, we can find a combinatorially equivalent flag $(V_k)_{k=0,\ldots,d-2}$ on H.

Now let us consider a hyperplane H in $\mathbb{R}^d$ that is orthogonal to the x_1-axis, and suppose we are given on it as above a valid flag $(V_k)_{k=0,\ldots,d-2}$ of general position subspaces for $\mathscr{A}_{d-1}$. By tilting H around V_{d-2} we can obtain a hyperplane H' that is in general position with respect to $\mathscr{A}$ and for which all points of $\mathscr{A} \cap H'$ are on the same side with respect to V_{d-2}, and for which V_0 lies in an unbounded chamber.

By setting $V_{d-1} := H'$, $V_d := \mathbb{R}^d$ we thus obtain a valid general flag for $\mathscr{A} = \mathscr{A}_d$. Define $\mathcal{P}^k(\mathscr{A}_d)$ as the points of $\mathscr{A}_d \cap V_k$ and analogously for $\mathcal{P}^k(\mathscr{A}_{d-1})$. The flag remains general by translating $H' = V_{d-1}$ in x_1-direction away from the origin: we can therefore suppose that there is $R \in \mathbb{R}$ such that for all k, $k = 1, \ldots, d-1$, every element of $\mathcal{P}^k(\mathscr{A}_{d-1})$ is contained in a ball of radius R centered in V_0, that contains no element of $\mathcal{P}^k(\mathscr{A}_d) \setminus \mathcal{P}^k(\mathscr{A}_{d-1})$.

Corollary 4.3.13. *Let $\mathscr{A}$ and $(V_k)_{k=1,\ldots,d}$ be as in the construction of Remark 4.3.12. Then, for every $k = 1, \ldots, d$, if $F_1 \in \mathcal{P}^k(\mathscr{A}_{d-1})$ and $F_2 \in \mathcal{P}^k(\mathscr{A}) \setminus \mathcal{P}^k(\mathscr{A}_{d-1})$ are both contained in the support of the same $F \in \mathcal{P}^{k-1}(\mathscr{A})$, then $F_1 \rightsquigarrow^k F_2$ in every special ordering of $\mathcal{P}^k(\mathscr{A})$.*

Proof. This is an immediate consequence of Remark 4.1.13 and 4.3.12. □ □

Theorem 4.3.14. *Any supersolvable complexified arrangement $\mathscr{A}$ is recursively orderable. Moreover, the recursively orderable special ordering $\rightsquigarrow$ can be chosen so that for all $i = 2, \ldots, d$ and all $k = 1, \ldots, i-1$, if $F_1 \in \mathcal{P}^k(\mathscr{A}_{i-1})$ and $F_2 \in \mathcal{P}^k(\mathscr{A}_i) \setminus \mathcal{P}^k(\mathscr{A}_{i-1})$ lie in the support of the same $k+1$-codimensional face, then $F_1 \rightsquigarrow F_2$.*

Proof. If $\mathscr{A}$ has rank one, there is nothing to prove. So let $d := \operatorname{rank}(\mathscr{A}) > 1$ and suppose the claim holds for all complexified supersolvable arrangements or rank strictly less than d - in particular, for $\mathscr{A}_{d-1}$.

The general flag $(V_k)_{k=0,\ldots,d}$ we will use is obtained via Remark 4.3.12 from a general flag for $\mathscr{A}_{d-1}$ that gives rise to a special ordering satisfying the claim of the theorem. In particular, there exists a special ordering of $\mathcal{P}(\mathscr{A}_{d-1})$ that satisfies the property required by the claim for every $i = 2, \ldots, d-2$ (and every $k = 0, \ldots, i-1$). By Corollary 4.3.13 and Remark 4.3.4, we only have to describe, for every k, a special ordering of $\mathcal{P}^k(\mathscr{A})$ that satisfies condition (ii)(a) of Definition 4.3.1. This will be done by a new induction on k.

For $k = 0$ there is nothing to prove, and for $k = 1$ the only possible special ordering will clearly do. Let then $k > 1$. Suppose that recursive special orderings $\rightsquigarrow^{k-2}, \rightsquigarrow^{k-1}$ have already been defined on $\mathcal{P}^{k-2}$ and $\mathcal{P}^{k-1}$, and write $\mathcal{P}^{k-1} = \{p_1, p_2, \ldots\}$ accordingly. Since $\mathscr{A}$ is supersolvable, every $F \in \mathcal{P}^k(\mathscr{A})$ is contained in the support of some element of $\mathcal{P}^{k-1}(\mathscr{A}_{d-1})$ that we will call $p(F)$. So what we have to show is the following.

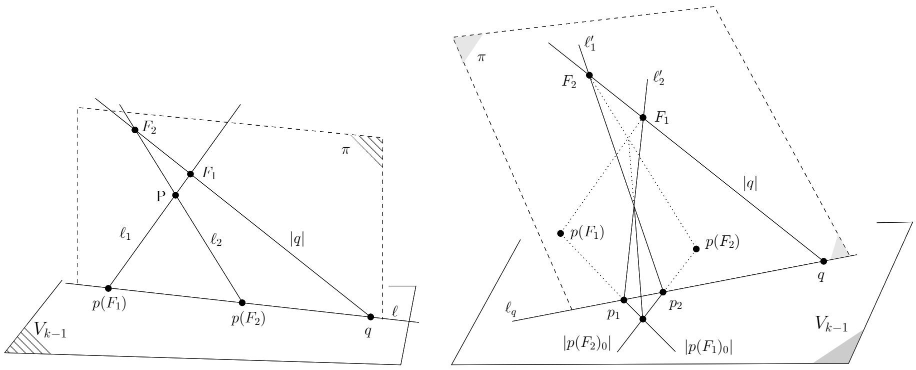
Claim 4.3.14.1. *The ordering on $\mathcal{P}^k(\mathscr{A})$ defined by*

$$F_1 \rightsquigarrow F_2 \Leftrightarrow \begin{cases} p(F_1) \rightsquigarrow^{k-1} p(F_2) \text{ or} \\ p(F_1) = p(F_2) \text{ and } F_1 \text{ is between } p(F_2) \text{ and } F_2 \text{ on } |p(F_2)| \end{cases}$$

is a special ordering.

Proof of the claim. Consider a special ordering of $\mathcal{P}^k(\mathscr{A})$ that agrees with the above ordering up to some face F_1, and suppose for contradiction that F_1 is not near the moving pseudohyperplane, i.e., that there is F_2 with $p(F_1) \rightsquigarrow^{k-1} p(F_2)$ which is on a line passing through F_1 between F_1 and the moving pseudohyperplane. By the inductive hypothesis on $\mathscr{A}_{d-1}$ we know that the above defined ordering is indeed special for the elements of $\mathcal{P}^k(\mathscr{A}_{d-1})$, and by Corollary 4.3.13 we conclude that F_1 cannot be in $\mathcal{P}(\mathscr{A}_{d-1})$.

Thus, the only obstruction to the construction of such a total ordering would come from the following situation: two faces $F_1, F_2 \in \mathcal{P}^k(\mathscr{A}) \setminus \mathcal{P}^k(\mathscr{A}_{d-1})$ lying on the support of the same $q \in \mathcal{P}^{k-1}(\mathscr{A}) \setminus \mathcal{P}^{k-1}(\mathscr{A}_{d-1})$ so that $p(F_1) \rightsquigarrow^{k-1} p(F_2)$ but F_2 lies between q and F_1 on $|q|$. We prove that this situation can indeed not occur.



(1) (2)

Figure 4.3:

Given any $p \in \mathcal{P}^{k-1}(\mathscr{A})$, let $p_0 := \min\{x \in \mathcal{P}^{k-2}(\mathscr{A}) \mid p \subset |x|\}$ as in Definition 4.2.1. Then we have two cases.

Case 1 (see Figure 4.3.(1)) $p(F_1)_0 = p(F_2)_0$. This means $p(F_1), p(F_2) \in \ell$, where $\ell := |p(F_1)_0|$. The line ℓ is the intersection $\pi \cap V_{k-1}$ of V_{k-1} with a plane π in V_k that contains also the lines $\ell_1 := |p(F_1)|$ and $\ell_2 := |p(F_2)|$. Then this plane must contain also the line $|q|$. Since $\mathscr{A}_{d-1}$ is central, ℓ_1 and ℓ_2 must intersect, and this gives a point $P \in \mathcal{P}^k(\mathscr{A}_{d-1})$ that, by Remark 4.1.13, lies between $p(F_i)$ and F_i for $i = 1, 2$. Again, by Remark 4.1.13 we know that on ℓ we have the sequence of points $q, p(F_2), p(F_1)$, so on $|q|$ we have the sequence q, F_1, F_2, and there is no obstruction.

Case 2 (see Figure 4.3.(2)). $p(F_1)_0 \rightsquigarrow p(F_2)_0$. Since $q \in \mathcal{P}(\mathscr{A}) \setminus \mathcal{P}(\mathscr{A}_{d-1})$, as above we have that the line $\ell_q := |q_0|$ intersects $|p(F_i)_0|$ in a point p_i between $p(F_i)$ and $p(F_i)_0$, for $i = 1, 2$. Consider now the plane π spanned by $|q|$ and ℓ_q (this might not be a flat of $\mathscr{A}$), and on it, for $i = 1, 2$ the line ℓ_i' spanned by p_i and F_i. The intersection $\ell_1' \cap \ell_2'$ lies on the segments $\overline{p_1F_1}$ and $\overline{p_2F_2}$ only if $|p(F_1)_0| \cap |p(F_2)_0|$ is between $p(F_i)_0$ and p_i Since the Theorem holds in V_{k-1} it is now a straightforward check to verify that $p(F_1) \rightsquigarrow p(F_2)$ implies that F_1 lies between F_2 and q on $|q|$ (Figure 4.3.(2) describes one of the two possible cases - namely, when $\overline{p_1F_1} \cap \overline{p_2F_2}$ is not empty). $\square$

This concludes the proof of Theorem 4.3.14. $\square$ $\square$

Part II

$\mathbb{C}$

5 Preliminaries: Unitary space

Ideally, this chapter would be a counterpart to Chapter 2 and introduce a successful combinatorial theory that characterizes in a natural way the algebraic, geometric and topological property of complex vector spaces. It is not, as development of a complex counterpart to oriented matroids has been limited. The question still lingers (see [98]):

What is a complex matroid?

In the next chapter we will outline the progress made so far on this question, and suggest a possible new answer. Here we start by a *tour d'horizon* of the landscape in which a theory of complex matroids should fit. This will include reviewing some peculiarities of unitary space and some of the problems motivating our quest.

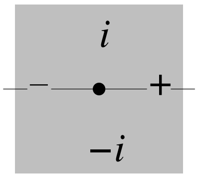
Where the bottom lines of the sections of Chapter 2 have been theorems and results, the considerations of this chapter will mainly end with questions and conjectures. The hope is that this will make it no less interesting.

5.1 Signs, convexity and orthogonality

Signs

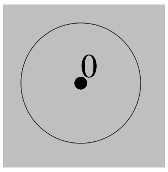
A notable feature of the field of complex numbers is the lack of a natural total ordering. A hyperplane (the kernel of a Hermitian form) does not disconnect complex space; in fact, if H is a hyperplane in $\mathbb{C}^d$, then $\mathbb{C}^d/H$ is isomorphic to the complex plane. The problem is to find a "natural" stratification of $\mathbb{C}$. This is a much more debatable question than its real analogue. On the real line the 'usual' stratification by sign gives a finite set of strata and thus lends itself very nicely to combinatorial study. In complex space, there are different 'natural' choices of what a 'complex sign' should be.

A first choice is to consider a set of *four* signs, $\{0, +, -, i, -i\}$. This is the approach taken by Björner and Ziegler [23]. We will see below that this choice has been topologically fruitful, especially in one of its variations, which consists in the stratification by nine cells that arises by additionally distinguishing the 'purely imaginary' numbers (that this is no essential gain of combinatorial information was shown by Ziegler in [98]). The success of this idea is largely based on the fact that, by applying it to an arrangement of (complex) hyperplanes, one obtains the structure of an arrangement of real hyperplanes in



$\mathbb{R}^{2d} \simeq \mathbb{C}^d$ that 'frames' the original arrangement, and to which the techniques of oriented matroid theory can be applied.

The other choice is to take $S^1 \cup \{0\}$ as a set of signs, assigning to every complex number its *phase*. We will make this formally precise in Section 6.1, but for the moment a picture will do.



This set of signs has been considered, with different motivations, in [11, 40, 46]. The next chapter will expand on the combinatorics of this choice of structure.

Convexity

Our starting point in Chapter 2 has been the notion of convexity, via Carathéodory's Theorem 2.1.1. So a natural question is whether there is some kind of generalization of that theorem to complex numbers.

It is not possible to give a negative answer to such a vague question. What we can show by an example is that the *consequence* of Carathéodory's theorem on which the validity of signed circuit and vector elimination rests does not hold in complex space - not even by taking into account the very refined information encoded by the infinite set of signs $S^1 \cup \{0\}$.

Example 5.1.1 (from [4])**.** Let $v_1, \ldots, v_7$ denote the columns of the following matrix:

$$M := \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & i & 1-i \\ 2 & -1 & 0 & -1 & 0 & -i & 3+i \\ -i & 0 & -i & 0 & 2i & -i & -2i \\ -1 & 0 & 0 & -i & i+1 & 0 & -2 \end{pmatrix}$$

The vectors $(1,1,1,1,1,0,0)$ and $(-1,0,0,1,1,1,1)$ are both elements of $\ker(M)$ of minimal support, giving rise to two phased circuits $X := (1,1,1,1,1,0,0)$ and $Y := (-1,0,0,1,1,1,1)$. Now, a "general" elimination axiom should describe the phases of the circuit obtained by eliminating v_1 from X and Y in terms of the phases of X and Y. This circuit should have support contained in $\{v_2, \ldots, v_7\}$, and below we list all circuits with such support (up to multiplication by a scalar).

$$
\begin{aligned}
-(1+i)v_2 + iv_3 + v_4 + \left(\frac{1}{2}+\frac{i}{2}\right)v_5 + v_6 &= 0,\\
(2+i)v_2 + (1-i)v_3 + v_4 + \left(\frac{3}{2}-\frac{1}{2}\right)v_5 + v_7 &= 0,\\
-\frac{5i}{2}v_2 + \left(-1+\frac{1}{2}\right)v_3 + v_4 + \left(1+\frac{i}{2}\right)v_6 - \frac{i}{2}v_7 &= 0,\\
\left(\frac{1}{2}+\frac{5}{2}i\right)v_2 + \left(\frac{3}{2}-\frac{i}{2}\right)v_3 + v_5 - \left(\frac{1}{2}+\frac{i}{2}\right)v_6 + \left(\frac{1}{2}+\frac{i}{2}\right)v_7 &= 0,\\
\left(\frac{3}{5}-\frac{4}{5}i\right)v_2 + v_4 + \left(\frac{7}{10}-\frac{i}{10}\right)v_5 + \left(\frac{3}{5}+\frac{i}{5}\right)v_6 + \left(\frac{2}{5}-\frac{i}{5}\right)v_7 &= 0,\\
\left(\frac{7}{13}+\frac{4}{13}i\right)v_3 + v_4 + \left(\frac{25}{26}+\frac{5}{26}i\right)v_5 + \left(\frac{8}{13}-\frac{1}{13}i\right)v_6 + \left(\frac{5}{13}+\frac{i}{13}\right)v_7 &= 0.
\end{aligned}
$$

But we could construct a matrix with, for instance, all real entries and with $(1,1,1,1,1,0,0)$ and $(-1,0,0,1,1,1,1)$ in its kernel, and thus with X and Y in the resulting phased circuit set. A general elimination axiom should give the same elimination of v_1 from X and Y in both of these complex matroids, but of course it will not.

Remark 5.1.2. There is an extensive literature in applied analysis which deals with convexity questions in complex space, but most of it seems not to be related with our needs. However, the line of research which strives to treat in unified way linear programming over real or complex vector spaces [12] seems to have the potential for an interesting combinatorial description. For instance, there are 'complex' versions of Farkas' Lemma [13], a result which, in the real case, links primal and dual Linear Programs and has a combinatorial reformulation leading to a cryptomorphic definition of oriented matroids.

Question 5.1.3. Is there a "combinatorial complex Farkas lemma" satisfied by complex matroids? If yes, this may lead to a good candidate for the (still lacking) notion of vectors (see Section 6.5).

Orthogonality

Two vectors $v, w \in \mathbb{C}^n$ are orthogonal if their (Hermitian) scalar product vanishes:

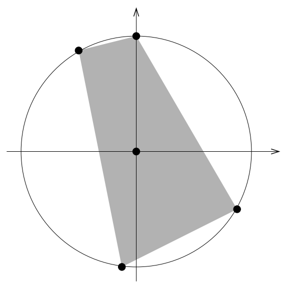
$$\langle v, w\rangle = \sum_i^n v_i \overline{w_i} = 0 \qquad (5.1.1)$$

The combinatorial translation of this condition in terms of the signs $\{0,+,-,i,-i\}$ is given in [98] and is inspired by orthogonality in oriented matroids.

To study orthogonality as expresed in Equation (5.1.1) in terms of the sign set $S^1 \cup \{0\}$, for all $j = 1, \ldots n$ write $v_j\overline{w_j} =: \lambda_j e^{i\alpha_j}$. By scaling the equation with a positive real number we can obtain that $\lambda_1 + \ldots + \lambda_n = 1$.

Since the λ_i are nonnegative real numbers, this is equivalent to say that, in the complex plane, the point 0 is in the interior of the convex hull of the points

$e^{i\alpha_j}$ for which $\lambda_j \neq 0$, as shown in the next figure. This idea is the motivation for the definition of orthogonality between 'phased vectors' given in the next chapter (Definition 6.1.10).



5.2 Arrangements over $\mathbb{C}$

As we already noted, the topology of the complement to arrangements in complex space is particularly interesting. In what follows we mention some aspects of this topic that may lead to future progress in understanding 'complex matroids'.

Unitary reflection groups

We have described in Section 2.3 how the study of the arrangements of mirrors of Coxeter groups has been a driving impetus for the study of the topology of complexified arrangements. As a natural extension, it has been natural from early on to consider finite groups of unitary transformations, first classified in 1954 by Shephard and Todd [88].

This subject is far less well understood than its real counterpart. The first textbook on the subject, to the best of our knowledge, appeared in 2010 [68], with the stated objective to give a self-contained proof of the classification.

Let now $\mathscr{A}$ be the arrangement of complex hyperplanes given by the fixed spaces of the reflections in a finite unitary reflection group G.

In analogy with the real case, one associates to G a *braid group* $B(G)$ and a *pure braid group* $P(G)$ as

$$B(G) := \pi_1(\mathscr{M}(\mathscr{A})/G), \qquad P(G) := \pi_1(\mathscr{M}(\mathscr{A}))$$

and the following sequence is exact

$$0 \to P(G) \to B(G) \to G \to 0.$$

The first topological question is then whether $\mathscr{M}(\mathscr{A})/G$ is a $K(\pi, 1)$ space. This had been conjectured since Deligne's result [38], but was proven only recently by David Bessis [14].

Bessis' work is an elegant generalization to the complex case of Deligne's argument, is case-free and includes the result in the real case by shedding a new light on it. Some of its aspects are worth a closer examining.

(1) *Combinatorics.* Even if the classification of finite unitary reflection groups does not build on a combinatorial argument (as happens in the real case), there are presentations of $B(G)$ that include only homogeneous relations between positive words of the same length and relations that assert the vanishing of powers of generators. Thus, also here one can define a *positive monoid* $B_+(G)$, which has a remarkable combinatorial structure.

    In fact, $B_+(G)$ is a *Garside monoid* (thus $B(G)$ is a Garside group [37]), and in particular it can be defined starting from an edge-labeled finite lattice $L(G)$.

    Deligne himself in [38] points out the role of Garside's ideas in his argument. However, the ('classical') Garside structure of Artin groups from which Deligne took inspiration for his theorem on complexified arrangements is different from the ('dual') Garside structure present in a general $B(G)$ (and exploited by Bessis in his proof). Where in the 'classical' structure $L(G)$ is the poset of regions of $\mathscr{A}_{\mathbb{R}}$ (or equivalently, in the case of a real reflection arrangement, the weak Bruhat order), in the 'dual' structure $L(G)$ is the so-called *generalized noncrossing partition lattice* (as yet defined only in terms of the groups, as the lower interval below a coxeter Element in the associated order by reflection length) [15, 82].

    These posets are defined in terms of the partial ordering of the group G by *reflection length* and can be combinatorially interpreted as posets of partitions of certain patterns of cyclically arranged objects with a 'noncrossing' condition of the blocks.

    The study of the combinatorics of the dual Garside structure has been very active in the last years. We refer to the introductory part of the Ph.D. Thesis of Vivien Ripoll [84] for a historical review and a comprehensive list of references. The general topic of Ripoll's thesis is a study of the combinatorics of these structures as related to the geometry of complex reflection arrangements.

    As yet, and to the best of our knowledge, no structural combinatorial relationship has been found between the classical (i.e., the weak order) and the dual (i.e., reflection order) Garside structures in the case where both are present (i.e., in the case of real reflection groups). The 'duality' terminology comes so far only from some enumerative evidence.

    Question 5.2.1. We ask whether a structural relationship exists. In particular, whether the one Garside structure can be defined combinatorially in terms of the other - maybe with some added information. One precise question in this respect could be: does the complex matroid of a complex reflection arrangement encode its 'dual Garside structure'?

(2) *Topology.* In the proof of asphericity for complex refleciton arrangements, the order complex of $L(G)$ ('dual' version) appears as the building block of a simplicial complex $X(G)$ which is then shown (1) to be a model of the universal covering space of $\mathscr{M}(\mathscr{A})$ and (2) contractible. The same can be said for the 'classical' $L(G)$ in Deligne's proof.

    We note that the construction of $X(G)$ can be carried out in general with the combinatorial analogue of the 'classical' $L(G)$, i.e., the poset of regions of real arrangements. It has been shown that $X(G)$ models the universal cover of $\mathscr{M}(\mathscr{A})$ for all complexified arrangements $\mathscr{A}$ [41]. If $\mathscr{A}_{\mathbb{R}}$ is simplicial, then $X(G)$ is contractible.

Topology of complex arrangements

The goal of this section is to complete the discussion of Section 1.3 by some remarks regarding non-matroidal invariants that might be captured by complex matroids.

Let $\mathscr{A}$ be an arrangement of complex hyperplanes. If $\mathscr{A}$ is complexified, then a presentation for the fundamental group $\pi_1(\mathscr{M}(\mathscr{A}))$ which uses only oriented matroid data is known by work of Salvetti [86] and, indipendently, Randell [78, 79]. For the general case, a presentation has been given by Arvola [6] and Dung and Vui [47].

Question 5.2.2. Can the fundamental group of a complex arrangement be presented in terms of its complex matroid?

One can look for refinement of matroid invariants and, in view of Randell's Lattice Isotopy Theorem 1.3.11, ask the following question.

Question 5.2.3 (Asked by Michael Falk)**.** How does a complex matroid change during a lattice isotopy? Is it possible to characterize 'lattice isotopic' components of the space of complex matroids?

Combinatorial models

The topological questions of the preceding section can be answered by giving a combinatorial model for $\mathscr{M}(\mathscr{A})$ from which to compute, in principle, everything. Even better if this combinatorial model is obtained as the 'topological representation' of some sort of complex matroid.

Discrete, combinatorially defined complexes that model the homotopy type of $\mathscr{M}(\mathscr{A})$ have been given by Björner and Ziegler [17] and Orlik [74]. They both use essentially the discrete stratification of $\mathbb{C}$ introduced at the beginning of the chapter.

In particular, Björner and Ziegler's models turn out to generalize nicely Salvetti's construction and allow a direct computation of the Orlik-Solomon algebra (up to a sign in the differential).

For our purposes, however, the discrete stratification has the disadvantage of being very sensitive to scaling of the linear forms defining the arrangement (an operation that has no effect on the arrangement's geometry). Indeed, the complexity of such models can explode by effect of the 'wrong' scalar choice on the defining forms. We ask for more naturality.

Question 5.2.4. Is there a Topological Realization Theorem for complex matroids that produces a 'correctly stratified' sphere with some sort of S^1-action on it (and does not depend on reorientation)?

Some work on this last question is being done jointly with Rade Zivaljević. In the next chapter we will see that any answer will have to use some substantially new idea with respect to its real counterpart.

6 Complex matroids

This chapter reproduces, with slight notational and editorial modifications, the paper [4], written jointly with Laura Anderson.

Why

Much of the motivation for pursuing the quest of a theory of complex matroids today is essentially the same as that listed by Ziegler in his paper [98], which carries our starting question as a title.

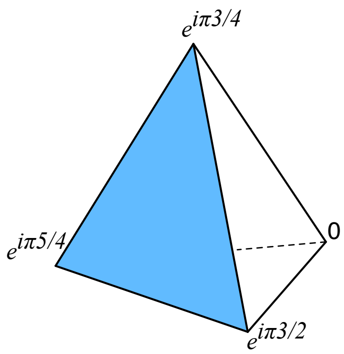
Topologically (1), one would like a combinatorial model to study the topology of complex hyperplane arrangements.

Topologically (2), MacPherson's idea of Combinatorial Differentiable (CD) manifolds and of matroid bundles should be pursued also for complex manifolds and vector bundles (see [3], [1]).

Combinatorially, the question about characterizing "complex structure" is fundamental.

In view of last section we may add that,

Algebraically, a successful theory (with a topological representation theorem) would probably enhance the understanding of the dual Garside structure (see Question 5.2.1).



$$\begin{pmatrix} 0 & 0 & i & i+1 \\ 0 & -1 & 0 & i+1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 6.1: A matrix with entries in $\mathbb{C}$ and 'its' complex matroid. Vertices are labeled by corresponding values of the phirotope.

In the introduction to [98], Ziegler also outlines three main possible approaches for developing a theory of complex matroids. He distinguishes between an approach 'parallel' to oriented matroid theory (i.e., abstract axiomatic development mimicking [20]), one 'on top' of it (i.e., *using oriented matroid theory*) and one 'in general', aiming for a theory that covers uniformly oriented matroids and complex matroids.

What has been done

Ziegler [98] adopted the 'on top of oriented matroids' point of view and defined a notion of complex matroid with extra structure given by the structure set $\{0, +, -, i, -i\}$. That is, where the set of covectors of a matroid realized by a matrix M over a field $\mathbb{K}$ says whether various elements of $\mathbb{K}$ are zero or nonzero, and the corresponding data set for oriented matroids realized over $\mathbb{R}$ describes whether these elements of $\mathbb{R}$ are zero, positive, or negative, the corresponding data set for Ziegler's complex matroids realized over $\mathbb{C}$ describes whether these elements of $\mathbb{C}$ are zero, positive real, negative real, have positive imaginary part, or have negative imaginary part. Ziegler's complex matroids have a Topological Representation Theorem [98, Theorem 3.5]. However, they are only known to have one axiomatization, in terms of covectors [98, Definition 1.3 and 4.1]. Ziegler's theory is completely discrete, which can be seen as either a strength or a weakness – his complex matroids lack any symmetry analogous to the action of $\mathbb{C}^*$ on complex linear objects.

Below, Krummeck, and Richter-Gebert [11], motivated by questions of developed another notion of complex matroid, with structure set $S^1 \cup \{0\}$, where S^1 is the set of unit elements in $\mathbb{C}$, and with axiomatization only in terms of bases with structure, or *phirotopes*. That is, where the set of bases of a matroid realized by a matrix M over a field $\mathbb{K}$ says whether various maximal minors of M are zero or nonzero, the corresponding data set for phirotopes realized by a matrix over $\mathbb{C}$ additionally describes the phase θ of each nonzero maximal minor $re^{i\theta}$. Below, Krummeck, and Richter-Gebert gave an axiomatization for phirotopes and proved various interesting properties in rank 2, in particular about realizability. Delucchi [39] developed a notion of orthogonality for this context, leading to dual phirotopes, and defined circuits and cocircuits associated to a phirotope (although he did not find circuit axioms).

Taking the point of view of the theory of *matroids with coefficients* developed by Dress and Wenzel, phirotopes correspond to basis orientations over the fuzzy ring $\mathbb{C}//\mathbb{R}^+$ [46], of which $S^1 \cup \{0\}$ is a subset. Within this framework, Dress and Wenzel show phirotopes to be cryptomorphic to what can be roughly taken to be an axiomatization for "signed flats" (with coefficients in the full fuzzy ring), and one can prove that dual pairs of matroids with coefficients have "orthogonal" signatures. However, Dress and Wenzel's work gives no cryptomorphic axiomatization of matroids with coefficients in terms of dual pairs, nor in terms of circuits.

What we do

We ask (and, to some extent, answer) how much of the foundations of oriented matroids can be paralleled with the structure set $S^1 \cup \{0\}$.

Our approach incorporates the idea of phirotope and unites the first and third categories outlined by Ziegler, in that we aim for a general theory that encompasses oriented matroid theory as a special case, and we attempt to develop it axiomatically staying as much as possible parallel to matroid and oriented matroid theory. This choice is motivated by the desire to 'isolate' the peculiarities of complex matroids (the "complex struxture") in the points where the theory must enhance, or depart from, the (oriented) matroid case. A good example of this is the necessity to restrict to modular elimination.

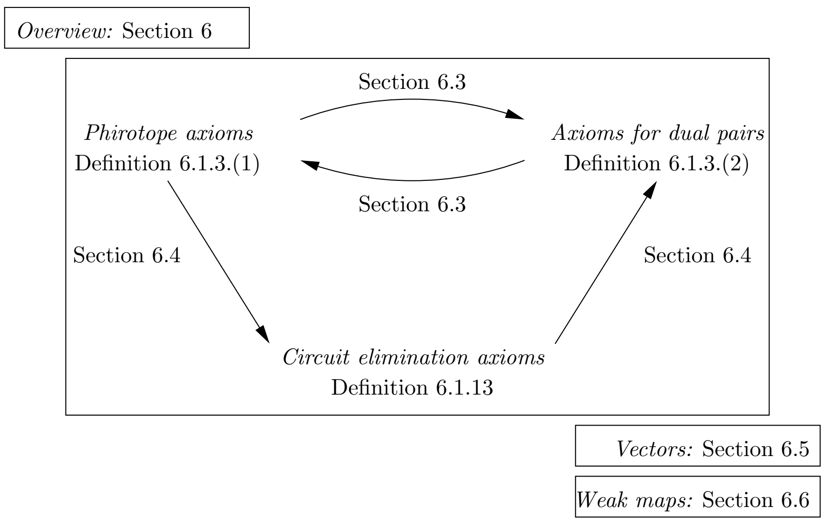
Ziegler observes that Dress and Wenzel's matroids with coefficients are too close to the algebraic world to encompass the required geometry. Even if we use phirotopes, we avoid 'over-algebraization' by not adopting the concept of 'composition of vectors'.

The following question is open, and its understanding could lead to an (at least partial) Topological Representation Theorem for complex matroids.

Question 6.0.5. Is there a 'forgetful' map from a complex matroid (in our sense) to a complex matroid (in Ziegler's sense)? Precisely: is it possible to associate a set of covectors (in Ziegler's sense) to a complex matroid as defined below, so to give rise to a correspondence between the 'correct' complex matroids in the realizable case?

In this chapter we give two different axiomatizations for circuits and cocircuits of a complex matroid and show them to be cryptomorphic to the phirotope axioms. We then turn to the issue of vectors and covectors and show that there is no "good" set of vector axioms. Finally, we briefly discuss weak maps of complex matroids.

The structure of the chapter is summarized in the following chart.



6.1 Overview

This section outlines our main results and should serve the reader as a road map through the remainder of the chapter. We start by defining complex phases and putting some notation in place. Then, we present our cryptomorphic axiomatizations for complex matroids. We close by sketching the discussion about covectors, complexification and weak maps that will take place in the last sections of the chapter.

Complex phases

Definition 6.1.1 (Phase vectors)**.** Given a finite ground set E, a *phase vector* (or *"phased set"*) is any

$$X \in (S^1 \cup \{0\})^E$$

where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle in the Gauss plane of the complex numbers. We will denote by $X(e)$ the e-th component of X. We define a partial order on phases by setting $0 < \mu$ for all $\mu \in S^1$ and declaring any two elements of S^1 as incomparable. This extends to a partial order on phase vectors defined componentwise. The minimal phase vector with respect to this ordering is the zero vector, which has value 0 on every component and will be denoted by $\widehat{0}$.

The *phase* $\mathrm{ph}(x)$ of $x \in \mathbb{C}$ is defined to be 0 if $x = 0$ and $\frac{x}{|x|}$ otherwise. For $v \in \mathbb{C}^E$, $\mathrm{ph}(v)$ is defined to be the vector with components $(\mathrm{ph}(v))_e = \mathrm{ph}(v_e)$.

Definition 6.1.2. Define the *phase convex hull* $\mathrm{pconv}(S)$ of a finite $S \subset S^1 \cup \{0\}$ to be the set of all phases of (real) positive linear combinations of S. Thus

- $\mathrm{pconv}(\emptyset) = \emptyset$,
- $\mathrm{pconv}(\{\mu\}) = \{\mu\}$ for all μ,
- $\mathrm{pconv}(\{\mu, -\mu\}) = \{0, \mu, -\mu\}$ for all μ,
- if $S = \{e^{i\alpha_1}, \ldots, e^{i\alpha_k}\}$ with $k \geq 2$ and $\alpha_1 < \cdots < \alpha_k < \alpha_1 + \pi$, then

$$\mathrm{pconv}(S) = \mathrm{pconv}(S \cup \{0\}) = \{e^{i\gamma} \mid \alpha_1 < \gamma < \alpha_k\}$$

- if $S = \{e^{i\alpha_1}, \ldots, e^{i\alpha_k}\}$ with $k \geq 3$ and $\alpha_1 < \cdots < \alpha_k = \alpha_1 + \pi$, then

$$\mathrm{pconv}(S) = \mathrm{pconv}(S \cup \{0\}) = \{e^{i\gamma} \mid \alpha_1 < \gamma < \alpha_k\},$$

- otherwise (i.e., if the nonzero elements of S do not lie in a closed half-circle of S^1) $\mathrm{pconv}(S) = S^1 \cup \{0\}$.

Axioms for complex matroids

Definition 6.1.3 (Complex matroids)**.**

1. (Phirotope axioms, compare [11]) A function $\varphi : E^d \to S^1 \cup \{0\}$ is called a *rank d phirotope* if

(φ 1) φ is nonzero

(φ 2) φ is alternating

(φ 3) For any two subsets $x_1, \ldots, x_{d+1}$ and $y_1, \ldots, y_{d-1}$ of E,

$$0 \in \mathrm{pconv}(\{(-1)^k \varphi(x_1, x_2, \ldots, \hat{x_k}, \ldots, x_{d+1}) \varphi(x_k, y_1, \ldots, y_{d-1})\}).$$

2. (Elimination axioms) A set $\mathcal{C} \subseteq (S^1 \cup \{0\})^E$ is the *set of phased circuits of a complex matroid* if and only if it satisfies

($\mathcal{C}0$) for all $X \in \mathcal{C}$ and all $\alpha \in S^1$, $\alpha X \in \mathcal{C}$ *(Symmetry)*

($\mathcal{C}1$) for all $X, Y \in \mathcal{C}$ with $\mathrm{supp}(X) = \mathrm{supp}(Y)$, $X = \alpha Y$ for some $\alpha \in S^1$ *(Incomparability)*

(ME) for all $X, Y \in \mathcal{C}$ such that $X \neq \mu Y$ for all $\mu \in S^1$ and such that $\mathrm{supp}(X)$, $\mathrm{supp}(Y)$ is a modular pair in $\{\mathrm{supp}(X) \mid X \in \mathcal{C}\}$, and given $e, f \in E$ with $X(e) = -Y(e) \neq 0$ and $X(f) \neq -Y(f)$, there is $Z \in \mathcal{C}$ with
– $f \in \mathrm{supp}(Z) \subseteq (\mathrm{supp}(X) \cup \mathrm{supp}(Y)) \setminus e$, and
– $\begin{cases} Z(f) \in \mathrm{pconv}(\{X(f), Y(f)\}) & \text{if } f \in \mathrm{supp}(X) \cap \mathrm{supp}(Y) \\ Z(f) \leq \max\{X(f), Y(f)\} & \text{else} \end{cases}$
(Modular Elimination).

Remark 6.1.4.

- The phirotope axioms imply that the support of φ is the set of bases of a matroid M_φ.
- The term "modular pair" in (ME) is defined according to Definition 1.2.6. Hence, by Lemma 1.2.9, property ($\mathcal{C}0$), ($\mathcal{C}1$) and (ME) together show that the set $\{\mathrm{supp}(X) \mid X \in \mathcal{C}\}$ is the set of circuits of a matroid $M_{\mathcal{C}}$.
- Based on Example 5.1.1, our feeling is that any "general" elimination axiom that is weak enough to hold for all complex matroids will not be strong enough to define the corresponding cocircuit signature (i.e., to prove Proposition 6.4.7).
- It is easily seen that, if M is a rank d matrix over $\mathbb{C}$ with columns indexed by E, the function $E^d \to S^1 \cup \{0\}$ taking each d-tuple to the phase of the determinant of the corresponding submatrix of M is a phirotope. In this case Property ($\varphi 3$) follows from the Grassmann-Plücker relations. We call M a *realization* of φ.

Definition 6.1.5. If M is a matroid and $\mathcal{C}$ is the set of phased circuits of a complex matroid such that $M_{\mathcal{C}} = M$, we say $\mathcal{C}$ is a *complex circuit orientation* of M.

Definition 6.1.6. For a rank d phirotope φ on the ground set E, we say that a subset $\{e_1, \dots, e_k\} \subseteq E$ is *φ-independent* if it is an independent set of the matroid M_φ. By a *φ-completion* of any φ-independent set $\{a_1, \dots, a_k\}$ we mean a set $\{a'_1, \dots, a'_{d-k}\}$ such that $\{a_1, \dots, a_k, a'_1 \dots, a'_{d-k}\}$ is a basis of M_φ.

Definition 6.1.7. We say two phirotopes $\mathcal{C}_1$, $\mathcal{C}_2$ are *equivalent* if $\mathcal{C}_1 = \alpha\mathcal{C}_2$ for some $\alpha \in S^1$.

Theorem 6.1.8. *There is a bijection between the set of all equivalence classes of phirotopes on a set E and the set of all sets of phased circuits of complex matroids on E, determined as follows. For a phirotope φ and the corresponding set $\mathcal{C}$ of phased circuits,*

- *The set of all supports of elements of $\mathcal{C}$ is the set of minimal nonempty φ-dependent sets, and*
- *The phases of $X \in \mathcal{C}$ are determined by the rule*

$$\frac{X(x_i)}{X(x_0)} = (-1)^{i-1} \frac{\varphi(x_0, \ldots, \widehat{x_i}, \ldots, x_d)}{\varphi(x_1, \ldots, x_d)}$$

for all $i = 0, \ldots, k$, where we have written $\{x_0, \ldots, x_k\} = \operatorname{supp}(X)$ and $\{x_{k+1}, \ldots, x_d\}$ is any φ-completion of $\operatorname{supp}(X) \setminus \{x_0\}$.

Thus we can refer to "the complex matroid with phirotope φ and phased circuit set $\mathcal{C}$". We will also refer to a matrix over $\mathbb{C}$ "realizing a complex matroid" (not just realizing a phirotope).

Corollary 6.1.9. *With the notation introduced in Remark 6.1.4, if $\mathcal{M}$ is a complex matroid with phirotope φ and phased circuit set $\mathcal{C}$, then $M_\varphi = M_{\mathcal{C}}$.*

We call this matroid the *underlying matroid* of $\mathcal{M}$. The rank of $\mathcal{M}$ is the rank of its underlying matroid.

To make the discussion of Section 5.1 precise in view of the definition of orthogonality, consider two vectors $v, w \in \mathbb{C}^E$. By definition, they are orthogonal if their (Hermitian) scalar product equals zero: $\langle v, w\rangle = \sum v_e \overline{w_e} = 0$. Now, $\operatorname{ph}(v_e \overline{w_e}) = \operatorname{ph}(v_e) \operatorname{ph}(w_e)^{-1}$ and if complex numbers with such phases must add up to zero, then the point 0 in the complex plane must be contained in

$$\operatorname{pconv}(\{\operatorname{ph}(v_e) \operatorname{ph}(w_e)^{-1} \mid e \in E\}).$$

This suggests the following definition.

Definition 6.1.10 (Orthogonality)**.** Let $S, T \in (S^1 \cup \{0\})^E$ be two phased sets for some finite set E. Let

$$P_{S,T} = \left\{ \frac{S(e)}{T(e)} \,\middle|\, e \in \operatorname{supp}(S) \cap \operatorname{supp}(T) \right\}.$$

We say S and T are *orthogonal*, written $S \perp T$, if

$$0 \in \operatorname{pconv}(P_{S,T}).$$

Two sets $\mathcal{S}, \mathcal{T} \subseteq (S^1 \cup \{0\})^E$ are called orthogonal, written $\mathcal{S} \perp \mathcal{T}$, if $S \perp T$ for all $S \in \mathcal{S}$ and all $T \in \mathcal{T}$. The set of all phased sets orthogonal to S is denoted $S^\perp$.

The notion of orthogonality introduced above behaves naturally with respect to duality.

Theorem 6.1.11. *If $\mathcal{M}$ is a complex matroid with ordered ground set E, phirotope $\varphi : E^d \to S^1 \cup \{0\}$, and circuit set $\mathcal{C}$, then there is a complex matroid $\mathcal{M}^*$ with ground set E and*

(1) phirotope $\varphi^* : E^{|E|-d} \to S^1 \cup \{0\}$ given by

$$\varphi^*(x_1, \ldots, x_{n-d}) = \varphi(y_1, \ldots, y_d) \operatorname{sign}(x_1, \ldots, x_{n-d}, y_1, \ldots, y_d),$$

where $\{y_1, \ldots, y_d\} = E \setminus \{x_1, \ldots, x_{n-d}\}$ and sign *denotes the sign of the indicated permutation of E,*

(2) circuit set $\mathcal{C}^* = \min(\mathcal{C}^\perp \setminus \{\underline{0}\})$*,*
where min *denotes support inclusion minimality.*

The underlying matroid of $\mathcal{M}^*$ *is the dual of the underlying matroid of* $\mathcal{M}$*. If* $\mathcal{M}$ *is realized by a vector space* $W \subset \mathbb{C}^E$ *then* $\mathcal{M}^*$ *is realized by* $W^\perp$*.*

Remark 6.1.12. The reader will perhaps notice a "missing item" in the statement of Theorem 6.1.11 as compared to its counterpart for oriented matroids, Theorem 2.2.7. We will show in Section 6.5 that there can be no axiomatic description of the phases of the row space of a matrix with complex coefficients that is cryptomorphic to the other axiomatizations.

Definition 6.1.13 (Axioms for dual pairs)**.** Let M be a matroid with ground set E. Two subsets $\mathcal{C}, \mathcal{D}$ of $(S^1 \cup \{0\})^E$ are the *dual pair of complex circuit signatures* of M if

(S1) for all $X \in \mathcal{C}$ and all $\alpha \in S^1$, $\alpha X \in \mathcal{C}$,

(S1*) for all $X \in \mathcal{D}$ and all $\alpha \in S^1$, $\alpha X \in \mathcal{D}$,

(S2) for all $X, Y \in \mathcal{C}$ with $\mathrm{supp}(X) = \mathrm{supp}(Y)$, $X = \alpha Y$ for some $\alpha \in S^1$,

(S2*) for all $X, Y \in \mathcal{D}$ with $\mathrm{supp}(X) = \mathrm{supp}(Y)$, $X = \alpha Y$ for some $\alpha \in S^1$,

(S3) the set $\{\mathrm{supp}(X) \mid X \in \mathcal{C}\}$ is the set of circuits of M and the set $\{\mathrm{supp}(X) \mid X \in \mathcal{D}\}$ is the set of cocircuits of M,

(S4) $\mathcal{C} \perp \mathcal{D}$.

Definition 6.1.14. If $\mathcal{C} \subset (S^1 \cup \{0\})^E$ satisfies S1 and S2 and $\{\mathrm{supp}(X) \mid X \in \mathcal{C}\}$ is the set of circuits of a matroid M, we say $\mathcal{C}$ is a *complex circuit signature* of M. Similarly, if $\mathcal{D} \subset (S^1 \cup \{0\})^E$ satisfies (S1*) and (S2*) and $\{\mathrm{supp}(X) \mid X \in \mathcal{D}\}$ is the set of cocircuits of M, we say $\mathcal{D}$ is a *complex cocircuit signature* of M. In particular, the set of phased circuits of a complex matroid is a complex circuit signature of the underlying matroid.

Theorem 6.1.15. *Let* $\mathcal{C}$ *be a complex circuit signature and* $\mathcal{D}$ *be a complex cocircuit signature of a matroid* M*. Then* $\mathcal{C}$ *and* $\mathcal{D}$ *are the set of phased circuits and cocircuits of a complex matroid with underlying matroid* M *if and only if*

$$\mathcal{C} \perp \mathcal{D}.$$

Minors

Definition 6.1.16. For $X \in (S^1 \cup \{0\})^E$ and $A \subseteq E$ let $X_{\setminus A} \in (S^1 \cup \{0\})^{E \setminus A}$ be the restriction of X to $E \setminus A$. For $\mathcal{U} \subseteq (S^1 \cup \{0\})^E$ define

(1) the *deletion* of A from $\mathcal{U}$ as

$$\mathcal{U} \setminus A = \{X_{\setminus A} \mid X \in \mathcal{U}, \mathrm{supp}(X) \cap A = \emptyset\}.$$

(2) the *contraction* of A in $\mathcal{U}$ as

$$\mathcal{U}/A := \min\{X_{\setminus A} \mid X \in \mathcal{U}\},$$

where min denotes support minimality.

Theorem 6.1.17. *Let $\mathcal{C}$ be the set of phased circuits of a complex matroid $\mathcal{M}$ on the ground set E with underlying matroid M. If $A \subseteq E$, then $\mathcal{C} \setminus A$ is the set of phased circuits of a complex matroid with underlying matroid $M \setminus A$, and $\mathcal{C}/A$ is the set of phased circuits of a complex matroid with underlying matroid M/A. Further:*

- *With the notation of Definition 6.1.16 and Theorem 6.1.11,*

$$\mathcal{C}^*/A = (\mathcal{C} \setminus A)^*.$$

- *After replacing real signs with complex phases, Definition 2.2.11 gives the phirotopes associated to $\mathcal{C} \setminus A$ and $\mathcal{C}/A$ in terms of the phirotope associated to $\mathcal{C}$.*

The complex matroids associated to $\mathcal{C} \setminus A$ and $\mathcal{C}/A$ are denoted $M \setminus A$ and M/A and called respectively the *deletion* of A from $\mathcal{M}$ and the *contraction* of A in $\mathcal{M}$.

6.2 Phirotopes, duality and minors

This section deals with phirotopes as defined in Definition 6.1.3. Its goal is to establish some basic facts about duality and minors in terms of phirotopes.

Duality

Recall from Section 6 that given a phirotope φ on the ground set E, the set $\mathbf{B}_\varphi := \{\{b_1, \dots, b_d\} \mid \varphi(b_1, \dots, b_d) \neq 0\}$ is the set of bases of the underlying matroid M_φ.

Definition 6.2.1. Given a rank d phirotope φ, choose a total ordering of E, and for all $(x_1, x_2, \dots, x_{n-d}) \in E^{n-d}$ let $(x'_1, \dots x'_d)$ be a permutation of $E \setminus \{x_1, \dots, x_{n-d}\}$. Define the *dual* of φ as

$$\varphi^*(x_1, \dots, x_{n-d}) := \varphi(x'_1, \dots, x'_d)^{-1} \operatorname{sign}(x_1, \dots, x_{n-d}, x'_1, \dots, x'_d).$$

Notice that, up to a global change of sign, φ^* is independent of the choice of orderings on E and $\{x'_1, \dots x'_d\}$.

Lemma 6.2.2. *φ^* is a rank $(n-d)$ phirotope, and the underlying matroid M_{φ^*} is the dual $(M_\varphi)^*$ to M_φ.*

Proof. By definition, $\mathbf{B}_{\varphi^*} = \{E \setminus B \mid B \in \mathbf{B}_\varphi\}$ which, by Theorem 1.1.7, is the set of bases of $(M_\varphi)^*$. Thus, to prove the lemma it suffices to prove that φ^* is indeed a phirotope.

Axioms $(\varphi\,1)$ and $(\varphi\,2)$ are clear from the definition. For $(\varphi\,3)$, consider two sets $X := \{x_0, \dots, x_{n-d}\}$ and $Y := \{y_1, \dots, y_{n-d-1}\}$, numbered such that $X \cap Y = \{x_{n-d-l}, \dots x_{n-d}\} = \{y_1, \dots, y_l\}$. Without loss of generality we can assume that the total ordering of E is given by

$$x_0, \dots, x_{n-d}, y_{l+1}, \dots, y_{n-d-1}, A,$$

where A is any total ordering of $E \setminus (X \cap Y)$.

Then we have

$$\varphi^*(x_0 \ldots, \hat{x_k}, \ldots, x_{n-d})\varphi^*(x_k, y_1, \ldots, y_{n-d-1}) =$$

$$\varphi(x_k, y_{l+1}, \ldots y_{n-d-1}, A)^{-1} \underbrace{\operatorname{sign}(x_0 \ldots, \hat{x_k}, \ldots, x_{n-d}, x_k, y_{l+1}, \ldots y_{n-d-1}, A)}_{\sigma_1}$$

$$\varphi(x_0, \ldots, \hat{x_k}, \ldots x_{n-d-l}, A)^{-1} \underbrace{\operatorname{sign}(x_k, y_1, \ldots, y_{n-d-1}, x_0, \ldots, \hat{x_k}, \ldots x_{n-d-l}, A)}_{\sigma_2}$$

where the sign

$$\sigma_1\sigma_2 =$$

$$(-1)^{n-d-k} \operatorname{sign}(x_0, \ldots, x_{n-d}, y_{l+1}, \ldots, y_{n-d-1}, A)$$

$$(-1)^{n-d+k} \operatorname{sign}(y_1, \ldots, y_{n-d-1}, x_0, \ldots, x_{n-d-l}, A)$$

$$= \operatorname{sign}(x_0, \ldots, x_{n-d}, y_{l+1}, \ldots, y_{n-d-1}, A) \operatorname{sign}(y_1, \ldots, y_{n-d-1}, x_0, \ldots, x_{n-d-l}, A)$$

does not depend on k. Then,

$$\{(-1)^k\varphi^*(x_0 \ldots, \hat{x_k}, \ldots, x_{n-d})\varphi^*(x_k, y_1, \ldots, y_{n-d-1}) \mid x_k \in X \setminus Y\} =$$

$$\sigma_1\sigma_2\{(-1)^k\varphi(x_k, y_{l+1}, \ldots y_{n-d-1}, A)^{-1}\varphi(x_0, \ldots, \hat{x_k}, \ldots x_{n-d-l}, A)^{-1} \mid x_k \in X\backslash Y\}.$$

We now have to prove that 0 is in the relative interior of the convex hull of the latter set. Equivalently, we want to show that there are positive real numbers λ_k such that

$$\sum_k \lambda_k(-1)^k\varphi(x_k, y_{l+1}, \ldots y_{n-d-1}, A)^{-1}\varphi(x_0, \ldots, \hat{x_k}, \ldots x_{n-d-l}, A)^{-1} = 0. \tag{6.2.1}$$

Because φ is a phirotope, we know that there are positive real numbers λ_k with

$$\sum_k \lambda_k(-1)^k\varphi(x_k, y_{l+1}, \ldots y_{n-d-1}, A)\varphi(x_0, \ldots, \hat{x_k}, \ldots x_{n-d-l}, A) = 0. \tag{6.2.2}$$

Since Equation (6.2.1) is the complex conjugate of Equation (6.2.2), the claim follows. □

Deletion and contraction

The following two lemmas prove the last part of Theorem 6.1.17.

Lemma 6.2.3. *Let $A \subset E$ be given, and choose a maximal φ-independent subset $\{a_1, a_2, \ldots, a_l\}$ of A. Then*

$$(\varphi/A)(x_1, \ldots, x_{d-l}) := \varphi(x_1, \ldots x_{d-l}, a_1 \ldots, a_l)$$

is a phirotope, and $M_{\varphi/A} = M_\varphi/A$. Up to global multiplication by a constant $c \in S^1$, φ/A is independent of the choice of $\{a_1, a_2, \ldots, a_l\}$.

Proof. The phirotope axioms for φ/A are easy to check. That $M_{\varphi/A} = M_\varphi/A$ follows by Definition 1.5.1.(1) because

$$\begin{aligned}\mathbf{B}_{\varphi/A} &= \{\{x_1, \ldots, x_{d-l}\} \mid \varphi(x_1, \ldots, x_{d-l}, a_1, \ldots, a_l) \neq 0\} \\ &= \{B \subseteq E \mid B \cup \{a_1, \ldots, a_l\} \in \mathbf{B}_\varphi\}.\end{aligned}$$

$\square$

Lemma 6.2.4. *Let $A \subset E$ be given, and let r be the rank of $E \setminus A$ in M_φ. If $r < d$, choose $\{a_1, \ldots, a_{d-r}\} \subseteq A$ such that $(E \setminus A) \cup \{a_1, \ldots, a_{d-r}\}$ spans M_φ. Define a function $\varphi \setminus A : E \setminus A \to S^1 \cup \{0\}$ as follows:*

$$(\varphi \setminus A)(x_1, \ldots, x_r) := \begin{cases} \varphi(x_1, \ldots x_r) & \text{If } r = d \\ \varphi(x_1, \ldots, x_r, a_1, \ldots, a_{d-r}), & \text{if } r < d. \end{cases}$$

Then, up to global multiplication by a nonzero constant, $\varphi \setminus A$ is independent of the choice of $a_1, \ldots, a_{d-r}$ and $(\varphi \setminus A)^* = \varphi^*/A$ - in particular, it is a phirotope – and $M_{\varphi \setminus A} = M_\varphi \setminus A$.

Proof. We prove the case where $A = \{a\}$, and we fix a linear ordering of E where a is the biggest element.

If $r < d$, then a is in every basis of M_φ. Thus

$$\varphi^*(x_1, \ldots, x_t) \neq 0 \text{ only if } a \notin \{x_1, \ldots, x_t\},$$

hence

$$\begin{aligned}(\varphi^*/a)(x_1, \ldots, x_t) &= \varphi^*(x_1, \ldots, x_t) \\ &= \varphi(x_{t+1}, \ldots, x_{n-1}, a)^{-1} \operatorname{sign}(x_1, \ldots, x_{n-1}, a) \\ &= (\varphi \setminus a)(x_{t+1}, \ldots, x_{n-1})^{-1} \operatorname{sign}(x_1, \ldots, x_{n-1}) \\ &= (\varphi \setminus a)^*(x_1, \ldots, x_t).\end{aligned}$$

If on the other hand $r = d$, then

$$\begin{aligned}(\varphi^*/a)(x_1, \ldots, x_t) &= \varphi^*(x_1, \ldots, x_t, a) \\ &= \varphi(x_{t+1}, \ldots, x_{n-1})^{-1} \operatorname{sign}(x_1, \ldots, x_t, a, x_{t+1}, \ldots, x_{n-1}) \\ &= \varphi(x_{t+1}, \ldots, x_{n-1})^{-1} (-1)^{n-t-2} \operatorname{sign}(x_1, \ldots, x_{n-1}, a) \\ &= (\varphi \setminus a)(x_{t+1}, \ldots, x_{n-1})^{-1} (-1)^{n-t-2} \operatorname{sign}(x_1, \ldots, x_{n-1}, a) \\ &= (-1)^{n-t-2} (\varphi \setminus a)^*(x_1, \ldots, x_t).\end{aligned}$$

$\square$

6.3 Cryptomorphism from phirotopes to dual pairs

Dual pairs from phirotopes

The point of this section is to prove Proposition 6.3.3, asserting that every phirotope φ induces a dual pair of complex circuit and cocircuit signatures on M_φ.

Lemma 6.3.1. *Let φ be a phirotope and M_φ its underlying matroid. Let $C = \{e, f, x_2, \dots, x_k\}$ be a circuit of M_φ and $\{x_{k+1}, \dots, x_d\}$ a φ-completion of $C \setminus e$. Then the number*

$$\frac{\varphi(e, x_2, \dots, x_d)}{\varphi(f, x_2, \dots, x_d)}$$

does not depend on the choice of $x_{k+1}, \dots, x_d$.

Proof. Let $\{x_{k+1}, \dots, x_{d-1}, x'_d\}$ be a φ-completion of $C \setminus e$. Then axiom (φ 3) for φ applied to $\{e, f, x_2, \dots, x_d\}$ and $\{x_2, \dots, x_{d-1}, x'_d\}$ reduces to

$$\varphi(f, x_2, \dots, x_d)\varphi(e, x_2, \dots, x_{d-1}, x'_d) - \varphi(e, x_2, \dots, x_d)\varphi(f, x_2, \dots, x_{d-1}, x'_d) = 0$$

and proves the claim for pairs of choices of φ-completions of $C \setminus e$ that differ by one element. The full claim follows by induction on the number of elements by which any two choices of completion differ. ☐

Definition 6.3.2. Given a phirotope φ, let $\mathcal{C}_\varphi$ be the family of all phased sets X such that $C := \mathrm{supp}(X)$ is a circuit of M_φ and for all $e, f \in \mathrm{supp}(X)$ we have

$$\frac{X(f)}{X(e)} = -\frac{\varphi(e, x_2, \dots, x_d)}{\varphi(f, x_2, \dots, x_d)}.$$

Notice that for any $c \in S^1$ we have $\mathcal{C}_{c\varphi} = \mathcal{C}_\varphi$. Thus, it makes sense to talk about $\mathcal{C}_{\varphi^*}$, $\mathcal{C}_{\varphi \setminus e}$, and $\mathcal{C}_{\varphi / e}$. Let $\mathcal{D}_\varphi := \mathcal{C}_{\varphi^*}$.

Proposition 6.3.3. *For every phirotope φ the sets $\mathcal{C}_\varphi$ and $\mathcal{D}_\varphi$ satisfy Definition 6.1.13 and are thus a dual pair of complex circuit signatures of the matroid M_φ. Moreover, given an element e of the ground set we have*

(1) $\mathcal{C}_{\varphi \setminus e} = \mathcal{C}_\varphi \setminus e$

(2) $\mathcal{C}_{\varphi / e} = \mathcal{C}_\varphi / e$

Proof. All of the properties in the definition of phased circuits and cocircuits are clear except (S4).

To see (S4), let $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. If $\mathrm{supp}(X) \cap \mathrm{supp}(Y) = \emptyset$, then $X \perp Y$ by definition. Otherwise, let $\mathrm{supp}\,(X) = \{x_1, \dots, x_k\}$ and $\mathrm{supp}(Y) = \{y_1, \dots, y_l\}$, with the elements of $\mathrm{supp}(X) \cap \mathrm{supp}(Y)$ written first. Thus, $x_i = y_i$ for all i less than some value m.

We can extend $\mathrm{supp}(X)$ to $\{x_1, \dots, x_{d+1}\}$ so that every $\{x_1, \dots, \hat{x}_k, \dots x_{d+1}\}$ with $x_k \in \mathrm{supp}(X)$ is a basis for M_φ. Similarly, we extend $\mathrm{supp}(Y)$ to $\{y_1, \dots, y_{n-d+1}\}$ so that every $\{y_1, \dots, \hat{y}_k, \dots y_{n-d+1}\}$ with $y_k \in \mathrm{supp}(Y)$ is a basis for M^*_φ. Let $\{z_1, \dots, z_{d-1}\} = E \backslash \{y_1, \dots, y_{n-d+1}\}$.

The Grassmann-Plücker relations tell us that 0 is in the phase convex hull of

$$\{(-1)^k \varphi(x_1, \dots, \hat{x}_k, \dots, x_{d+1}) \varphi(x_k, z_1, \dots, z_{d-1}) \mid k = 1, \dots d+1\}.$$

Note that one of the factors of $\varphi(x_1, \dots, \hat{x}_k, \dots, x_{d+1})\varphi(x_k, z_1, \dots, z_{d-1})$ will be 0 unless $x_k \in \mathrm{supp}(X) \cap \mathrm{supp}(Y)$. Applying the definition of φ^*, we see that the above set can be written

$$\left\{ \frac{(-1)^k \varphi(x_1, \dots, \hat{x}_k, \dots, x_{d+1}) \varphi^*(y_1, \dots, \hat{y}_k, \dots, y_{n-d+1})^{-1}}{\mathrm{sign}(x_k, z_1, \dots, z_{d-1}, y_1, \dots, \hat{y}_k, \dots, y_{n-d+1})} \;\middle|\; \begin{array}{l} x_k = y_k, \text{ both in} \\ \mathrm{supp}(X) \cap \mathrm{supp}(Y) \end{array} \right\}.$$

Now note that

$$\operatorname{sign}(x_k, z_1, \ldots, z_{d-1}, y_1, \ldots, \hat{y}_k, \ldots, y_{n-d+1})$$

$$= (-1)^{d-1+k} \operatorname{sign}(z_1, \ldots, z_{d-1}, y_1, \ldots, y_{n-d+1})$$

and that if 0 is in the phase convex hull of a set A of complex numbers then 0 is in the phase convex hull of cA for any complex number c.

So, multiplying all elements of our set by

$$(-1)^{d-1} \operatorname{sign}(z_1, \ldots, y_1, \ldots, y_{n-d+1})\varphi(x_2, \ldots, x_{d+1})^{-1}\varphi^*(y_2, \ldots, y_{n-d+1}),$$

we see that 0 is in the phase convex hull of

$$\left\{ \frac{X(x_k)Y(x_k)}{X(x_1)Y(y_1)} \middle| x_k \in \operatorname{supp}(X) \cap \operatorname{supp}(Y) \right\}.$$

Multiplying all elements of this set by $X(x_1)Y(y_1)$, we see that $X \perp Y$.

That $\mathcal{C}_{\varphi \setminus e} = \mathcal{C}_\varphi \setminus e$ and $\mathcal{C}_{\varphi/e} = \mathcal{C}_\varphi/e$ follows immediately from the definition of $\mathcal{C}$. □

Corollary 6.3.4. *Given a phirotope φ, consider $X \in \mathcal{C}_\varphi$ and $Y \in \mathcal{D}_\varphi$ such that* $\operatorname{supp}(X) = \{x_0, \ldots, x_l\}$, $\operatorname{supp}(Y) = \{y_1, \ldots, y_h\}$. *Choose elements $x_{l+1}, \ldots, x_d$ such that $\{x_1, \ldots, x_d\} \in \mathbf{B}_\varphi$ and elements $z_2, \ldots, z_d$ that span the hyperplane $E \setminus \operatorname{supp}(Y)$ of M_φ. Then,*

(1) for every $x_i, x_j \in \operatorname{supp}(X)$,

$$\frac{X(x_i)}{X(x_j)} = (-1)^{i-j+1} \frac{\varphi(x_0, \ldots, \widehat{x_i}, \ldots, x_d)}{\varphi(x_0, \ldots, \widehat{x_j}, \ldots, x_d)},$$

(2) for every $y_i, y_j \in \operatorname{supp}(Y)$,

$$\frac{Y(y_i)}{Y(y_j)} = \frac{\varphi(y_i, z_2, \ldots, z_d)}{\varphi(y_j, z_2, \ldots, z_d)}.$$

In particular, $\mathcal{D}_\varphi$ can be defined alternatively as the family of all phased sets $Y \subset (S^1 \cup \{0\})^E$ satisfying (2).

Proof. The claim (1) follows because φ is alternating, and thus it is enough to keep track of the permutations involved.

For claim (2), consider $Z \in \mathcal{C}_\varphi$ such that $\operatorname{supp}(Z)$ is the basic circuit of y_i with respect to $\{y_j, z_2, \ldots, z_d\}$. Then, $\operatorname{supp}(Z) \cap \operatorname{supp}(Y) = \{y_i, y_j\}$, and thus since $Z \perp Y$ we must have

$$\frac{Y(y_i)}{Y(y_j)} = -\frac{Z(y_i)}{Z(y_j)} = \frac{\varphi(y_i, z_2, \ldots, z_d)}{\varphi(y_j, z_2, \ldots, z_d)}.$$

□

Phirotopes from dual pairs

Recall from Definition 1.5.4 the notion of *basis graph* of a matroid. Moreover, recall from Lemma 1.5.3 that if B is a basis of a matroid M on the ground set E, and $x \in E \setminus B$, then there is a unique circuit $C(B, x)$ contained in $B \cup \{x\}$, called the basic circuit of x with respect to B.

To construct a phirotope from a dual pair $\mathcal{C}$, $\mathcal{D}$ of circuit orientations we will follow the strategy of [20, Proposition 3.5.2 (2. proof)], which proves a similar result for oriented matroids. The gist of the proof is as follows.

- We arbitrarily choose one ordered basis $(b_1, \dots, b_d)$ to have $\varphi(b_0, \dots, b_d) = 1$. This defines the phirotope on any permutation of this basis.

- Given a definition of the phirotope on all permutations of a basis $B_1 = \{e, x_2, \dots, x_d\}$, consider an adjacent basis $B_2 = \{f, x_2, \dots, x_d\}$ in the basis graph. Let $X \in \mathcal{C}$ with $\mathrm{supp}(X) = C(B_1, f)$. Then the relation

$$\varphi(f, x_2, \dots, x_d) = -\frac{X(e)}{X(f)}\varphi(e, x_2, \dots, x_d)$$

  (from Definition 6.3.2) determines $\varphi(f, x_2, \dots, x_d)$.

- Thus, for each edge $\{B_1, B_2\}$ in the edge graph, we associate the fraction $\frac{X(f)}{X(e)}$ to the direction from B_1 to B_2. To find the phirotope on permutations of some basis B, we find a path from $\{b_1, \dots, b_d\}$ to B and multiply the appropriate quotients along this path.

  The hard work of the proof is showing that the definition at B is independent of the path chosen.

We first need a preliminary lemma that investigates the values of the signatures of the circuits involved in the basis exchanges of "triangles" and "squares" of basis graphs.

Lemma 6.3.5. *Let $\mathcal{C}, \mathcal{D}$ be the set of phased circuits resp. cocircuits of a complex matroid with underlying matroid M.*

(1) Given three distinct elements $e, f, g \in E$ with bases B_e, B_f, B_g of M and $A \subset E$ such that $B_e = A \cup e$, $B_f = A \cup f$, $B_g = A \cup g$, and for all $x, y \in \{e, f, g\}$ consider $X_{x,y} \in \mathcal{C}$ with $\mathrm{supp}(X_{x,y}) = C(A \cup x, y)$,

$$\frac{X_{e,f}(e)}{X_{e,f}(f)}\frac{X_{f,g}(f)}{X_{f,g}(g)} = -\frac{X_{e,g}(e)}{X_{e,g}(g)}.$$

(2) Given three distinct elements $e, f, g \in E$ with bases

$$B_{e,f} = A \cup \{e, f\},\ B_{f,g} := A \cup \{f, g\},\ B_{e,g} := A \cup \{e, g\}$$

of M for some $A \subset E$, choose any $X \in \mathcal{C}$ with $\mathrm{supp}(X) = C(B_{e,f}, g)$. Then,

$$\frac{X(g)}{X(e)}\frac{X(e)}{X(f)} = \frac{X(g)}{X(f)}.$$

(3) *Consider an independent set $A \subset E$ and distinct elements $e, f, g, h \in E$ such that*

$$B_1 := A \cup \{f, g\},\ B_2 := A \cup \{e, g\},\ B_1' := A \cup \{f, h\},\ B_2' := A \cup \{e, h\}$$

are bases of M, with

$$f \in C_1 := C(B_1, e),\ f \in C_1' := C(B_1', e),$$

$$g \in C_2 := C(B_1, h),\ g \in C_2' := C(B_2, h).$$

Then for any $X_1, X_2, X_1', X_2' \in \mathcal{C}$ with $\mathrm{supp}(X_1) = C_1$, $\mathrm{supp}(X_2) = C_2$, *supp*$(X_1') = C_1'$, $\mathrm{supp}(X_2') = C_2'$,

$$\frac{X_1(e)}{X_1(f)} \frac{X_2(h)}{X_2(g)} = \frac{X_1'(e)}{X_1'(f)} \frac{X_2'(h)}{X_2'(g)}.$$

The following diagrams illustrate the three cases of the lemma.

$$\begin{tikzcd}
B_e \arrow[r, "\frac{X_{e,f}(f)}{X_{e,f}(e)}"] \arrow[dr, "\frac{X_{e,g}(g)}{X_{e,g}(f)}"'] & B_f \arrow[d, "\frac{X_{f,g}(g)}{X_{f,g}(f)}"] \\
& B_g
\end{tikzcd}
\qquad
\begin{tikzcd}
B_{e,f} \arrow[r, "\frac{X(g)}{X(e)}"] \arrow[dr, "\frac{X(g)}{X(f)}"'] & B_{f,g} \arrow[d, "\frac{X(e)}{X(f)}"] \\
& B_{e,g}
\end{tikzcd}
\qquad
\begin{tikzcd}
B_2 \arrow[r, "\frac{X_2'(h)}{X_2'(g)}"] & B_2' \\
B_1 \arrow[u, "\frac{X_1(e)}{X_1(f)}"] \arrow[r, "\frac{X_2(h)}{X_2(g)}"] & B_1' \arrow[u, "\frac{X_1'(e)}{X_1'(f)}"']
\end{tikzcd}$$

Case (1) Case (2) Case (3)

Proof. (1) For the cocircuit $D := E \setminus \mathrm{cl}(A)$, we have $D \cap C(A \cup x, y) = \{x, y\}$ for all $x, y \in \{e, f, g\}$. therefore, for any $Y \in \mathcal{D}$ with $\mathrm{supp}(Y) = D$ we have $Y \perp X_{x,y}$ for all $x, y \in \{e, f, g\}$ and thus

$$\frac{X_{e,f}(e)}{X_{e,f}(f)} \frac{X_{f,g}(f)}{X_{f,g}(g)} = \Big(- \frac{Y(e)}{Y(f)} \Big) \Big(- \frac{Y(f)}{Y(g)} \Big) = \frac{Y(e)}{Y(g)} = - \frac{X_{e,g}(e)}{X_{e,g}(g)}.$$

(2) is evident.
(3) The claim is trivial when $C_1 = C_1'$ and $C_2 = C_2'$. If this is not the case, then without loss of generality suppose that $g \in C_1$. Then we can use C_1 to eliminate g from B_1 (or from B_2), and we obtain that $B := A \cup \{e, f\}$ is a basis. Since $g \in C_1$ implies $h \in C_1'$ (for else one could eliminate e and obtain a circuit contained in B_1), we can use C_1' to eliminate h from B_1' (or from B_2'). Then the basis graph of the matroid contains

$$\begin{tikzcd}
B_2 \arrow[rr, dash] \arrow[dd, dash] \arrow[dr, dash] & & B_2' \arrow[dd, dash] \arrow[dl, dash] \\
& B \arrow[l, phantom, "T"] \arrow[u, phantom, "T'"] \arrow[r, phantom, "T''"] \arrow[d, phantom, "T'''"] & \\
B_1 \arrow[rr, dash] \arrow[ur, dash] & & B_1' \arrow[ul, dash]
\end{tikzcd}$$

and we can apply part (1) to the “triangles” T, T', T'', T''' to conclude. $\square$

Proposition 6.3.6. *If $\mathcal{C}$ and $\mathcal{D}$ are the phased circuits resp. phased cocircuits of a complex matroid, then $\mathcal{C} = \mathcal{C}_\varphi$ and $\mathcal{D} = \mathcal{D}_\varphi$ for a phirotope φ. Moreover, φ is uniquely determined up to a nonzero constant.*

Proof. In this proof we fix a total ordering $>$ of the ground set E of the underlying matroid M. We will often identify a subset $A \subseteq E$ with the corresponding sequence ordered by $>$.
1. Labeling the basis graph. Consider the basis graph G of M. We define a function γ on ordered pairs of adjacent vertices of G. Given two bases B_1, B_2 of M corresponding to a pair of adjacent vertices of G we define

$$\gamma(B_1, B_2) := (-1)^{i-j+1} \frac{X(x_j)}{X(x_i)},$$

where $B_1 \cup B_2 = \{x_0, \ldots, x_d\}$, $B_1 \setminus B_2 = \{x_i\}$, $B_2 \setminus B_1 = \{x_j\}$, the x_l are numbered in increasing order with respect to $>$, and $X \in \mathcal{C}$ is any phased circuit with $\mathrm{supp}(X) = C(B_1, x_j)$. Clearly, $\gamma(B_1, B_2) = \gamma(B_2, B_1)^{-1}$.

Given any closed path $A = B_0, B_1, B_2, \ldots, B_k = A$ in G,

$$\prod_{i=0}^{k-1} \gamma(B_i, B_{i+1}) = 0.$$

To see this note that by Theorem 1.5.5 it is enough to check the cases $k = 3, 4$, which is easy to do using Lemma 6.3.5 and keeping track of the signs.
2. Construction of the phirotope associated with $\mathcal{C}, \mathcal{D}$. If we fix a "basepoint" $B \in V(G)$, Step 1 above tells us there is a well-defined quantity associated to every $B' \in V(G)$ and given by

$$\overline{\varphi_{\mathcal{C}}}(B') := \prod_{i=0}^{k-1} \gamma(B_i, B_{i+1})$$

where $B = B_0, B_1, \ldots, B_k = B'$ is any path from B to B' in G, and the empty product equals 1.

Now we are ready to define a function $\varphi_{\mathcal{C}} : E^d \to S^1 \cup \{0\}$ as follows. Given $x_1 < x_2 < \ldots < x_d \in E$, let

$$\varphi'_{\mathcal{C}}(x_1, \ldots x_d) := \begin{cases} 0 & \text{if } \{x_1, \ldots, x_d\} \notin V(G), \\ \overline{\varphi_{\mathcal{C}}}(\{x_1, \ldots, x_d\}) & \text{else.} \end{cases}$$

This function can be extended to any ordered d-tuple of elements of E by setting

$$\varphi_{\mathcal{C}}(x_1, \ldots, x_d) := \mathrm{sign}(\sigma) \varphi'_{\mathcal{C}}(x_{\sigma(1)}, \ldots, x_{\sigma(d)}),$$

where σ is a permutation such that $x_{\sigma(i)} < x_{\sigma(j)}$ if $i < j$. For every $X \in \mathcal{C}$ let $\mathrm{supp}(X) = \{x_0, x_1, \ldots, x_l\}$ be numbered, as usual, in increasing order with respect to $>$. For all $0 \leq i, j \leq l$ we can complete $\mathrm{supp}(X) \setminus x_i$ to a basis of M by a set $\{a_1, \ldots a_m\}$. Then $\mathrm{supp}(X) = C(A_i, x_i)$, where $A_i := \{x_0, \ldots, \hat{x}_i, \ldots, x_l, a_1, \ldots a_m\}$. We have

$$\frac{X(x_i)}{X(x_j)} = (-1)^{i-j+1} \gamma(A_j, A_i) = (-1)^{i-j+1} \overline{\varphi_{\mathcal{C}}}(A_j)^{-1} \overline{\varphi_{\mathcal{C}}}(A_i) \tag{6.3.1}$$

$$= (-1)^{i-j+1} \frac{\varphi_{\mathcal{C}}(x_0, \ldots, \hat{x}_i, \ldots, x_l, a_1, \ldots, a_m)}{\varphi_{\mathcal{C}}(x_0, \ldots, \hat{x}_j, \ldots, x_l, a_1, \ldots, a_m)}$$

For any pair of adjacent vertices $B_1, B_2 \in V(G)$ with $\{e\} = B_2 \setminus B_1$, $\{f\} = B_1 \setminus B_2$, the basic circuit $C = C(B_1, e)$ of M intersects the basic circuit $D = C^*(E \setminus B_2, f)$ of M^* in the set $\{e, f\}$. Choose $X \in \mathcal{C}$, $Y \in \mathcal{D}$ with $\mathrm{supp}(X) = C$, $\mathrm{supp}(Y) = D$. By $\mathcal{C} \perp \mathcal{D}$ we have

$$\frac{X(e)}{Y(e)} = -\frac{X(f)}{Y(f)}$$

and so

$$\frac{Y(e)}{Y(f)} = -\frac{X(e)}{X(f)} = \gamma(B_1, B_2).$$

For every $Y \in \mathcal{D}$ and $e, f \in \mathrm{supp}(Y)$, choose a basis T of the hyperplane H of M defined by $H := E \setminus \mathrm{supp}(Y)$. Then, $T \cup \{e, f\}$ contains a circuit C with $C \cap \mathrm{supp}(Y) = \{e, f\}$. Writing $T_e = T \cup e$, $T_f = T \cup f$ we have, as above,

$$\begin{aligned}\frac{Y(e)}{Y(f)} = (-1)^{i-j+2}\gamma(T_f, T_e) &= (-1)^{i-j}\overline{\varphi_{\mathcal{C}}}(T_f)^{-1}\overline{\varphi_{\mathcal{C}}}(T_e) \\ &= \frac{\varphi_{\mathcal{C}}(e, t_2, \ldots, t_d)}{\varphi_{\mathcal{C}}(f, t_2, \ldots, t_d)}\end{aligned} \qquad (6.3.2)$$

where e and f are respectively i-th and j-th in the $>$-ordering of $T \cup \{e, f\}$, and $t_2, \ldots t_d$ is any total ordering of T. In view of Corollary 6.3.4, equations (6.3.1) and (6.3.2) show that $\mathcal{C} = \mathcal{C}_{\varphi_{\mathcal{C}}}$, $\mathcal{D} = \mathcal{D}_{\varphi_{\mathcal{C}}}$.
3. Verification of the axioms for phirotopes The function $\varphi_{\mathcal{C}}$ we constructed so far is an alternating, nonzero function $E^d \to S^1 \cup 0$. We now prove that $\varphi_{\mathcal{C}}$ satisfies (φ3). To this end, consider any two subsets $S := \{x_0, \ldots x_d\} \subset E$, $T := \{y_2 \ldots y_d\} \subset E$. If for some j the set $S \setminus x_j$ is a basis of the underlying matroid M, then $S \setminus x_i$ is a basis of M only if x_i is in the basic circuit C_S of x_j with respect to $S \setminus x_j$. Also, $T \cup x_j$ is a basis only if T is an independent set and x_j is in the cocircuit D_T given by the complement of the hyperplane spanned by T.

We may from now on suppose that T is independent and $S \setminus x_j$ is a basis of M for some j. Then, the product

$$\varphi_{\mathcal{C}}(x_0, \ldots, \hat{x}_i, \ldots, x_d)\varphi_{\mathcal{C}}(x_i, y_2, \ldots, y_d)$$

is nonzero if and only if $x_i \in C_S \cap D_T$.

We thus have to consider the set

$$Q := \{(-1)^i\varphi_{\mathcal{C}}(x_0, \ldots, \hat{x}_i, \ldots, x_d)\varphi_{\mathcal{C}}(x_i, y_2, \ldots, y_d) \mid x_i \in C_S \cap D_T\}$$

and show that $0 \in \mathrm{relint}\,\mathrm{conv}\,Q$.

Let us suppose without loss of generality that $x_0 \in C_S \cap D_T$. Take $X \in \mathcal{C}$ such that $\mathrm{supp}(X) = C_S$ and $X(x_0) = 1$, $Y \in \mathcal{D}$ such that $\mathrm{supp}(Y) = D_T$ and $Y(x_0) = 1$.

Then we may consider the rotated set μQ for $\mu = \varphi_{\mathcal{C}}(x_1, \ldots, x_d)^{-1}\varphi_{\mathcal{C}}(x_0, y_2, \ldots, y_d)^{-1}$. By equations (6.3.1) and (6.3.2)

$$\begin{aligned}
\mu Q &= \left\{ (-1)^i \frac{\varphi_{\mathcal{C}}(x_0, \ldots, \hat{x}_i, \ldots, x_d)}{\varphi_{\mathcal{C}}(x_1, \ldots, x_d)} \frac{\varphi_{\mathcal{C}}(x_i, y_2, \ldots, y_d)}{\varphi_{\mathcal{C}}(x_0, y_2, \ldots, y_d)} \;\middle|\; x_i \in C_S \cap D_T \right\} \\
&= \left\{ \frac{\varphi_{\mathcal{C}}(x_0, x_1 \ldots, \hat{x}_i, \ldots, x_d)}{\varphi_{\mathcal{C}}(x_i, x_1, \ldots, \hat{x}_i, \ldots, x_d)} \frac{\varphi_{\mathcal{C}}(x_i, y_2, \ldots, y_d)}{\varphi_{\mathcal{C}}(x_0, y_2, \ldots, y_d)} \;\middle|\; x_i \in C_S \cap D_T \right\} \\
&= \left\{ \frac{X(x_0)Y(x_i)}{X(x_i)Y(x_0)} \;\middle|\; x_i \in C_S \cap D_T \right\} = \left\{ \frac{Y(x_i)}{X(x_i)} \;\middle|\; x_i \in C_S \cap D_T \right\},
\end{aligned}$$

thus $0 \in \operatorname{relint}\operatorname{conv} Q$ if and only if $0 \in (\operatorname{relint}\operatorname{conv} \mu Q) = P_{X,Y}$ - but the latter is the case because, by assumption, $X \perp Y$. $\square$

6.4 Elimination axioms

Our next goal is the statement of a set of axioms governing the behavior of complex phases in circuit elimination. Example 5.1.1 shows that one cannot hope for a general elimination axiom similar to the standard one for oriented matroids.

Deletion and contraction

In the following we will often argue by induction on the size of the ground set of the complex matroid. As a preparation, we prove that our notion of complex circuit orientation (Definition 6.1.5) behaves well with respect to the operations of deletion and contraction as introduced in Definition 6.1.16.

Lemma 6.4.1. *Let M be a matroid on the ground set E, and let $e \in E$. Then*

(1) if C_1, C_2 is a modular pair of circuits of $M \setminus e$ then it is a modular pair of circuits of M,

(2) if C_1, C_2 is a modular pair of circuits of M/e then $C_1 \cup e, C_2 \cup e$ is a modular pair of circuits of M.

Proof. In view of Definition 1.2.8 we show the equivalent statements about the dual M^*. In what follows, r^*, $r^*_{\setminus e}$, $r^*_{/e}$ are the rank functions of M^*, $M^* \setminus e$, M^*/e respectively.
(1) Let H_1, H_2 be a modular pair of hyperplanes of M^*/e. Then for $i = 1, 2$, $H_i' := H_i \cup e$ is a hyperplane of M^*,

$$r^*_{/e}(H_1 \cap H_2) = r^*((H_1 \cap H_2) \cup e) - r^*(e) = r^*(H_1' \cap H_2') - 1$$

and since by assumption $r^*_{/e}(H_1 \cap H_2) = r^*_{/e}(E \setminus e) - 2 = r^*(E) - 3$, we have $r^*(H_1' \cap H_2') = r^*(E) - 2$. So H_1, H_2 is a modular pair.
(2) Let H_1, H_2 be a modular pair of hyperplanes of $M^* \setminus e$. For $i = 1, 2$ let $H_i' \subset H_i \cup e$ denote a hyperplane of M^* containing H_i. If $r^*_{\setminus e}(E \setminus e) = r^*(E)$, then

$$r^*_{\setminus e}(E \setminus e) - 2 = r^*_{\setminus e}(H_1 \cap H_2) = r^*(H_1 \cap H_2) \le r^*(H_1' \cap H_2') \le r^*(E) - 2$$

and H'_1, H'_2 are a modular pair. If however $r^*_{\setminus e}(E \setminus e) < r^*(E)$, then e is in every basis of M^*, and $e \in H'_1 \cap H'_2$. Then

$$r^*(E) - 3 = r^*_{\setminus e}(E \setminus e) - 2 = r^*(H_1 \cap H_2) = r^*(H'_1 \cap H'_2) - 1$$

and H'_1, H'_2 are a modular pair in M^*. $\square$

Proposition 6.4.2. *If $\mathcal{C}$ is a complex circuit orientation of a matroid M on E, then for all $e \in E$*

(1) $\mathcal{C}/e$ is a complex circuit orientation of the matroid M/e, and

(2) $\mathcal{C} \setminus e$ is a complex circuit orientation of the matroid $M \setminus e$.

Proof. Let $\mathcal{C}$ be as in the statement. For (1) note that the elements of $\mathcal{C} \setminus e$ are all oriented circuits in $\mathcal{C}$ not containing e in their support, and so (ME) holds in $\mathcal{C} \setminus e$ because, by Lemma 6.4.1.(1), a modular pair of circuits in $M \setminus e$ is modular in M too, and the result of modular elimination between them in M is again an element of $M \setminus e$.

For (2), remember first that every element of $X \in \mathcal{C}/e$ is a subset of some element $X' \in \mathcal{C}$ with $\mathrm{supp}(X') = \mathrm{supp}(X) \cup e$, and in particular $X'(x) = X(x)$ for all $x \in \mathrm{supp}(X)$. Lemma 6.4.1.(2) ensures that for every modular pair X, Y in $\mathcal{C}/e$ the corresponding $X', Y' \in \mathcal{C}$ defined as above also define a modular pair. As above, the element Z' obtained by modular elimination of f between X' and Y' restricts to $Z \in \mathcal{C}/e$ with $f \in \mathrm{supp}(Z) \subset \mathrm{supp}(X) \cup \mathrm{supp}(Y)$. By the uniqueness of modular elimination we are done. $\square$

From phirotopes to circuit orientations

In this section we prove that the set $\mathcal{C}_\varphi$ of circuits induced by a phirotope φ satisfies the conditions of Definition 6.1.3.(2) for phased circuits. Conditions $(\mathcal{C}0)$ and $(\mathcal{C}1)$ are clear; we have to prove that (ME) holds in $\mathcal{C}_\varphi$, and as a stepping stone we prove the following "special elimination" property.

Lemma 6.4.3 (SE)**.** *Let φ be a phirotope on the ground set E. For all $X, Y \in \mathcal{C}_\varphi$ and $e, f \in \mathrm{supp}(X) \cap \mathrm{supp}(Y)$ such that $X(e) = Y(e)$ and $X(f) \neq Y(f)$, there is $Z \in \mathcal{C}$ with $f \in \mathrm{supp}(Z) \subseteq \mathrm{supp}(X) \cup \mathrm{supp}(Y)$.*

Proof. Suppose by way of contradiction that there are $X, Y \in \mathcal{C}_\varphi$, $e, f \in E$ so that the claim does not hold and let $A := \mathrm{supp}(X) \setminus \{e, f\}$, $B := \mathrm{supp}(Y) \setminus \{e, f\}$. Then $f \notin \mathrm{cl}(A \cup B)$ and we can extend A to A' and B to B', where A' and B' are bases of the hyperplane H containing $\mathrm{cl}(A \cup B)$ but not e (and thus not f either). Then let $D := E \setminus H$. It follows that $D \cap \mathrm{supp}(X) = D \cap \mathrm{supp}(Y) = \{e, f\}$. If we fix a total ordering of the ground set E we can think of any subset of E as representing an ordered tuple of elements. With $D' := D \setminus \{e, f\}$ we can write

$$\frac{X(f)}{X(e)} = -\frac{\varphi(e, A')}{\varphi(f, A')} = \frac{\varphi^*(e, D', H \setminus A')}{\varphi^*(f, D', H \setminus A')}.$$

But since this value does not depend on how we complete the set D' to a complement of a basis of M_φ, we have

$$\frac{X(f)}{X(e)} = \frac{\varphi^*(e, D', H \setminus B')}{\varphi^*(f, D', H \setminus B')} = -\frac{\varphi(e, B')}{\varphi(f, B')} = \frac{Y(f)}{Y(e)},$$

contradicting the assumption.

□

Proposition 6.4.4 (ME)**.** *Let φ be a phirotope. For all $X, Y \in \mathcal{C}_\varphi$ with $X \neq \mu Y$ for all $\mu \in S^1$ and such that* $\operatorname{supp}(X)$, $\operatorname{supp}(Y)$ *is a modular pair of circuits of M_φ, given $e, f \in E$ with $X(e) = -Y(e) \neq 0$ and $X(f) \neq Y(f)$, there is $Z \in \mathcal{C}_\varphi$ with $f \in \operatorname{supp}(Z) \subseteq \operatorname{supp}(X) \cup \operatorname{supp}(Y) \setminus \{e\}$, and*

$$\begin{cases} Z(f) \in \operatorname{pconv}(\{X(f), Y(f)\}) & \textit{if } f \in \operatorname{supp}(X) \cap \operatorname{supp}(Y), \\ Z(f) \leq \max\{X(f), Y(f)\} & \textit{else.} \end{cases}$$

Proof. For ease of notation and terminology, let us prove this for the dual matroid – that is, when $X, Y \in \mathcal{C}_{\varphi^*}$ are cocircuits of the complex matroid defined by φ.

Since the supports of X, Y form a modular pair, we have $x, y \in E$ and $A \subset E$ such that $\operatorname{supp}(X) = E \setminus \operatorname{cl}(A \cup \{x\})$, $\operatorname{supp}(Y) = E \setminus \operatorname{cl}(A \cup \{y\})$. Then it follows that $x \in \operatorname{supp}(Y)$ and $y \in \operatorname{supp}(X)$, for otherwise $\operatorname{supp}(X) = \operatorname{supp}(Y)$ and $X = \mu Y$ for some $\mu \in S^1$, which cannot be. From now on we fix a total ordering $a_2, \dots, a_d$ of A and, when appropriate, write A for $a_2, \dots, a_d$.

Let D be the (unique) cocircuit complementary to the hyperplane $E \setminus \operatorname{cl}(A \cup \{e\})$. By definition, the sign vector defined by $Z(x) := Y(x)$ and

$$\frac{Z(f)}{Z(x)} := \frac{\varphi(f, e, A)}{\varphi(x, e, A)} \text{ for all } f \neq x$$

is a signature of D. We will prove that it satisfies the requirements.

First of all, consider the element $y \in \operatorname{supp}(X) \cap \operatorname{supp}(Z)$. We have

$$\frac{Z(y)}{Z(x)} := \frac{\varphi(y, e, A)}{\varphi(x, e, A)} = \frac{\varphi(y, e, A)}{\varphi(x, y, A)} \frac{\varphi(x, y, A)}{\varphi(x, e, A)} = -\frac{Y(e)}{Y(x)} \frac{X(y)}{X(e)} = \frac{X(y)}{Y(x)}$$

and therefore, since we set $Z(x) = Y(x)$, we obtain $Z(y) = X(y)$.

Now let us consider an element $f \in \operatorname{supp}(Z) \setminus \operatorname{supp}(X)$. Then $f \notin \operatorname{supp}(X)$, and since $f \notin \operatorname{cl}(A)$ (for otherwise $f \notin \operatorname{supp}(Z)$) we conclude that we can exchange f for x in the base $A \cup \{x\}$ of the hyperplane $E \setminus \operatorname{supp}(X) = \operatorname{cl}(A \cup \{x\}) = \operatorname{cl}(A \cup \{f\})$. Therefore we can compute

$$\frac{Z(f)}{Z(x)} \frac{Y(x)}{Y(f)} = \frac{\varphi(f, e, A)}{\varphi(x, e, A)} \frac{\varphi(x, y, A)}{\varphi(f, y, A)} = \frac{\varphi(e, f, A)}{\varphi(y, f, A)} \frac{\varphi(y, x, A)}{\varphi(e, x, A)} = \frac{X(e)}{X(y)} \frac{X(y)}{X(e)} = 1,$$

hence $Z(f) = Y(f)$. By a similar argument we obtain $Z(f) = X(f)$ for every $f \in \operatorname{supp}(Z) \setminus \operatorname{supp}(Y)$.

As the last case, we consider an element $f \in \operatorname{supp}(Z) \cap \operatorname{supp}(X) \cap \operatorname{supp}(Y)$. Because the set $B := \{e, y\} \cup A$ is a basis of M_φ and f is not an element of $\operatorname{cl}(A \cup \{y\})$ nor of $\operatorname{cl}(A \cup \{e\})$, the basic circuit $C(f, B)$ of f with respect to B contains e, y, f, and thus $C(f, B) \cap \operatorname{supp}(X) = \{e, y, f\}$. In order to compute $Z(f)$, we apply the axiom (b) for chirotopes to the tuples of elements y, f, e, A

and x, A and conclude that 0 must be in the relative interior of the phase convex hull of

$$\{\varphi(f,e,A)\varphi(y,x,A),\ -\varphi(y,e,A)\varphi(f,x,A),\ \varphi(y,f,A)\varphi(e,x,A)\}.$$

This condition does not depend upon rotation - i.e., multiplication by an element of S^1. Thus, after multiplication by $(\varphi(y,e,A)\varphi(y,x,A))^{-1}$, equivalently we may say

$$0 \in \operatorname{pconv}\left\{ \frac{\varphi(f,e,A)}{\varphi(y,e,A)},\ -\frac{\varphi(f,x,A)}{\varphi(y,x,A)},\ \frac{\varphi(y,f,A)}{\varphi(y,x,A)} \underbrace{\frac{\varphi(e,x,A)}{\varphi(y,e,A)}}_{=-\frac{Z(x)}{Z(y)}} \right\}$$

which, by Corollary 6.3.4, can be rewritten as

$$0 \in \operatorname{pconv}\left\{ \frac{Z(f)}{Z(y)},\ -\frac{X(f)}{X(y)},\ -\frac{Y(f)}{Y(x)}\frac{Y(x)}{X(y)} \right\}$$

We already established that $Z(y) = X(y)$, and thus multiplying everything by this number we conclude that

$$0 \in \operatorname{pconv}\big\{Z(f),\ -X(f),\ -Y(f)\big\}$$

or, equivalently, $Z(f) \in \operatorname{pconv}(\{X(f), Y(f)\})$.

□

From circuit orientations to dual pairs

The goal of this section is to "close the circle" and show that the axiomatization in terms of circuit elimination given in Definition 6.1.3 is equivalent to the axiomatization for dual pairs of Definition 6.1.13. We will do so by showing that the set of circuits of a complex matroid induces a (unique) orthogonal complex signature of the cocircuits of the underlying matroid.

We first need a fact from matroid theory that we summarize in the following lemma.

Lemma 6.4.5. *Let M be a matroid on the ground set E. Consider a circuit C and a cocircuit D of M such that $|C \cap D| \geq 3$. Then there are elements $e, f \in D \cap C$ and a cocircuit D' of M such that*
(1) D and D' are a modular pair,
(2) $e \in (D' \cap C) \subseteq (D \cap C) \setminus f$.

Proof. Let D and C be as above, and let r be the rank of M. Then $C \setminus D$ is an independent set of rank at most $r-2$ and can be completed to a basis B of the hyperplane $H := E \setminus D$.

For every $e \in C \cap D$, the set $B \cup e$ is a basis of M. The basic circuit of f with respect to this basis cannot be contained fully in $(C \setminus D) \cup e$, and thus it contains an element $x \in B \setminus (C \setminus D)$. Let $A := B \setminus x$. Then we have $H = \operatorname{cl}(A \cup x)$ and we can define

$$H' := \operatorname{cl}(A \cup f), \qquad D' := E \setminus H'.$$

Clearly, $(D' \cap C) \subseteq (D \cap C) \setminus f$. To prove $e \in D' \cap C$, it is enough to show $e \notin H'$. But if e were in H', then there would be a circuit contained in the set $A \cup \{e, f\}$, and by the uniqueness of basic circuits, this would be also the basic circuit of f with respect to $B \cup e$ - contradicting the definition of x. □

As a first step, we prove the analog of Lemma 6.4.3.

Lemma 6.4.6. *Let $\mathcal{C}$ be a circuit orientation of a complex matroid. Then*

(SE) for all $X, Y \in \mathcal{C}$, $e, f \in E$ with $X(e) = -Y(e) \neq 0$ and $Y(f) \neq X(f)$, there is $Z \in \mathcal{C}$ with $f \in \operatorname{supp}(Z) \subseteq \operatorname{supp}(X) \cup \operatorname{supp}(Y) \setminus e$.

Proof. By Lemma 1.2.9 the set $\mathbf{C} := \{\operatorname{supp}(X) \mid X \in \mathcal{C}\}$ is the set of circuits of a matroid M.

We argue by induction on the rank of the M. The claim is trivial in rank 0 and 1, and every pair of circuits is modular in rank 2. So let $\mathcal{C}$ be a circuit orientation of a complex matroid of rank $d > 2$ and suppose the claim holds for all complex matroids of smaller rank.

By way of contradiction, let $X, Y \in \mathcal{C}$ and $e, f \in E$ be such that for all $C \in \mathbf{C}$ with $C \subseteq \operatorname{supp}(X) \cup \operatorname{supp}(Y)$, $f \notin C$. The case where $X(f)Y(f) = 0$ is covered by matroid elimination (Definition 1.1.2.($\mathbf{C}$2)). So suppose $e, f \in \operatorname{supp}(X) \cap \operatorname{supp}(Y)$ and choose $a \in \operatorname{supp}(Y) \setminus \operatorname{supp}(X)$. By Proposition 6.4.2, $\mathcal{C}/a$ is again a complex orientation of the circuits of the rank $d-1$ matroid M/a. By definition there are $X', Y' \in \mathcal{C}/a$ with $X'(g) \leq X(g)$, $Y'(g) \leq Y(g)$ for all $g \in E \setminus a$, and with $f \in \operatorname{supp}(X') \cap \operatorname{supp}(Y')$. With the notation of Definition 6.1.16, $Y' = Y_{\setminus a}$ and thus $e \in \operatorname{supp}(Y')$.

Now, if $e \notin \operatorname{supp}(X')$ we reach a contradiction by taking $C := \operatorname{supp}(X') \cup a$. Otherwise $e, f \in \operatorname{supp}(X') \cap \operatorname{supp}(Y')$ so $X'(e) = X(e) = -Y(e) = -Y'(e)$ and $X'(f) = X(f) \neq Y(f) = Y'(f)$. We apply induction hypothesis to the rank-$(d-1)$ complex matroid $\mathcal{C}/a$ and find $Z' \in \mathcal{C}/a$ with $f \in \operatorname{supp}(Z') \subseteq \operatorname{supp}(X') \cup \operatorname{supp}(Y') \setminus e$. Then we reach a contradiction by taking $C := \operatorname{supp}(Z') \cup a \in \mathbf{C}$. □

Proposition 6.4.7. *For any complex circuit orientation $\mathcal{C}$ with underlying matroid M there is a unique complex circuit signature $\mathcal{D}$ of M^* such that $\mathcal{D} \perp \mathcal{C}$.*

Proof. Let $\mathcal{C}$ be a complex circuit orientation with underlying matroid M.
Definition of $\mathcal{D}$: For every cocircuit D of M, choose a maximal independent subset A of the hyperplane D^c. Then for every $e, f \in D$, there is a unique circuit $C_{D,e,f}$ of M with support contained in $A \cup \{e, f\}$. (Namely, $C_{D,e,f}$ is the basic circuit of f with respect to $A \cup e$.) Choose $X_{D,e,f} \in \mathcal{C}$ with $\operatorname{supp}(X_{D,e,f}) = C_{D,e,f}$.

$$\mathcal{D} := \left\{ W \in (S^1 \cup \{0\})^E \;\middle|\; \begin{array}{l} D := \operatorname{supp}(W) \in \mathbf{C}(M^*), \\ \forall e, f \in \operatorname{supp}(W),\ \frac{W(e)}{W(f)} = -\frac{X_{D,e,f}(e)}{X_{D,e,f}(f)} \end{array} \right\}$$

Certainly this $\mathcal{D}$ is the unique candidate for a complex circuit signature of M^* orthogonal to $\mathcal{C}$. It remains to see that $\mathcal{D}$ is, in fact, a well-defined complex circuit signature.

Claim 1. *$\mathcal{D}$ is well-defined and independent of the choice of the $X_{D,e,f}$.*

Proof. First we prove independence of the choice of the $X_{D,e,f}$. Given $D \in \mathbf{C}^*(M)$ and $e, f \in D$, let Y and Y' be two candidates for $X_{D,e,f}$. Multiplying Y by an element of S^1, we may assume $Y(e) = -Y'(e)$. If $Y(e)/Y(f) \neq Y'(e)/Y'(f)$, then by Lemma 6.4.6 there is $Z \in \mathcal{C}$ with $\operatorname{supp}(Z) \cap D = \{f\}$, contradicting Lemma 1.1.9.

To conclude that $\mathcal{D}$ is well-defined, it is enough to prove that, given $D \in \mathbf{C}^*(M)$ and $e, f, g \in D$,

$$-\frac{X_{D,f,g}(f)}{X_{D,f,g}(g)} = \Big(-\frac{X_{D,e,f}(f)}{X_{D,e,f}(e)}\Big)\Big(-\frac{X_{D,e,g}(e)}{X_{D,e,g}(g)}\Big).$$

The circuits $C_{D,e,f}$ and $C_{D,e,g}$ form a modular pair, because their complements both contain the corank 2 coflat $\operatorname{cl}(E\backslash(A\cup\{f,g\}))$. Then (modular) elimination of e from $X_{D,e,f}$ and $\frac{-X_{D,e,f}(e)}{X_{D,e,g}(e)}X_{D,e,g}$ gives $Y \in \mathcal{C}$ with $f, g \in \operatorname{supp}(Y)$ and $\frac{Y(f)}{Y(g)} = \frac{X_{D,e,f}(f)}{-X_{D,e,f}(e)}\frac{X_{D,e,g}(e)}{X_{D,e,g}(g)}$. So

$$\frac{X_{D,f,g}(f)}{X_{D,f,g}(g)} = \frac{Y(f)}{Y(g)} = -\frac{X_{D,e,f}(f)}{X_{D,e,f}(e)}\frac{X_{D,e,g}(e)}{X_{D,e,g}(g)}$$

and the claim follows.

Claim 2. *Fix $W \in \mathcal{D}$. For all $X \in \mathcal{C}$ with $|\operatorname{supp}(W)\cap\operatorname{supp}(X)| \leq 3$, $W \perp X$.*

Proof. The claim is either trivial or clear by definition if $|\operatorname{supp}(W)\cap\operatorname{supp}(X)| \leq 2$. So consider $X \in \mathcal{C}$ with $|\operatorname{supp}(X)\cap\operatorname{supp}(W)| = 3$, and by way of contradiction let $\operatorname{supp}(W) \cap \operatorname{supp}(X) = \{e, f, g\}$ so that $P_{X,W}$ is contained in a closed half-circle and includes a point in the interior of this half-circle.

By Lemma 6.4.5 applied to M^*, there is a circuit $X' \in \mathcal{C}$ and two elements of $\{e, f, g\}$ (say, e, f) such that $\operatorname{supp}(X')$ and $\operatorname{supp}(X)$ are a modular pair in M, and $e \in \operatorname{supp}(X') \cap \operatorname{supp}(W) \subseteq \operatorname{supp}(X) \cap \operatorname{supp}(W) \setminus f$, and since $\operatorname{supp}(X')\cap\operatorname{supp}(W)| \geq 2$, we know $\operatorname{supp}(X')\cap\operatorname{supp}(W) = \{e, g\}$. Multiplying by an element of S^1, we may assume $X'(e) = -X(e)$. Thus

$$\frac{X'(g)}{W(g)} = -\frac{X'(e)}{W(e)} = \frac{X(e)}{W(e)},$$

and

$$P_{X,W} = \Big\{\frac{X'(g)}{W(g)}, \frac{X(f)}{W(f)}, \frac{X(g)}{W(g)}\Big\}.$$

In particular, these three points lie in the unit circle as described before. Modular elimination of e between X' and X gives a circuit $Y \in \mathcal{C}$ with $\operatorname{supp}(Y)\cap\operatorname{supp}(W) = \{f, g\}$, $Y(f) = X(f)$, and $Y(g) \in \operatorname{pconv}(\{X(g), X'(g)\})$. Thus $P_{Y,W}$ lies in a half-open half-circle of S^1, contradicting $Y \perp W$.

Claim 3. *$\mathcal{D} \perp \mathcal{C}$.*

Proof. Induction on the rank of M. If M has rank 2, then all circuits have size 3, and we conclude with Claim 2. Assume that M has rank $r > 2$ and the claim holds for all matroids of rank $r - 1$ or less.

Suppose by way of contradiction that there is $X \in \mathcal{C}$ and $W \in \mathcal{D}$ with $X \not\perp W$. Choose $e \in E \setminus \operatorname{supp}(W)$. Then, $\mathcal{C}/e$ is a complex circuit orientation of the matroid M/e and $\mathcal{D}$ is a circuit signature of the matroid $M^* \backslash e$ satisfying $X' \perp W'$ for all $X' \in \mathcal{C}/e$ and $W' \in \mathcal{D}\backslash e$ with $|\operatorname{supp}(X')\cap\operatorname{supp}(W')| \leq 2$. Since the rank of M/e is $r-1$, by induction hypothesis $X' \perp W'$ for all $X' \in \mathcal{C}/e$, $W' \in \mathcal{D} \setminus e$.

Now look at our X, W and choose $f \in \mathrm{supp}(X) \cap \mathrm{supp}(W)$. By the definition of contraction and deletion, $W \in \mathcal{D} \setminus e$ and there is $X' \in \mathcal{C}/e$ with $X' \subseteq X$ and $f \in \mathrm{supp}(X')$. The vertices of $P_{X',W}$ are a subset of the vertices of $P_{X,W}$ - thus $X \not\perp W$ forces $X' \not\perp W$, contradicting the induction hypothesis. $\square$

At last, we can justify Theorem 6.1.8 and Theorem 6.1.15.

Corollary 6.4.8. *The definition of complex matroids in terms of their oriented circuits obtained from axioms ($\mathcal{C}0$), ($\mathcal{C}1$), (ME) is equivalent to the definition in terms of phirotopes (and, in turn, with the one in terms of dual pairs).*

Proof. This is just a combination of Proposition 6.4.4, Proposition 6.4.7 and Proposition 6.3.6. $\square$

Duality

Given the set $\mathcal{C}$ of phased circuits of a complex matroid, the corresponding set of phased cocircuits can be defined by orthogonality.

Proposition 6.4.9. *Let $\mathcal{C} \subseteq (S^1 \cup \{0\})^E$ be a complex circuit orientation of M. Then the set of elements of $\mathcal{C}^\perp \setminus \{\widehat{0}\}$ of minimal support is exactly the complex signature $\mathcal{D}$ of M^* given by Proposition 6.4.7.*

Proof. Recall

$$\mathcal{C}^\perp = \{W \in (S^1 \cup \{0\})^E \mid W \perp X \text{ for all } X \in \mathcal{C}\}.$$

For any collection of phased sets, $\mathcal{T}$, let $\lfloor \mathcal{T} \rfloor$ denote the elements of $\mathcal{T}^\perp \setminus \{\widehat{0}\}$ with minimal support.

By Proposition 6.4.7, we have $\mathcal{D} \subset \mathcal{C}^\perp$. Since $\mathrm{supp}(\mathcal{D}) := \{\mathrm{supp}(X) \mid X \in \mathcal{D}\}$ is the set of circuits of the underlying matroid, by [77, Proposition 2.1.20] it can be written as $\mathrm{supp}(\mathcal{D}) = \lfloor \mathcal{S} \rfloor$, where

$$\mathcal{S} := \{A \subseteq E \mid |A \cap \mathrm{supp}(X)| \neq 1 \forall X \in \mathcal{C}\}.$$

Now, $\mathrm{supp}(\mathcal{C}^\perp) \subset \mathcal{S}$ (since $X \perp W$ forbids $|\mathrm{supp}(X) \cap \mathrm{supp}(W)| = 1$), and so
(1) $\mathcal{D} \subseteq \lfloor \mathcal{C}^\perp \rfloor$ because for every $W \in \mathcal{C}^\perp$ there is $Y \in \mathcal{D}$ with $\mathrm{supp}(Y) \subseteq \mathrm{supp}(W)$,
(2) $\mathcal{D} \supseteq \lfloor \mathcal{C}^\perp \rfloor$ because every $W \in \lfloor \mathcal{C}^\perp \rfloor$ has the same support as some $Y_W \in \mathcal{D}$, and one sees as in the proof of Proposition 6.4.7 that for any $X \in S^1 \cup \{0\}$ with $\mathrm{supp}(W) \in \mathrm{supp}(\mathcal{D})$ the condition $W \perp \mathcal{C}$ determines the ratios $W(f)/W(e)$ uniquely for every pair $e, f \in \mathrm{supp}(W)$. Thus, $Y_W = W$. $\square$

6.5 Vectors

Sadly, there is no vector axiomatization for complex matroids that is cryptomorphic to the other axiomatizations and has the property that, for complex subspaces W of $\mathbb{C}^n$, the complex matroid of W has vector set $\{\mathrm{ph}(v) : v \in W\}$. In this section we give an example to show that the circuits of a complex matroid with realization W do not determine $\{\mathrm{ph}(v) : v \in W\}$.

Let W_1 be the row space of

$$\begin{pmatrix} 1 & 1+i & 1 & 0 \\ 1+i & 3i & 0 & 1 \end{pmatrix}$$

and let W_2 be the row space of

$$\begin{pmatrix} 1 & 1+i & 1 & 0 \\ 1+i & 4i & 0 & 1 \end{pmatrix}.$$

We shall verify that W_1 and W_2 have the same complex matroid, but that there is a $v \in W_1$ such that $\mathrm{ph}(v) \neq \mathrm{ph}(w)$ for every $w \in W_2$.

For each W_i, the underlying matroid is uniform, of rank 2, with 4 elements, so has 4 (unphased) circuits. Thus each complex matroid has circuit set consisting of four S^1 orbits. We can read two of the orbits for each W_i directly from the presentation above: each of the two complex matroids has $\mathrm{ph}(1, 1+i, 1, 0)$ and $\mathrm{ph}(1+i, 3i, 0, 1) = \mathrm{ph}(1+i, 4i, 0, 1)$ as circuits. To see the remaining two orbits, we perform Gauss-Jordan elimination on the two matrices:

$$\begin{pmatrix} 1 & 1+i & 1 & 0 \\ 1+i & 3i & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & -1+i \\ 0 & 1 & i-1 & -i \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1+i & 1 & 0 \\ 1+i & 4i & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & \frac{1}{2}(-1+i) \\ 0 & 1 & \frac{1}{2}(i-1) & \frac{-i}{2} \end{pmatrix}.$$

So, the two W_i give the same complex matroid.

On the other hand, note that $(2+i, 1+4i, 1, 1) \in W_1$. Assume by way of contradiction that some $w \in W_2$ has $\mathrm{ph}(w) = \mathrm{ph}(2+i, 1+4i, 1, 1)$. Then $w = k(1, 1+i, 1, 0) + l(1+i, 4i, 0, 1)$ for some k and l. To have the correct signs on the last two components, k and l must both be positive real numbers. However, one easily checks that no such k and l give the correct sign on the first two components.

6.6 Weak maps and strong maps

Intuitively, a weak map of matroids is the combinatorial analog to moving a subspace of a vector space $\mathbb{K}^n$ into more special position with respect to the coordinate hyperplanes. The same intuition motivates the definition of weak maps for oriented matroids, although the intuition is known to be somewhat problematic in this case: there are weak maps of realizable oriented matroids which do not arise from geometrically "close" realizations (cf. Proposition 2.4.7 in [20]).

A strong map of (oriented) matroids is the combinatorial analog to taking a subspace of a vector space. In the case of oriented matroids, this analogy has a beautifully straightforward interpretation via the Topological Representation Theorem. The covectors of a rank r oriented matroid $\mathcal{M}$ label the cells in a regular cell decomposition of S^{r-1}, and the covectors of any rank k strong map image of $\mathcal{M}$ label the cells in the intersection of this cell complex with a $(k-1)$-dimenrsional "pseudoequator". For details of this, see [20, Section 7.7]. The big point is that strong maps of oriented matroids have a straightforward definition in terms of covectors (and hence also in terms of vectors), but it is not so clear how to see strong maps directly in terms of circuits, cocircuits, or chirotopes. As far as we know there is no definition of strong maps of oriented matroids in terms of circuits, cocircuits, or chirotopes without involving composition somehow. From the perspective of the Topological Representation Theorem, such a definition seems unlikely: the cocircuits of an oriented matroid represent only the vertices in the cell decomposition of S^{r-1}, and without referring to composition it's not clear how to describe how arbitrary pseudoequators intersect the entire cell decomposition. For the same reasons, it seems unlikely that we can define strong maps of complex matroids without vector axioms.

On the other hand, this section will develop a notion of weak maps of complex matroids that behaves much like weak maps of oriented matroids.

Recall the partial order on $(S^1 \cup \{0\})^E$: we order $S^1 \cup \{0\}$ to have unique minimum 0 and all other elements maximal, and then order $(S^1 \cup \{0\})^E$ componentwise. Also recall [77, Proposition 7.3.11] that for matroids M_1 and M_2 on the same ground set E, there is a weak map from M_1 to M_2 if and only if every circuit in M_1 contains a circuit of M_2.

Definition 6.6.1. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be complex matroids on the set E with circuit sets $\mathcal{C}_1$ resp. $\mathcal{C}_2$. We say there is a *weak map* from $\mathcal{M}_1$ to Mo_2, and write $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$, if for every $X \in \mathcal{C}_1$ there exists $Y \in \mathcal{C}_2$ such that $X \geq Y$.

Proposition 6.6.2. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$.*

1. *If $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$ then $\underline{\mathcal{M}_1} \rightsquigarrow \underline{\mathcal{M}_2}$.*
2. *If $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$ then $\operatorname{rank}(\mathcal{M}_1) \geq \operatorname{rank}(\mathcal{M}_2)$.*

Proof. The first statement is clear from the definition of weak maps, and the second statement follows from the first. □

Proposition 6.6.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be complex matroids of the same rank and on the same ground set. Let φ_1 and φ_2 be phirotopes for $\mathcal{M}_1$ and $\mathcal{M}_2$, and let φ_1 and φ_2 be their duals. The following are equivalent.*

1. *$\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$.*
2. *For some $c \in S^1$, $\varphi_1 \geq c\varphi_2$.*
3. *For some $c \in S^1$, $\varphi_1^* \geq c\varphi_2^*$.*

Proof. The equivalence of the latter two statements is clear from Theorem 6.1.11. Let M_1 and M_2 denote the underlying matroids of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively.

If $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$ then by Lemma 6.6.2.1 we know that every basis of M_2 is also a basis of M_1. In particular, we have the following.

1. There exists B_0 an ordered basis of both M_1 and M_2. Without loss of generality assume $\varphi_1(B_0) = \varphi_2(B_0)$.
2. The basis graph of M_2 is a subgraph of the basis graph of M_1.

For any ordered sequence S, let $\underline{S}$ denote the set of elements of S.

We will induct on distance from $\underline{B_0}$ in the basis graph of M_2 to see that φ_1 and φ_2 coincide on every ordered basis B of M_2. If $\underline{B} \neq \underline{B_0}$, by basis exchange we can find B_1 a basis closer to $\underline{B_0}$ such that $\underline{B} = \{e, x_2, \ldots, x_r\}$ and $B_1 = \{f, x_2, \ldots, x_r\}$ for some $e, f, x_2, \ldots, x_r$. Then by Theorem 6.1.8, any signature $X \in \mathcal{C}_{\varphi_1}$ on the basic circuit of f with respect to B satisfies

$$\frac{X(e)}{X(f)} = -\frac{\varphi_1(f, x_2, \ldots, x_r)}{\varphi_1(e, x_2, \ldots, x_r)} = -\frac{\varphi_2(f, x_2, \ldots, x_r)}{\varphi_1(e, x_2, \ldots, x_r)}.$$

But $X \geq Y$ for some $Y \in \mathcal{C}_{\varphi_2}$, and Y is a circuit signature in $\mathcal{M}_2$ on the basic circuit of f with respect to B in M_2. So

$$-\frac{\varphi_2(f, x_2, \ldots, x_r)}{\varphi_2(e, x_2, \ldots, x_r)} = \frac{Y(e)}{Y(f)} = \frac{X(e)}{X(f)}$$

and thus $\varphi_2(f, x_2, \ldots, x_r) = \varphi_1(f, x_2, \ldots, x_r)$.

Our proof that the second statement implies the first is adapted from [20] and is by induction on $|E|$.

Recall that a loop of a matroid is an element e such that $\{e\}$ is a circuit, and a coloop is an element e such that $\{e\}$ is a cocircuit. Loops and coloops of complex matroids are loops or coloops of the underlying matroid. Write $\mathcal{C}_1 := \mathcal{C}_{\varphi_1}$ and $\mathcal{C}_2 := \mathcal{C}_{\varphi_2}$ for the sets of circuits of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. First note:

- If $\mathcal{M}_1$ has a loop e_0, then e_0 is also a loop of $\mathcal{M}_2$, and the induction hypothesis tells us that $\mathcal{M}_1 \setminus e_0 \rightsquigarrow \mathcal{M}_1 \setminus e_0$, hence $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$.
- If $\mathcal{M}_1$ has no loops but $\mathcal{M}_2$ has a coloop e_0, then $\varphi_1/e_0 \rightsquigarrow \varphi_2/e_0$ and $\{e_0\} \notin \mathcal{C}_1$, so for every $X \in \mathcal{C}_1$ there is a $Y \in \mathcal{C}_2$ such that $X \setminus e_0 \geq Y \setminus e_0$. Since e_0 is a coloop, this implies $Y(e_0) = 0$, so $X \geq Y$.

So consider the case when $\varphi_1 \geq \varphi_2$ and $\mathcal{M}_2$ has no coloops. Let $X \in \mathcal{C}_1$. Let A be a maximal subset of $\mathrm{supp}(X)$ that's independent in $\mathcal{M}_2$, and extend A to a basis B of $\mathcal{M}_2$. Let $\tilde{\varphi_1}$, $\tilde{\varphi_2}$ be the restrictions of φ_1 and φ_2 to $(\mathrm{supp}(X) \cup B)^r$. Then $\tilde{\varphi_1} \geq \tilde{\varphi_2}$.

If $A := E \setminus (\mathrm{supp}(X) \cup B) \neq \emptyset$ then, since $X \in \mathcal{C}_1 \setminus A$, the induction hypothesis tells us that there is a $Y \in \mathcal{C}_2 \setminus A \subseteq \mathcal{C}_2$ such that $X \geq Y$.

If $\mathrm{supp}(X) \cup B = E$, we can see that $B \subsetneq \mathrm{supp}(X)$. Otherwise, any $b \in B \setminus \mathrm{supp}(X)$ satisfies $\mathrm{rank}_{\mathcal{M}_2}(\mathrm{supp}(X) \cup (B \setminus b)) < \mathrm{rank}_{\mathcal{M}_2}(\mathrm{supp}(X) \cup B)$. Thus b is a coloop of $\mathcal{M}_2$, but $\mathcal{M}_2$ has no coloops. Thus $\mathrm{supp}(X)$ is a circuit of $\underline{\mathcal{M}_2}$.

An easy induction on the rank shows that whenever M_1 and M_2 are matroids of the same rank such that every circuit of M_1 is a circuit of M_2, then $M_1 = M_2$.

We conclude $\underline{\mathcal{M}_1} = \underline{(}\mathcal{M}_2)$, and so $\varphi_1 \geq \varphi_2$ implies $\varphi_1 = \varphi_2$. Thus $\mathcal{M}_1 = \mathcal{M}_2$. $\square$

As with realizable oriented matroids, weak maps of realizable complex matroids can arise from moving subspaces into more special position with respect to the coordinate hyperplanes. To make this precise, we give here the complex version of the same argument for oriented matroids (cf. [3]). Consider the complex Grassmannian $G(r, \mathbb{C}^n)$, the topological space of all rank r subspaces of $\mathbb{C}^n$. For any $W \in G(r, \mathbb{C}^n)$, let $\mu(W)$ be the corresponding rank r complex matroid. Thus, if $W = \mathrm{row}(M)$, the function $\varphi_M : [n]^r \to S^1 \cup \{0\}$ taking each $(e_1, \ldots, e_r)$ to the sign of the minor of M with columns indexed by $(e_1, \ldots, e_r)$ is a phirotope for $\mu(W)$.

The following is our central result on the realizable interpretation of weak maps:

Theorem 6.6.4. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be rank r complex matroids on the ground set $[n]$. If $\overline{\mu^{-1}(\mathcal{M}_1)} \cap \mu^{-1}(\mathcal{M}_2) \neq \emptyset$ then $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$.*

Proof. For any r-subset B of $[n]$, let $U_B \subset G(r, \mathbb{C}^n)$ be the set of all row spaces of $r \times n$ complex matrices such that the square submatrix with column set indexed by B is the identity. Then $U_B \cong \mathbb{C}^{r \times (n-r)}$, and the set of all U_B is an atlas on $G(r, \mathbb{C}^n)$. Thus $U_B \cap \overline{\mu^{-1}(\mathcal{M}_1)} \cap \mu^{-1}(\mathcal{M}_2) \neq \emptyset$ for some B. Without loss of generality assume $B = [r]$. Thus we can (and will) identify U_B with the set of $r \times n$ matrices M of the form $(I|M')$, where I is the $r \times r$ identity matrix, and M' is a $r \times (n-r)$ matrix.

Now consider the two maps

$$U_B \xrightarrow{d} \mathbb{C}^{[n]^r} \xrightarrow{\mathrm{ph}} (S^1 \cup \{0\})^{[n]^r}$$

where $d(M)(e_1, \ldots, e_r)$ is the $(e_1, \ldots, e_r)$ minor of M (that is, the determinant of the submatrix of M with columns indexed by $(e_1, \ldots, e_r)$, in that order). The composition of these two maps takes each W to the phirotope for $\mu(W)$ with value 1 on $(1, 2, \ldots, r)$.

The map d is continuous, hence the hypothesis gives

$$\overline{d(\mu^{-1}(\mathcal{M}_1))} \cap d(\mu^{-1}(\mathcal{M}_2)) \neq \emptyset.$$

But for each i, $d(\mu^{-1}(\mathcal{M}_i)) \subseteq \mathrm{ph}^{-1}(\varphi_{M_i})$, so $\overline{\mathrm{ph}^{-1}(\varphi_{\mathcal{M}_1})} \cap \mathrm{ph}^{-1}(\varphi_{\mathcal{M}_2}) \neq \emptyset$. In particular, for every $X \in [n]^r$, we have

$$\overline{\mathrm{ph}^{-1}(\varphi_{\mathcal{M}_1}(X))} \cap \mathrm{ph}^{-1}(\varphi_{\mathcal{M}_2}(X)) \neq \emptyset.$$

Notice that, for every $c \in S^1 \cup \{0\}$, $\mathrm{ph}^{-1}(c) = \mathbb{R}_+ c$. Thus, for any $c_1, c_2 \in S^1 \cup \{0\}$,

$$\overline{\mathrm{ph}^{-1}(c_1)} \cap \mathrm{ph}^{-1}(c_2) \neq \emptyset \textit{ if and only if } c_1 \geq c_2.$$

So $\varphi_{\mathcal{M}_1} \geq \varphi_{\mathcal{M}_2}$, and by Proposition 6.6.3 this means $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$. $\square$

Part III

$\mathbb{Z}$

7 Preliminaries: Integral lattices

The exposition in this chapter richly benefited from many conversations with Luca Moci.

Having thought about a combinatorial abstraction of linear dependency in vector spaces, one can try to generalize to a finitely generated free $\mathbb{Z}$-module.

In this chapter we consider integer vectors $v_1, \ldots, v_n$, say as columns of a matrix M of rank d over $\mathbb{Z}$. Every linear dependency over $\mathbb{Z}$ gives rise to a linear dependency over $\mathbb{Q}$, and vice versa. Therefore it is natural to associate to this set of vectors the matroid which is represented (over $\mathbb{Q}$) by M. However, this matroid does not at all exhaust the combinatorial information encoded by M such as, for example, the number of lattice points that are contained in the associated *zonotope*

$$Z(M) := \{t_1 v_1 + \ldots + t_n v_n \mid 0 \leq t_i \leq 1 \text{ for all } i\}.$$

The impulse to consider this situation, however, comes from more than the apparent naturality of the generalization. Recently, some common structure has been brought to light between the following objects associated to M as above:

1. the zonotope $Z(M)$;
2. the *toric arrangement* defined by M;
3. the *Dahmen-Micchelli space* associated to the box spline function defined by M;
4. the *partition function* associated to M and the corresponding discrete Dahmen-Micchelli space.

The connections between these objects have been investigated and described recently in work by many authors, including De Concini, Procesi and Vergne. This represents the core of the material of the forthcoming book of De Concini and Procesi [36], to which we refer for definitions and basics about the partition function and Dahmen-Micchelli spaces. A quick primer in toric arrangements is given in our Section 8.2.

7.1 The multiplicity Tutte polynomial

From a combinatorial point of view even more hope (and suspence) has been raised by the results of Luca Moci who introduced an analogue of the Tutte polynomial of a matroid (the "universal Tutte-Grothendieck invariant" [31], a two-variable polynomial which, for instance, specializes to the characteristic polynomial of Definition 1.3.2), called the *multiplicity Tutte polynomial*.

Definition 7.1.1 (Section 2.2 of [72])**.** Let $v_1, \ldots, v_n \in \mathbb{Z}^d$ as above and for $J \subseteq \{1, \ldots, n\}$ let $M(J)$ be the matrix with columns v_j with $j \in J$. Then

$$T_M(x, y) := \sum_{J \subset \{1,\ldots,n\}} m(J)(x-1)^{d-\mathrm{rank}(M(J))}(y-1)^{|J|-\mathrm{rank}(M(J))},$$

where the rank function is the one of the associated matroid and the *multiplicity* of J is defined as $m(J) := |\det M(J)|$.

Moci proves that this polynomial satisfies a deletion-contraction recursion of the type

$$T_M(x, y) = T_{M \setminus v_i}(x, y) + T_{M/v_i}(x, y)$$

for a suitable definition of 'contraction' of a list of integer vectors. We refer to [72, Section 3.2] for a precise statement, and content ourselves of pointing out a decisive point which arises here: matroid contraction is represented by (orthogonal) projection onto a subspace and duality theory rests on the fact that for every linear subspace W of a vector space there is another linear subspace U (its orthogonal complement) such that $W \oplus U$ is a direct sum decomposition of the whole space. This is no longer true if one replaces 'linear subspace' by 'sublattice', for torsion may (and will) appear. Thus it is natural for Moci to give the definition of $T_M(x, y)$ as a polynomial associated to a 'list of elements of finitely generated abelian groups'.

Having commented briefly about the definition of $T_M(x, y)$, we return to our situation and state some of Moci's results.

1. $T_M(x, 1)$ computes the integer points in the zonotope $Z(M)$
2. $t^n T_M(\frac{2t+1}{t}, 1)$ is the Poincaré polynomial of the complement to the toric arrangement in the complex d-torus.
3. $T_M(1, y)$ is the Hilbert series of the associated discrete Dahmen-Micchelli space.

This should be motivation enough to seek for the common combinatorial structure shared by this different subjects and giving rise to these enumerative results. It could be, again, an 'enrichment' of the underlying matroid structure by some data abstracting the 'arithmetical' data of the multiplicities $m(J)$.

The small size of this chapter is due to the youth of the subject testifies for the urgent need of a deeper understanding of the situation. As a contribution to the understanding of the combinatorics of toric arrangements we offer the next chapter.

8 A Salvetti complex for toric arrangements

This chapter reproduces the paper [34], written jointly with Giacomo D'Antonio.

Introduction

A *toric arrangement* is, roughly speaking, a family of subtori of a complex torus $(\mathbb{C}^*)^n$. The study of the topology and the combinatorics of such objects is a fairly new, yet thriving topic. As the very first attempt in this direction we can cite the work of Lehrer [67], where the representation theory on the cohomology of the configuration space $F(\mathbb{C}^*, n)$ of n points in the pointed complex plane is studied. This configuration space is indeed the complement of a toric arrangement. Its topology is already well known, since $F(\mathbb{C}^*, n) \simeq F(\mathbb{C}, n+1)$.

The foundation of the topic can be traced to the paper [35] by De Concini and Procesi. There the main objects are defined, the cohomology of the complement of a toric arrangement is studied (mainly from the point of view of algebraic geometry) and some applications of the theory are outlined. In particular, these authors treat the topic with the explicit goal of generalizing the theory of hyperplane arrangements, and they put all this in a wider context that encompasses applications in topics such as the study of integer points of Zonotopes and box splines. An extensive account of the work of De Concini and Procesi on this new subject can be found in their forthcoming book [36].

Ehrenborg, Readdy and Slone [52] take another point of view, studying toric arrangements on the "compact torus" $(S^1)^n$ and considering the problem of enumerating faces of the induced decomposition of the compact torus.

The next step is the work of Moci, in particular his papers [71], [72] and [73], developing the theory with a special focus on combinatorics. In particular, Moci introduces a two-variable polynomial that encodes enumerative invariants of many of the different objects populating the landscape outlined by De Concini and Procesi in [36]. The same author, in joint work with Settepanella [70], studied the homotopy type of the complement of a special class of toric arrangements (*thick* arrangements, see Section 8.2 below). In this work we will use a similar but more general approach, so that our results hold for a wider class of toric arrangements, which we call *complexified* because of structural affinity with the case of hyperplane arrangements.

Indeed, a rich and lively theory exists for arrangements of hyperplanes in affine complex space. An affine hyperplane is the (translate of the) kernel of a linear form. An affine arrangement is called complexified if the defining linear forms are real linear forms. Equivalently, a complexified arrangement induces an arrangement of real (affine) hyperplanes that determines it completely. It is from this equivalent formulation that we take inspiration for our definition of *complexified toric arrangements*: these are the arrangements that induce an arrangement in the compact torus and are determined by it. Every 'thick' arrangement in the sense of [70] is complexified, and there are nonthick complexified arrangements.

It is our explicit goal to try to present the theory and the results in a way that at once underlines the structural similarities with the theory of hyperplane arrangements and shows where (and why) the peculiarities of the toric theory are.

We will try to do so by using a combinatorial tool that aptly generalizes the idea of a poset and its order complex: *acyclic categories* and their nerves.

Our first main result shows that the combinatorial structure of a complexified toric arrangement can be used to construct an acyclic category whose nerve is homotopy equivalent to the complement of the arrangement. It is this acyclic category that we suggest to call *Salvetti category*. Accordingly, we suggest to call the complex obtained as the nerve of the Salvetti category the *Salvetti complex* of the toric arrangement. Our result specializes to the construction of [70] for the case of thick arrangements.

The second main result is the computation of a (finite) presentation for the fundamental group of the arrangement's complement, appearing here for the first time, to the best of our knowledge.

Our paper begins with a review of the relevant background facts about hyperplane arrangements and acyclic categories: this will be the content of Section 8.1. Then, in Section 8.2 we give a brief account of the theory of toric arrangements, with the special goal to set some notations, terminology and basic facts that will be relevant for the sequel. With Section 8.3 we will enter the core ouf our work, defining our combinatorial model (Definition 8.3.1) and proving our first main result (Theorem 8.3.3): the nerve of the Salvetti category models the homotopy type of any complexified toric arrangement. The computation of our presentation for the fundamental group will be carried out in Section 8.4, and the presentation itself will be given as our second main result, Theorem 8.4.21.

8.1 Background

Arrangements of hyperplanes

Before turning our attention to toric arrangements, let us briefly review some basics about hyperplane arrangements.

Let families of linear forms $l_1, \dots l_n \in \mathrm{Hom}(\mathbb{C}^d, \mathbb{C})$ and scalars $z_1, \dots z_n$ be given. For every $i = 1, \dots, n$ we have then an affine hyperplane

$$H_i := \{z \in \mathbb{C}^d : l_i(z) = z_i\}.$$

The *(affine) hyperplane arrangement* in $\mathbb{C}^d$ defined by the given linear forms and scalars is the set

$$\mathscr{A} = \{H_1, \dots, H_n\}.$$

The arrangement is called *complexified* if its defining forms are real, i.e., $l_i \in \mathrm{Hom}(\mathbb{R}^d, \mathbb{R})$ for all i.

There are several descriptions of the homotopy type of the complement of a set of hyperplanes in complex space. In this paper we will take inspiration by the work of Salvetti [86], where a regular polytopal complex which embeds in the complement of a complexified real arrangement as a deformation retract is constructed: the *Salvetti complex*.

Definition 8.1.1. Let $\mathscr{A}$ be a complexified real arrangement in $\mathbb{C}^n$. We write $\mathcal{D} = \mathcal{D}(\mathscr{A})$ for the cellular decomposition induced by $\mathscr{A}$ on $\mathbb{R}^n$ and $\mathcal{F} = \mathcal{F}(\mathscr{A})$ for its face poset (ordered by inclusion[1]). The maximal elements of $\mathcal{F}$ are called *chambers*.

Given a face $F \in \mathcal{F}$, we can consider the affine subspace $|F|$ it generates, say $|F| = y + L$ for a linear subspace L. The projection map $\pi_F : \mathbb{R}^n \to \mathbb{R}^n/L$ maps chambers of $\mathscr{A}$ on chambers of the arrangement

$$\mathscr{A}_F = \{\pi_F(H) : F \subseteq H\}. \tag{8.1.1}$$

We define the *Salvetti poset* $\mathrm{Sal}(\mathscr{A})$ on the element set

$$\{[F, C] : F, C \in \mathcal{D} \text{ and } F \leq C \text{ in } \mathcal{F}\}$$

by the order relation

$$[F_1, C_1] \leq [F_2, C_2] \iff F_2 \leq F_1 \text{ in } \mathcal{F} \text{ and } \pi_{F_1}(C_2) = \pi_{F_1}(C_1). \tag{8.1.2}$$

Definition 8.1.2. Let $\mathscr{A}$ be a complexified real arrangement in $\mathbb{C}^n$; the *Salvetti complex* of $\mathscr{A}$ is the simplicial complex $\mathcal{S} = \mathcal{S}(\mathscr{A}) := \Delta(\mathrm{Sal}(\mathscr{A}))$.

Proposition 8.1.3 (Salvetti [86])**.** *The complex $\mathcal{S}(\mathscr{A})$ is a deformation retract of the arrangement's complement, i.e., of the space $\mathbb{C}^d \setminus \bigcup_{i=1}^n H_i$*

The simplicial complex $\mathcal{S}$ is the barycentric subdivision of a regular polytopal complex that we want now to describe.

Consider the graph $\mathcal{G}(\mathscr{A})$ with the set of chambers of $\mathscr{A}$ as vertex set and edge set given by

$$E = \{e_{[F,C]} = (C, D) : F \in \mathcal{D},\, \mathrm{codim}\,(F) = 1,\, F \leq C,\, \mathrm{op}(C, F) = D\}$$

where $\mathrm{op}(C, F)$ is the *opposite chamber* of C with respect to F. We can assign a direction to an edge $e_{[F,C]}$ by thinking it oriented from C to $\mathrm{op}(C, F)$. We say that every edge $e_{[F,C]}$ of $\mathcal{G}(\mathscr{A})$ 'crosses' the hyperplane which supports F. A hyperplane H separates two chambers C and D if a straight line segment from any point in the interior of C to any point in the interior of D intersects H.

A path in $\mathcal{G}(\mathscr{A})$ from a vertex (chamber) C to a vertex (chamber) D is *positive minimal* if it is directed and if it never crosses any hyperplane more than once.

Definition 8.1.4. The *unsubdivided Salvetti complex* is the polytopal complex

[1] The reader should be aware that this is in contrast to some of the existing literature.

(i) whose 1-skeleton is the realization of the graph $\mathcal{G}(\mathscr{A})$;

(ii) whose k-cells corresponds to the pairs $[F, C]$ with $F \in \mathcal{F}(\mathscr{A})$ a face of codimension k and C a chamber with $F \leq C$;

(iii) where the 1-skeleton of a k-cell $e_{[F,C]}$ is attached along the *minimal positive directed paths* in $\mathcal{G}(\mathscr{A})$ from C to $\mathcal{OP}(C, F)$.

The reader can now easily convince her- or himself that condition (8.1.2) states exactly when a cell $e_{[F_1,C_1]}$ lies in the boundary of the cell $e_{[F_2,C_2]}$ in the unsubdivided Salvetti complex. In other words, the poset $\mathrm{Sal}(\mathscr{A})$ is the face poset of the unsubdivided Salvetti complex (and hence $\mathcal{S}$ is its barycentric subdivision).

We close this section by noting that the coarser structure of the unsubdivided complex has been used already in the seminal paper by Salvetti [86] to compute the fundamental group of the complement of a complexified hyperplane arrangement. We will return to this topic and review the techniques introduced by Salvetti when we will compute our presentation for the fundamental group of complexified toric arrangements.

Acyclic categories

Let us now introduce the idea of acyclic categories. We can think of acyclic categories as posets in which more than one relation between two elements is allowed. Our main general reference for this topic is Kozlov's book [65] and, for specifics about actions of infinite groups, Babson and Kozlov's paper [9].

Definition 8.1.5. An *acyclic category* is a small category C, such that:

(i) the only morphisms that have inverses are the identities;

(ii) the only morphism from an object to itself is the identity.

We will write $\mathcal{O}(C)$ for the objects of C and $\mathcal{M}(C)$ for its morphisms.

Acyclic categories occur sometimes in the literature as "loop-free categories" or "scwol"s (small category without loops, cfr. [25]).

The nerve

To an acyclic category we can associate its *nerve*. This is the generalization of the order complex of a poset. Meaning that, if the category is indeed a poset (that is, between two arbitrary objects there is at most a morphism), then its nerve is indeed its order complex. In general, however, the nerve of an acyclic category will not be a simplicial complex. Instead it will be a *regular trisp*. Trisps –also called Δ-*complexes* in [60]– are a generalization of simplicial complexes.

To define trisps we start with the notion of a *polytopal complex*. This is, roughly speaking, a complex obtained gluing polytopal cells. We will follow Kozlov's book ([65, Definition 2.39]), except that we don't require polytopal complexes to be *regular*. More precisely:

Definition 8.1.6. A *polytopal complex* is a topological space X obtained with the following construction:

(i) Start with the 0-skeleton X_0, a discrete set of points.

(ii) At the k-th step we attach all the k-dimensional faces. These are convex polytopes $P \subseteq \mathbb{R}^k$, attached along the maps $f : \partial P \to X_{k-1}$. The *attaching maps* are required to be *cellular*. Furthermore, the interior of each face of P has to be attached homeomorphically to the interior of a face in X_{k-1}. The k-skeleton is defined as

$$X_k = \left(\bigsqcup P \sqcup X_{k-1}\right) /_{x\sim f(x)}$$

(iii) We define $X = \cup_{k\in\mathbb{N}} X_k$.

A *trisp* can be described then as a polytopal complex in which every cell is a simplex. For more details about trisps and for the precise definition we refer to [65].

Having introduced trisps, we can now define the nerve of an acyclic category.

Definition 8.1.7. Let C be an acyclic category; the *nerve* $\Delta(C)$ is the trisp

(i) whose k-dimensional simplexes are k-length chains of composable morphisms
$$\sigma = a_0 \stackrel{m_1}{\to} a_1 \stackrel{m_2}{\to} a_2 \stackrel{m_3}{\to} \cdots \stackrel{m_k}{\to} a_k,$$

(ii) where the boundary simplexes of a simplex σ as above are defined as:

$$\begin{aligned}
\partial_0\sigma &= a_1 \stackrel{m_2}{\to} a_2 \stackrel{m_3}{\to} \cdots \stackrel{m_k}{\to} a_k \\
\partial_j\sigma &= a_0 \stackrel{m_1}{\to} \cdots \stackrel{m_{j-1}}{\to} a_{j-1} \stackrel{m_{j+1}\circ m_j}{\to} a_{j+1} \stackrel{m_{j+2}}{\to} \cdots \stackrel{m_k}{\to} a_k \\
\partial_k\sigma &= a_0 \stackrel{m_1}{\to} a_1 \stackrel{m_2}{\to} a_2 \stackrel{m_3}{\to} \cdots \stackrel{m_{k-1}}{\to} a_{k-1}
\end{aligned}$$

Face category

Acyclic categories can be used to describe the topology of a polytopal complex. For this section we refer to [25, III $\mathcal{C}$.1].

Definition 8.1.8. Let X be a polytopal complex; its *face category* is the acyclic category $\mathcal{F}(X)$

(i) whose set of objects $\mathcal{O}(\mathcal{F}(X))$ corresponds to the set of cells of X,

(ii) where for every cell P of X and for every face F of the polytope P there is a morphism $m_{P,F} : Q \to P \in \mathcal{M}(\mathcal{F}(X))$, where Q is the face of X upon which F is glued,

(iii) where if $P_3 \stackrel{m_{P_2,F_2}}{\to} P_2 \stackrel{m_{P_1,F_1}}{\to} P_1$ is a composable chain of morphisms in $\mathcal{F}(X)$, then
$$m_{P_1,F_1} \circ m_{P_2,F_2} = m_{P_1,F'}$$
(here F' is the face of F_1 which is glued upon $F_2 \subseteq P_2$, and hence upon P_3).

Remark 8.1.9. We notice that in point (iii) of definition 8.1.8 the face F' is uniquely determined, since the (restriction of the) gluing map $F_2 \to P_2$ is a cellular homeomophism.

Definition 8.1.10. The *barycentric subdivision* of a polytopal complex X, is the regular trisp $\mathcal{B}(X) = \Delta(\mathcal{F}(X))$: the nerve of the face category.

The face category describes the topology of a polytopal complex in the following sense:

Proposition 8.1.11. *Let X be a polytopal complex, then the geometric realization of $\mathcal{B}(X)$ is homeomorphic to X.*

These concepts have been already used in metric geometry and especially in geometric group theory. There acyclic categories are called *scwol*s, the nerve of a category is called the *geometric realization* and the face category of a polytopal complex is called the *barycentric subdivision.* More details can be found in [25, III$\mathcal{C}$].

8.2 Toric arrangements

We will now introduce toric arrangements together with some construction that will be needed in the following.

The n-dimensional *complex* torus is the space $(\mathbb{C}^*)^n$; the n-dimensional *compact* torus is $(S^1)^n$. A *character* of a complex torus T is an *affine homomorphism* $\chi : T \to \mathbb{C}^*$, i.e., a *Laurent polynomial* in $\mathbb{C}[x_1^{\pm 1}, \ldots x_n^{\pm 1}]$ that is also a group homorphism with respect to the complex multiplication. One can easily see that, then, χ is a Laurent *monomial* and for $x \in T$ we have

$$\chi(x) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad \text{with} \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n.$$

The correspondence between a character $\chi \in \Lambda$ and the associated integer vector α_χ makes the set of characters into a lattice $\Lambda \cong \mathbb{Z}^n$ with the operation defined by pointwis multiplication of characters.

The above, "concrete" definitions suffice for many purposes. It is however convenient for us and common in the literature to give a more abstract definition, starting with any (finitely generated) lattice Λ, which will be our character lattice. We then define the corresponding torus to be

$$T_\Lambda := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*).$$

Choosing a basis for Λ gives an isomorphism $T_\Lambda \cong (\mathbb{C}^*)^{\mathrm{rk}\,\Lambda}$ whose components are the evaluation maps on the elements of the basis. Analogously, the *compact* torus on the lattice Λ is defined as $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, S^1)$.

Definition 8.2.1. A *complexified toric arrangement* is a finite collection

$$\mathscr{A} = \{(\chi, a) : \chi \in \Lambda,\, a \in S^1\},$$

where Λ is a finitely generated lattice. We may think of $\mathscr{A}$ as the arrangement of the hypersurfaces $H_{\chi,a} = \{x \in T_\Lambda : \chi(x) = a\}$, where (χ, a) runs over $\mathscr{A}$.

The *complement* of $\mathscr{A}$ is then

$$M(\mathscr{A}) := (\mathbb{C}^*)^n \setminus \bigcup_{(\chi,a)\in\mathscr{A}} H_{\chi,a}.$$

Remark 8.2.2. Toric arrangements were first defined in [35] as sets of pairs (χ, a) with $a \in \mathbb{C}^*$. Restricting the constants to S^1 allows for the same $\mathscr{A}$ to define an arrangement of subtori on the compact torus $(S^1)^n$ (since a Laurent monomial maps $(S^1)^n$ on S^1). The analogy with the case of complexified hyperplane arrangements motivates our terminology.

Definition 8.2.3. Let $\mathscr{A}$ be a complexified toric arrangement. With $\mathcal{D} = \mathcal{D}(\mathscr{A})$ we will denote the induced cell-decomposition of the compact torus $(S^1)^n$.

Remark 8.2.4. On the other hand, [72] and [70] define a toric arrangement as an arrangement of kernels of characters (thus requiring $a = 1$). This cuts out a whole class of arrangements (e.g. $\mathscr{A} = \{t = -1, s = -1\}$ in $(\mathbb{C}^*)^2$). Moreover one can have hypersurfaces with many connected components, which are not in general kernels of characters (e.g. $t^2 = 1$).

Definition 8.2.5. A toric arrangement $\mathscr{A}$ on a k-dimensional torus T_Λ is called *essential* if

$$\operatorname{rk} \mathscr{A} := \operatorname{rk} \left\langle \chi \in \Lambda : \, (\chi, a) \in \mathscr{A} \text{ for some } a \in S^1 \right\rangle = k.$$

This can be stated equivalently by saying that the layers of maximal codimension are points.

Remark 8.2.6. Consider a (non essential) arrangement $\mathscr{A} = \{(\chi_1, a_1), \ldots, (\chi_n, a_n)\}$ with $\operatorname{rk} \mathscr{A} = l < k$. Then there exists an essential arrangement $\mathscr{A}'$ (the *essentialisation* of $\mathscr{A}$) such that

$$M(\mathscr{A}) = M(\mathscr{A}') \times (\mathbb{C}^*)^{k-l}.$$

With the notation of Definition 8.2.10, $\mathscr{A}' = \mathscr{A}_\Gamma$ where

$$\Gamma = \{\chi \in \Lambda : \quad \exists k \in \mathbb{Z} : \, \chi^k \in \langle \chi_1, \ldots, \chi_n \rangle\}.$$

In other words, it is not restrictive to consider essential arrangements.

Assumption 1. Unless otherwise stated, we will always assume our arrangement to be *complexified* and *essential*.

Remark 8.2.7. As is the case in the theory of hyperplane arrangements, one of the goals of the study of toric arrangements is to relate topological properties of the complement $M(\mathscr{A})$ to the *combinatorics* of the arrangement $\mathscr{A}$. In the hyperplane case, the combinatorics is expressed by the *poset of intersections* $\mathcal{L}(\mathscr{A})$ of elements of $\mathscr{A}$. In the case of toric arrangements, the results of [35] suggest that the right combinatorial invariant may be the *poset of layers* $\mathcal{C}(\mathscr{A})$, where a *layer* is a connected component of an intersection of hypersurfaces $H_{\chi,a}$, and the partial order is given by inclusion.

In the case of hyperplane arrangements, $\mathcal{L}(\mathscr{A})$ does not suffice to determine the homotopy type of the complement: indeed, there are explicit examples of arrangements with isomorphic intersection poset, whose complements are not homeomorphic (see [85]). In the case of a complexified real hyperplane arrangement, the homeomorphism type of the complement is determined instead by the *face poset* of the induced (regular CW) decomposition $\mathcal{D}(\mathscr{A})$ of $\mathbb{R}^n$.

In general, the homotopy type of a complexified toric arrangement cannot be described in terms of the face poset of the induced decomposition of the compact torus. Indeed Moci and Settepanella in [70] characterize exactly the arrangements for which this poset describes the homotopy type of $M(\mathscr{A})$: these are the arrangements $\mathscr{A}$ for which $\mathcal{D}(\mathscr{A})$ is a regular cell-complex or, in the terminology of [70], *thick* arrangements.

In our take at this matter we would like to keep full generality and therefore suggest to replace the poset of faces with the following more general object.

Definition 8.2.8. Let $\mathscr{A}$ be a complexified toric arrangement. Then $\mathcal{F}(\mathscr{A})$ will denote the *face category* of the complex $\mathcal{D}(\mathscr{A})$ (see Definition 8.2.3).

Remark 8.2.9. Thick arrangements are precisely those arrangement for which the face category $\mathcal{F}(\mathscr{A})$ is a poset. For such arrangements the construction of the Salvetti complex in the affine case translates almost literally to the toric case (see [70] for the details).

Our construction is more general in the sense that it does not assume thickness and, moreover, in the thick case it specializes to the complex considered by Moci and Settepanella.

Restriction

The operation of passing to sub arrangements, while intuitive and elementary in the case of hyperplane arrangements, needs some careful consideration in the toric case.

Let Γ be a subgroup of the lattice Λ. Then $T_\Gamma := \mathrm{Hom}_{\mathbb{Z}}(\Gamma, S^1)$ is a compact $(\mathrm{rk}\,\Gamma)$-torus and the inclusion $i_\Gamma : \Gamma \to \Lambda$ induces a surjection $\pi_\Gamma : T_\Lambda \to T_\Gamma$ given by restriction: $\pi_\Gamma(p) = p_{|\Gamma}$.

Definition 8.2.10. Given a subgroup $\Gamma \subseteq \Lambda$ and an arrangement $\mathscr{A}$ in T_Λ, we define the arrangement

$$\mathscr{A}_\Gamma = \{(\chi, a) \in \mathscr{A} : \chi \in \Gamma\}.$$

Proposition 8.2.11. *The map* $\pi_\Gamma : T_\Lambda \to T_\Gamma$ *induces a* cellular map π_Γ^{cell} : $\mathcal{D}(\mathscr{A}) \to \mathcal{D}(\mathscr{A}_\Gamma)$.

Proof. We can choose a basis $x_1, \dots, x_n$ for Λ such that $\Gamma = \langle x_1^{k_1}, \dots, x_l^{k_l} \rangle$. The isomorphism $T_\Lambda \simeq \mathbb{C}^n$ is given by evaluation on the chosen basis: $p \mapsto (p(x_1), \dots p(x_n))$. Therefore the projection $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^l$ is given by the map $(y_1, \dots, y_n) \mapsto (y_1^{k_1}, \dots, y_l^{k_l})$. This map is continuous and maps hypersurfaces (of $\mathscr{A}_\Gamma \subseteq \mathscr{A}$ in $(\mathbb{C}^*)^n$) onto hypersurfaces (of $\mathscr{A}_\Gamma$ in $(\mathbb{C}^*)^l$), hence is cellular. $\square$

The construction of $\mathscr{A}_\Gamma$ is to be thought of as the analogue of the quotient construction in (8.1.1). In particular, given any face $F \in \mathcal{F}(\mathscr{A})$ we can let Γ be the lattice

$$\Lambda_F := \{\chi \in \Lambda \mid \chi \text{ is constant on } F\}.$$

Correspondingly, we obtain a toric subarrangement with an associated cellular map:

$$\mathscr{A}_F := \mathscr{A}_{\Lambda_F}, \qquad \pi_F := \pi_{\Lambda_F}^{\mathrm{cell}} : \mathcal{D}(\mathscr{A}) \to \mathcal{D}(\mathscr{A}_F). \tag{8.2.1}$$

The fact that π_F is cellular implies that π_F induces a morphism of acyclic categories $\pi_F : \mathcal{F}(\mathscr{A}) \to \mathcal{F}(\mathscr{A}_F)$.

Covering spaces

In order to connect the theory of toric arrangements to that of hyperplane arrangements, we will look at a particular *covering space* of a toric arrangament complement. Again, for our purposes it is convenient to work with abstract tori.

Consider the following covering map

$$p : \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*)$$
$$\varphi \mapsto \exp \circ \varphi$$

where $\exp : \mathbb{C} \to \mathbb{C}^*$ is the exponential map, i.e., $z \mapsto e^{2\pi i z}$. Notice that $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) \cong \mathbb{C}^n$ and, through this isomorphism, p is just the universal covering map

$$(t_1, \ldots, t_n) \mapsto (e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})$$

of the torus T_Λ. Furthermore, p restricts to a universal covering map

$$\mathbb{R}^n \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, S^1) \cong (S^1)^n$$

of the compact torus, under which the preimage of a toric arrangement $\mathscr{A}$ is the (infinite) affine hyperplane arrangement

$$\mathscr{A}^\upharpoonright = \{(\chi, a') \in \Lambda \times \mathbb{R} \mid (\chi, e^{2\pi i a'}) \in \mathscr{A}\},$$

or, in coordinates:

$$\mathscr{A}^\upharpoonright = \{\langle \alpha, x \rangle = a' \mid (x^\alpha, e^{2\pi i a'}) \in \mathscr{A}\}.$$

Here $\alpha \in \mathbb{Z}^n$ and x^α is the associated character $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. With this definition p induces a cellular map $p : \mathcal{D}(\mathscr{A}^\upharpoonright) \to \mathcal{D}(\mathscr{A})$.

The arrangement $\mathscr{A}^\upharpoonright$ is a locally finite complexified affine hyperplane arrangement and therefore admits a Salvetti complex

$$\mathcal{S}^\upharpoonright = \mathcal{S}^\upharpoonright(\mathscr{A}) := \mathcal{S}(\mathscr{A}^\upharpoonright).$$

The character lattice Λ acts cellulary on $\mathcal{S}^\upharpoonright$ and continously on the covering space $M(\mathscr{A})$. These two actions are compatible, meaning that the embedding $\mathcal{S}^\upharpoonright \to M(\mathscr{A}^\upharpoonright)$ constructed in [86] is Λ-equivariant (more precisely, it can be so constructed).

Example 8.2.12. Figure 8.1 shows the Salvetti complex for the arrangement $\mathscr{A}^\upharpoonright$, with $\mathscr{A} = \{(ts, 1), (ts^{-1}, 1)\}$. The green cells belong to the same Λ-orbit.

With the previous constructions in mind, we can now restate a key result of [70].

Proposition 8.2.13 ([70, Lemma 1.1]). *Let $\mathscr{A}$ be an essential toric arrangement; the embedding $\mathcal{S}^\upharpoonright \to M(\mathscr{A}^\upharpoonright)$ induces an embedding*

$$\mathcal{S}^\upharpoonright/\Lambda \to M(\mathscr{A})$$

of the quotient $\mathcal{S}^\upharpoonright$ in the complement $M(\mathscr{A})$ as a deformation retract.

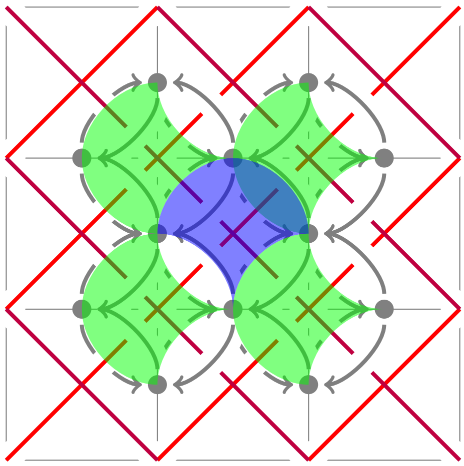


Figure 8.1: Salvetti Complex for $\mathscr{A}^{\upharpoonright}$

Remark 8.2.14. In the proof of Proposition 8.2.13 given in [70] the hypotesis of essentiality is required. Indeed the construction of the homotopy inverse $\psi : \mathcal{S}^{\upharpoonright}/\Lambda \to M(\mathscr{A})$ does not work for non-essential arrangements.

8.3 Toric Salvetti complex

We now head towards the first main theorem of this paper, introducing the notion of Salvetti complex for general complexified toric arrangements with a construction that specializes to the complex of [70] in the case of thick arrangements.

Definition 8.3.1 (Salvetti category)**.** Let $\mathscr{A}$ be a toric arrangement on $(\mathbb{C}^*)^n$. The *Salvetti Category* of $\mathscr{A}$ is the acyclic category $\zeta = \zeta(\mathscr{A})$ defined as follows:

(i) the objects are the morphisms in $\mathcal{F}(\mathscr{A})$ between faces and chambers

$$\mathcal{O}(\zeta) = \{m : F \to C : \quad m \in \mathcal{M}(\mathcal{F}(\mathscr{A})),\, C \text{ chamber}\};$$

(ii) for every morphism $n : F_2 \to F_1$ in $\mathcal{F}(\mathscr{A})$, and for every pair $m_1 : F_1 \to C_1$, $m_2 : F_2 \to C_2$ in $\mathcal{O}(\zeta)$ there is a morphism $(n, m_1, m_2) : m_1 \to m_2$ if and only if

$$\pi_{F_1}(m_1) = \pi_{F_1}(m_2); \tag{8.3.1}$$

where π_{F_1} is the morphism of face categories induced by the cellular map in (8.2.1);

(iii) let $m_i : F_i \to C_i$ for $i = 1, 2, 3$ be elements in $\mathcal{O}(\zeta)$, suppose the pairs (m_1, m_2) and (m_1, m_3) satisfy condition (8.3.1), then the pair (m_1, m_3) satisfies the same condition and we can define for morphisms $n : F_2 \to F_1$, $n' : F_3 \to F_2$ the composition

$$(n', m_2, m_3) \circ (n, m_1, m_2) = (n \circ n', m_1, m_3).$$

Definition 8.3.2. Let $\mathscr{A}$ be a toric arrangement; its *Salvetti complex* is the nerve $\Delta(\zeta(\mathscr{A}))$.

We can now state the main theorem of this section.

Theorem 8.3.3. *Let Λ be a lattice and $\mathscr{A}$ be a complexified toric arrangement in T_Λ. The nerve $\Delta(\zeta(\mathscr{A}))$ embeds in $M(\mathscr{A})$ as a deformation retract.*

Remark 8.3.4. Being the nerve of an acyclic category, $\Delta(\zeta(\mathscr{A}))$ is a regular trisp.

Remark 8.3.5. In the case of affine arrangements of hyperplanes, the Salvetti poset defined in Section 8.1 is indeed the poset of cells of a regular CW-complex, of which the (simplicial) Salvetti complex is the barycentric subdivision. Earlier we have called this the "unsubdivided" Salvetti complex. Our goal now is to describe a CW complex of which the nerve $\Delta(\zeta)$ is the barycentric subdivision. This complex will not be regular in general, but the resulting economy in terms of cells will come in handy in the following considerations.

Let then $\mathscr{A}$ denote a toric arrangement. Every cell of the unsubdivided Salvetti complex of $\mathscr{A}^\upharpoonright$ corresponds to the topological closure of the star of a vertex $[F, C]$ of the subdivided complex. Because the projection $\mathrm{Sal}(\mathscr{A}^\upharpoonright) \to \zeta$ is a covering of categories, the interior of the star of any vertex of the nerve $\Delta(\mathrm{Sal}(\mathscr{A}^\upharpoonright))$ is mapped homeomorphically to the interior of the star of its image. This gives a canonical CW-structure on $\Delta(\zeta)$. The acyclic category ζ is precisely the face category of the resulting CW complex.

In particular, the explicit determination of the boundary maps of this complex is now reduced to a straightforward computation.

Before we can get to the proof, some preparatory considerations are in order.

Restriction vs. covering

In order to proceed with the argument we still need to spend a few words on the quotient construction of (8.1.1) and its toric analogue.

Let F be a face of $\mathcal{D}(\mathscr{A})$ and let Λ_F be the sublattice of characters in Λ that are constant on F. Every $\varphi \in \Lambda_F$ is then constant on the affine subspace spanned by F, which we write $y + L$ for $y \in \mathbb{R}^n$ and L a linear subspace of $\mathbb{R}^n$: therefore φ vanishes on L. Then we have an isomorphism

$$\rho : \; \mathbb{R}^n/L \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda_F, \mathbb{R}). \tag{8.3.2}$$

Recall from (8.2.1) the arrangement

$$\mathscr{A}_F = \{(\chi, a) \in \mathscr{A} : \chi \in \Lambda_F\} \subseteq \mathscr{A}$$

in $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_F, \mathbb{R})$. The isomorphism ρ from (8.3.2) does not map the arrangement $(\mathscr{A}^\upharpoonright)_F$ onto $(\mathscr{A}_F)^\upharpoonright$. Indeed $(\mathscr{A}_F)^\upharpoonright$ contains all the translates of the hyperplanes in $(\mathscr{A}^\upharpoonright)_F$. That is

$$(\mathscr{A}^\upharpoonright)_F \subseteq \mathscr{A}_F{}^\upharpoonright = \{(\chi, a + k) \mid (\chi, a) \in (\mathscr{A}^\upharpoonright)_F, \, k \in \mathbb{Z}\}$$

and therefore we have a natural cellular support map

$$s : \mathcal{D}(\mathscr{A}_F{}^\upharpoonright) \to \mathcal{D}(\mathscr{A}_F^\upharpoonright)$$

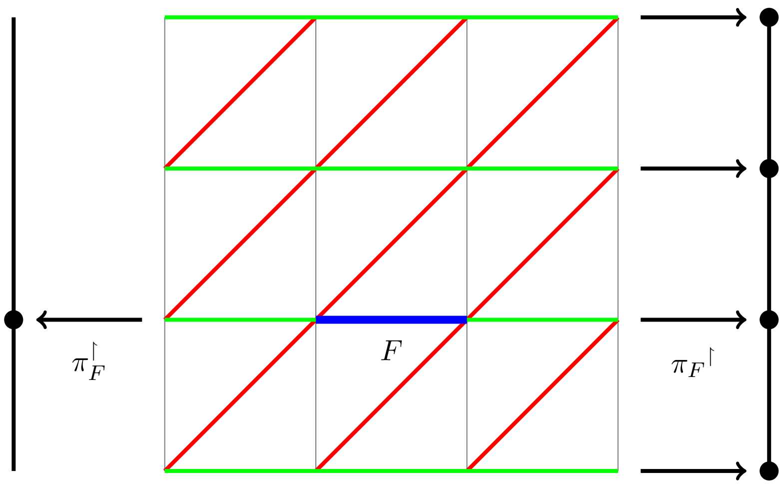


Figure 8.2: Restriction vs. Covering

The map π_F of (8.2.1) lifts (via p) to a map $\mathbb{R}^{rk\Lambda} \to \mathbb{R}^{rk\Lambda_F}$ which induces a cellular map $\pi_F{}^{\upharpoonleft} : \mathcal{D}(\mathscr{A}^{\upharpoonleft}) \to \mathcal{D}((\mathscr{A}_F)^{\upharpoonleft})$ and the following diagram commutes

$$\begin{tikzcd}
\mathcal{D}(\mathscr{A}^{\upharpoonleft}) \arrow[r, "\pi_F{}^{\upharpoonleft}"] \arrow[d, "p"'] & \mathcal{D}((\mathscr{A}_F)^{\upharpoonleft}) \arrow[d, "p"] \\
\mathcal{D}(\mathscr{A}) \arrow[r, "\pi_F"'] & \mathcal{D}(\mathscr{A}_F)
\end{tikzcd} \tag{8.3.3}$$

On the other hand, in $\mathrm{Hom}(\Lambda, \mathbb{R})$ we have the projection from (8.1.2), which we call $\pi^{\upharpoonleft}_F$ and in terms of which the Salvetti complex of $\mathscr{A}^{\upharpoonleft}$ is defined, which is

$$\pi^{\upharpoonleft}_F : \mathcal{D}(\mathscr{A}^{\upharpoonleft}) \to \mathcal{D}((\mathscr{A}^{\upharpoonleft})_F)$$

and is related to $\pi_F{}^{\upharpoonleft}$ via

$$\pi^{\upharpoonleft}_F = s \circ \pi_F{}^{\upharpoonleft}.$$

Figure 8.2 shows an example of projections $\pi^{\upharpoonleft}_F$ and $\pi_F{}^{\upharpoonleft}$.

Lemma 8.3.6. *Let $F_1, F_2, C_1, C_2 \in \mathcal{F}(\mathscr{A}^{\upharpoonleft})$ with C_1, C_2 chambers, $F_1 \leq C_1$ and $F_1 \leq F_2 \leq C_2$. Then*

$$\pi_{F_1}{}^{\upharpoonleft}(C_1) = \pi_{F_1}{}^{\upharpoonleft}(C_2) \iff \pi^{\upharpoonleft}_{F_1}(C_1) = \pi^{\upharpoonleft}_{F_1}(C_2).$$

Proof. The direction $\Rightarrow$ follows since $\pi^{\upharpoonleft}_F = s \circ \pi_F{}^{\upharpoonleft}$. For $\Leftarrow$: if $\pi^{\upharpoonleft}_{F_1}(C_1) = \pi^{\upharpoonleft}_{F_1}(C_2)$, then $\pi_{F_1}{}^{\upharpoonleft}(C_1) = \pi_{F_1}{}^{\upharpoonleft}(C_2 + \lambda)$, for some $\lambda \in \Lambda_F$. But since F_2 is a common face of C_1 and C_2, it has to be $\lambda = 0$. $\square$

Corollary 8.3.7. *Let $[F_1, C_1]$, $[F_2, C_2]$ denote two elements of $Sal\mathscr{A}^{\upharpoonleft}$, the Salvetti poset of $\mathscr{A}^{\upharpoonleft}$. Then*

$$[F_1, C_1] \leq [F_2, C_2] \iff F_1 \geq F_2 \text{ in } \mathcal{F}(\mathscr{A}) \text{ and } \pi_{F_1}{}^{\upharpoonleft}(C_1) = \pi_{F_1}{}^{\upharpoonleft}(C_2)$$

Quotients

Our strategy for the proof of Theorem 8.3.3 will be to prove that the toric Salvetti complex $\Delta(\zeta)$ is the quotient of the action $\Lambda \curvearrowright \mathcal{S}^{\upharpoonright}$ in the category of trisps. For this, we need first to take care of some ground work.

Lemma 8.3.8. *Let $\mathscr{A}$ be a complexified toric arrangement. Then there is a covering $q : \mathcal{F}(\mathscr{A}^{\upharpoonright}) \to \mathcal{F}(\mathscr{A})$ of acyclic categories with Galois group Λ and*

$$\mathcal{F}(\mathscr{A}) = \mathcal{F}(\mathscr{A}^{\upharpoonright})/\Lambda$$

as a quotient of acyclic categories.

Proof. Let $F \in \mathcal{D}(\mathscr{A}^{\upharpoonright})$ be a face of the affine arrangement $\mathscr{A}^{\upharpoonright}$. In particular F is a polytope and $p(F) \in \mathcal{D}(\mathscr{A})$ is a face of $\mathscr{A}$. We can then use F a polytopal model of $p(F)$ in Definition 8.1.8 and map a morphism $F' \leq F$ to the corresponding morphism $m_{F',F}$.

This defines a functor $q : \mathcal{F}(\mathscr{A}^{\upharpoonright}) \to \mathcal{F}(\mathscr{A})$. Furthermore q is a covering of categories in the sense of [25, Definition A.15] with Λ as automorphism group and Λ acts transitively on the fibers of q. It then follows that $\mathcal{F}(\mathscr{A}) \cong \mathcal{F}(\mathscr{A}^{\upharpoonright})/\Lambda$. □

In particular, we note the following consequence.

Corollary 8.3.9. *The morphisms in $\mathcal{F}(\mathscr{A})$ correspond to the orbits* $\{\Lambda(F_1 \leq F_2) \mid F_1, F_2 \in \mathcal{D}(\mathscr{A}^{\upharpoonright})\}$.

Now we can prove a key lemma, finally making sense of our definition of ζ.

Lemma 8.3.10. *The category ζ is the quotient Sal$(\mathscr{A}^{\upharpoonright})/\Lambda$ in the category of acyclic categories.*

Proof. We first need to construct a projection, i.e., a functor $\Pi : \mathrm{Sal}\,(\mathscr{A}^{\upharpoonright}) \to \zeta$. Recall that the objects of $\mathrm{Sal}\,(\mathscr{A}^{\upharpoonright})$ are of the form $[F, C]$ with $F, C \in \mathcal{F}(\mathscr{A}^{\upharpoonright})$, $F \leq C$, and C a chamber of $\mathscr{A}^{\upharpoonright}$. Also, from the proof of Lemma 8.3.8 we recall the projection $q : \mathcal{F}(\mathscr{A}^{\upharpoonright}) \to \mathcal{F}(\mathscr{A})$. It is now possible to define Π on the objects as follows:

$$\Pi([F, C]) = q(F \leq C) : q(F) \to q(C).$$

According to Corollary 8.3.7, relations in $\mathcal{F}(\mathscr{A}^{\upharpoonright})$ are of the form $[F_1, C_1] \leq [F_2, C_2]$ where $F_2 \leq F_1$ and $\pi_{F_1}{}^{\upharpoonright}(C_1) = \pi_{F_1}{}^{\upharpoonright}(C_2)$.

On the other hand, morphisms in $\zeta(\mathscr{A})$ are given by triples (n, m_1, m_2) where $m_1 : F_1 \to C_2$, $m_2 : F_2 \to C_2$ are objects of ζ, $n : F_2 \to F_1$ is a morphism in $\mathcal{F}(\mathscr{A})$ and the following condition holds:

$$\pi_{F_1}(m_1) = \pi_{F_1}(m_2).$$

Therefore, in order to able to map a relation $[F_1, C_1] \leq [F_2, C_2]$ to the morphism $(q(F_2 \leq F_1), \Pi([F_1, C_1]), \Pi([F_2, C_2]))$ and for this map to be surjective, we need to verify the following condition:

$$\pi_{F_1}{}^{\upharpoonright}(C_1) = \pi_{F_1}{}^{\upharpoonright}(C_2) \iff \pi_{q(F_1)}(\Pi([F_1, C_1])) = \pi_{q(F_1)}(\Pi([F_2, C_2])).$$

We go back to the diagram (8.3.3), and write the corresponding commutative diagram of face categories:

$$\begin{tikzcd} \mathcal{F}(\mathscr{A}\!\upharpoonright) \arrow[r, "\pi_{F_1}\upharpoonright"] \arrow[d, "q"'] & \mathcal{F}(\mathscr{A}_{F_1}\!\upharpoonright) \arrow[d, "q"] \\ \mathcal{F}(\mathscr{A}) \arrow[r, "\pi_{q(F_1)}"'] & F(\mathscr{A}_{q(F_1)}) \end{tikzcd}$$

Now $\pi_{F_1}\upharpoonright$ is a map of posets and since $\pi_{F_1}\upharpoonright(F_1) = \pi_{F_1}\upharpoonright(F_2)$ we have

$$\pi_{F_1}\upharpoonright(C_1) = \pi_{F_1}\upharpoonright(C_2) \iff \pi_{F_1}\upharpoonright(F_1 \le C_1) = \pi_{F_1}\upharpoonright(F_2 \le C_2).$$

Furthermore q is a covering of categories, in particular is injective on the morphisms incident on $\pi_{F_1}\upharpoonright(F_1)$. It then follows that

$$\begin{aligned} \pi_{F_1}\upharpoonright(F_1 \le C_1) = \pi_{F_1}\upharpoonright(F_2 \le C_2) &\Leftrightarrow q \circ \pi_{F_1}\upharpoonright(F_1 \le C_1) = q \circ \pi_{F_1}\upharpoonright(F_2 \le C_2) \\ &\Leftrightarrow \pi_{q(F_1)}(q(F_1 \le C_1)) = \pi_{q(F_1)}(q(F_2 \le C_2)). \end{aligned}$$

Concluding: the functor Π is well defined and it now follows easily from Lemma 8.3.8 that it is a Galois covering of acyclic categories with Λ as automorphism group. $\square$

We want to show that, in our particular case, the nerve construction commutes with the quotient. Babson and Kozlov in [9] give a necessary and sufficient condition for this:

Proposition 8.3.11 ([9, Theorem 3.4]). *Let $\mathcal{C}$ be an acyclic category equipped with a group action $G \curvearrowright \mathcal{C}$. A canonical isomorphism $\Delta(\mathcal{C})/G \cong \Delta(\mathcal{C}/G)$ exists if and only if the following condition is satisfied:*

> *Let $t \ge 2$ and let $(m_1, \dots, m_{t-1}, m_a)$, $(m_1, \dots, m_{t-1}, m_b)$ composable morphism chains. Let $Gm_a = Gm_b$, then ther exists some $g \in G$, such that $g(m_a) = m_b$ and $g(m_i) = m_i$, $\forall i \in \{1, \dots, t-1\}$.*

The next lemma ensures that we can apply the previous proposition to our case.

Lemma 8.3.12. *Let $\mathcal{C}$ be an acylic category and $G \curvearrowright \mathcal{C}$ act as the Galois group of a covering map. Then the condition of proposition 8.3.11 is satisfied.*

Proof. Consider two composable morphism chains as in the condition of proposition 8.3.11. Since $t \ge 2$ and the chains are composable, m_a and m_b must have the same domain, $m_a : p \to q$, $m_b : p \to r$. Furthermore there is a $g \in G$, such that $m_b = gm_a$.

Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a covering map with Galois group G. Then $\varphi(m_a) = \varphi(m_b) \Rightarrow m_a = m_b$ and the condition is trivially satisfied. $\square$

We finally get to the proof of Theorem 8.3.3, which now follows as an application of the previous considerations.

Proof of Theorem 8.3.3. According to proposition 8.2.13 the statement holds for the complex $S^{\upharpoonright}/\Lambda = \Delta(\mathrm{Sal}\,\mathscr{A}^{\upharpoonright})/\Lambda$. The lattice Λ acts on $S^{\upharpoonright}$ as the automorphism group of a covering map, in particular lemma 8.3.12 holds and we have:

$$S^{\upharpoonright}/\Lambda = \Delta(\mathrm{Sal}\,\mathscr{A}^{\upharpoonright})/\Lambda \cong \Delta(\mathrm{Sal}\,\mathscr{A}^{\upharpoonright}/\Lambda) \cong \Delta(\zeta).$$

□

8.4 The fundamental group

As an application of the results of the previous sections, and in a structural tribute to the seminal paper of Salvetti [86], we would like to give a presentation for the fundamental group of a complexified toric arrangement.

Product structure

First, note that the inclusion $M(\mathscr{A}) \to T_\Lambda$ induces an epimorphism of groups

$$\varepsilon : \pi_1(M(\mathscr{A})) \to \pi_1(T_\Lambda) \simeq \mathbb{Z}^n.$$

Lemma 8.4.1. *The map ε has a section ξ.*

Proof. Choose a point $y \in \mathbb{R}^n$ in a chamber of $\mathscr{A}^{\upharpoonright}$. Then for all choices of $x \in \mathbb{R}^n$ we have

$$x + iy \in M(\mathscr{A}^{\upharpoonright}).$$

Accordingly, for every choice of arguments $\theta_1, \ldots, \theta_n \in \mathbb{R}$,

$$(\lambda_1 e^{2\pi i \theta_1}, \ldots, \lambda_n e^{2\pi i \theta_n}) \in M(\mathscr{A})$$

where, for all $j = 1, \ldots, n$, $\lambda_j := e^{-2\pi y_j}$ This defines a map

$$f : T_\Lambda \to M(\mathscr{A}), \quad z \mapsto (\lambda_1 e^{2\pi i \arg z_1}, \ldots, \lambda_n e^{2\pi i \arg z_n})$$

that induces a homomorphism

$$\xi : \pi_1(T_\Lambda) \to \pi_1(M(\mathscr{A})).$$

Since f is a homotopy (right-) inverse to the inclusion $M(\mathscr{A}) \to T_\Lambda$, $\varepsilon\xi = id$ and ξ is the required section. □

Lemma 8.4.2. *The sequence*

$$0 \to p_*(\pi_1(\mathcal{S}^{\upharpoonright})) \xrightarrow{\iota} \pi_1(M(\mathscr{A})) \xrightarrow{\varepsilon} \pi_1(T_\Lambda) \to 0$$

is split exact. Therefore

$$\pi_1(M(\mathscr{A})) \simeq \pi_1(\mathcal{S}^{\upharpoonright}) \rtimes \pi_1(T_\Lambda).$$

Proof. We already showed that the map ε has a section, we then need only to prove $\iota(p_*(\pi_1(\mathcal{S}^\upharpoonright))) = \mathrm{Ker}\ \varepsilon$. It is clear that $\iota(p_*(\pi_1(\mathcal{S}^\upharpoonright))) \subseteq \mathrm{Ker}\ \varepsilon$. For the opposite inclusion we consider the sequence

$$0 \to p_*(\pi_1(M(\mathscr{A}^\upharpoonright))) \to \pi_1(M(\mathscr{A})) \to \pi_1(T_\Lambda) \to 0$$

Let $[\gamma] \in \pi_1(M(\mathscr{A}))$ be an element of Ker ε. Let j be the inclusion of $M(\mathscr{A})$ in the ambient torus T_Λ. Then $j \circ \gamma$ is a null homotopic loop in T_Λ and lifts therefore to a closed path γ' in the universal cover $\mathbb{C}^n$. Let $\gamma^\upharpoonright$ be the lift of γ to $M(\mathscr{A}^\upharpoonright)$ with base point x, then $\gamma' = j^\upharpoonright \circ \gamma^\upharpoonright$ and $\gamma^\upharpoonright$ is also a closed path. That is, $[\gamma] = p_*[\gamma^\upharpoonright] \in p_*(\pi_1(M(\mathscr{A}^\upharpoonright))) \cong p_*(\pi_1(\mathcal{S}^\upharpoonright))$. □

Presentation of $\pi_1(M(\mathscr{A}^\upharpoonright))$

As a stepping stone towards the computation of a presentation for the fundamental group of $M(\mathscr{A})$, we establish some notation and recall the presentation of $\pi_1(\mathcal{S}^\upharpoonright)$ given by Salvetti in [86].

Choose - and from now fix - a chamber C_0 of $\mathscr{A}^\upharpoonright$, and let x_0 be a generic point in C_0 - i.e. such that for all $i = 1, \ldots, d$ the straight line segment s_i from x_0 to $u_i x_0$ meets only faces of codimension at most 1.

Remark 8.4.3. In general, given a set $\mathcal{K}$ of cells of a complex, $\mathcal{K}_i$ will denote the subset of cells of codimension i.

Also, to streamline notation we will from now write $\mathcal{F}$, respectively $\mathcal{F}^\upharpoonright$ for $\mathcal{F}(\mathscr{A})$, $\mathcal{F}(\mathscr{A}^\upharpoonright)$.

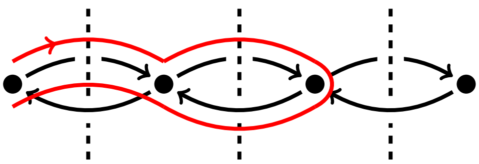


Figure 8.3: Generators, an example: $\beta_{F_2} = l_{F_1} l_{F_2}^2 l_{F_1}^{-1}$

8.4.0.1. Generators. Recall the graph $\mathcal{G}^\upharpoonright := \mathcal{G}(\mathscr{A}^\upharpoonright)$ of Definition 8.1.4. Here we will adopt a useful notational convention inspired by [86]: we will write edges of $\mathcal{G}^\upharpoonright$ as indexed by the face of codimension 1 they cross, and in writing a path we will write l_F for a crossing of F 'along the direction of the edge', l_F^{-1} for a crossing 'against the direction' of the edge. By specifying the first vertex of the path then there is no confusion about which edge is used, and in which direction.

A *positive* path then is a path of the form

$$l_{F_1} l_{F_2} \ldots l_{F_k}$$

for $F_1, \ldots F_k \in \mathcal{F}_1^\upharpoonright$. It is also *minimal* if the hyperplane supporting F_i is different from the hyperplane supporting F_j for all $i \neq j$.

Since any two positive minimal paths with same origin and same end are homotopic, given $C, C' \in \mathcal{F}_0^{\upharpoonleft}$ we will sometimes write $(C \to C')$ for the (class of) positive minimal paths starting at C and ending at C'.

For every $F \in \mathcal{F}_1^{\upharpoonleft}$ we define a path as follows:

$$\beta_F := (C_0 \to (C_0)_F) l_F^2 (C_0 \to (C_0)_F)^{-1}, \tag{8.4.1}$$

where, here and in the following, for a chamber C and a face F the expression C_F will denote the unique chamber in $\pi_F^{-1}(\pi_F(C))$ that contains F in its boundary.

Lemma 8.4.4 (p. 616 of [86]). *The group $\pi_1(\mathcal{S}^{\upharpoonleft})$ is generated by the set $\{\beta_F | F \in \mathscr{F}_1^{\upharpoonleft}\}$.*

Given a positive path $\nu = l_{F_1}, \dots, l_{F_k}$ define loops

$$\beta_{F_i}^{\nu} := l_{F_1} \cdots l_{F_{i-1}} l_{F_i}^2 l_{F_{i-1}}^{-1} \cdots l_{F_1}^{-1}. \tag{8.4.2}$$

Moreover, let $F_{j_1}, \dots, F_{j_l}$ be the sequence obtained from $F_1, \dots, F_k$ by recursively deleting faces F_j that are supported on a hyperplane which supports an odd number of elements of $F_{j+1}, \dots, F_k$ (compare [86, p. 614]) and define

$$\Sigma(\nu) := (F_{i_l}, \dots, F_{i_l}). \tag{8.4.3}$$

Lemma 8.4.5 (Lemma 12 in [86]). *Given a positive path $\nu = l_{F_1}, \cdots, l_{F_k}$ starting in the chamber C and ending in C' Then there is a homotopy*

$$\nu \simeq \Big(\prod_{G \in \Sigma(\nu)} \beta_G^{\nu}\Big)(C \to C').$$

From this Lemma another useful result follows.

Lemma 8.4.6 (Corollary 12 in [86]). *Let F,G be two faces of codimension* 1 *that are supported on the same hyperplane. Then β_F is homotopic to*

$$\Big(\prod_{i=1}^{h} \beta_{j_i}^{\nu}\Big) \beta_G \Big(\prod_{i=1}^{h} \beta_{j_i}^{\nu}\Big)^{-1},$$

where ν is a positive minimal path from C_0 to $\pi_G(C_0)$, and $j_1, \dots, j_h$ are the indices of the edges in ν that cross a hyperplane that does not separate C_0 from $\pi_F(C_0)$, in the order in which they appear in ν.

8.4.0.2. Relations. For every face $G \in \mathcal{F}_2^{\upharpoonleft}$ consider a chamber $C > G$ and let C' be its opposite chamber with respect to G. Consider a minimal positive path ω from C to C'. Let us then consider the set $h(G) := \{F_1, \dots F_k\}$ of the codimension 1 faces adjacent to G, indexed according to the order in which the positive minimal path ω 'crosses' them. This ordering is well defined up to cyclic permutation. Let now for $i = 1, \dots k$ F_{i+k} be the facet opposite to F_i with respect to G. Define a path

$$\alpha_G(C) := l_{F_1} l_{F_2} \dots l_{F_{2k}}. \tag{8.4.4}$$

Salvetti introduces a set of relations associated with G:

$$R_G: \quad \beta_{F_1} \dots \beta_{F_k} = \beta_{F_2} \dots \beta_{F_k} \beta_{F_1} = \dots$$

stating the equality of all cyclic permutations of the product. In fact, for every cyclic permutation σ of $\{1, \dots, k\}$

$$\beta_{F_{\sigma(1)}} \cdots \beta_{F_{\sigma(k)}} \simeq (C_0 \to \widetilde{C}) \alpha_G(\widetilde{C})(C_0 \to \widetilde{C})^{-1} \tag{8.4.5}$$

where $\widetilde{C} := (C_0)_G$ and $\simeq$ means homotopy.

8.4.0.3. Presentation. One of the results of [86] is that the fundamental group of $M(\mathscr{A}^{\upharpoonright})$ can be presented as

$$\pi_1(\mathcal{S}^{\upharpoonright}) = \langle \beta_F,\, F \in \mathcal{F}_1^{\upharpoonright} \mid R_G,\, G \in \mathcal{F}_2^{\upharpoonright} \rangle.$$

Generators

We describe the action of $u \in \Lambda$ on a path $\gamma \in \mathcal{G}^{\upharpoonright}$ by writing $u.\gamma$ for the path obtained by translation of γ with u.

Definition 8.4.7. Choose a basis $u_1, \dots u_n$ of Λ, and for $i = 1, \dots d$ let $\omega_i = \omega_i^{(1)}$ be the positive minimal path of $\mathcal{G}^{\upharpoonright}$ from C_0 to $u_i C_0$ obtained by crossing the faces met by the straight line segment s_i (which connects from x_0 to $u_i x_0$). Also, for $k \geq 1$ let $\omega_i^{(k)} = \omega_i(u_i.\omega_i^{(k-1)})$. Similarly, let $\omega_i^{(-1)} := \omega_i^{-1}$ and $\omega_i^{(-k)} := \omega_i^{(-1)}(u_i^{-1}.\omega_i^{(1-k)})$. Given any $u \in \Lambda$ write $u = u_1^{q_1} \cdots u_n^{q_n}$ and define

$$\omega_u := \omega_1^{(q_1)}\ u_1^{q_1}.\omega_2^{(q_2)}\ \cdots \big(\prod_{j=1}^{r-1} u_n^{q_n}\big).\omega_r^{(q_n)}. \tag{8.4.6}$$

Let then

$$\tau_i := p_*(\omega_i), \qquad \tau_u := p_*(\omega_u).$$

Notice that a path ω_u needs not be minimal, nor positive. In fact, it is positive if and only if u has nonnegative coordinates in Λ. Given i and k, the path $\omega_i^{(k)}$ is positive if and only if $k \geq 0$, and in this case it is also minimal.

Lemma 8.4.8. *In $\pi_1(\mathcal{M}(\mathscr{A}))$, $p(\omega_i^{(k)}) = \tau_i^k$ and $\tau_i \tau_j = \tau_j \tau_i$ for all i, j. The $\varepsilon_* \tau_i$ generate $\pi_1(T_\Lambda)$.*

Proof. Let $X = f(T_\Lambda)$ be the image of the map f in the proof of Lemma 8.4.1, where we now choose y to be a point of our base chamber C_0.

Let the straight line segment s_j be parametrized by

$$s_j(t) := t x_0 + (1-t) u_j x_0, \quad 0 \leq t \leq 1.$$

The Minkowski sum $X' := s_1 + \cdots + s_n \subset \mathbb{R}^n$ is a fundamental region for the action of Λ on $\mathbb{R}^n$. For $Y := X' + iy \subseteq M(\mathscr{A}^{\upharpoonright})$ we have $p(Y) = X$. In particular, the segments s_j map under ε to a system of generators of $\pi_1(T_\Lambda)$ - in fact, the one associated with the basis $u_1, \dots, u_n$ of Λ.

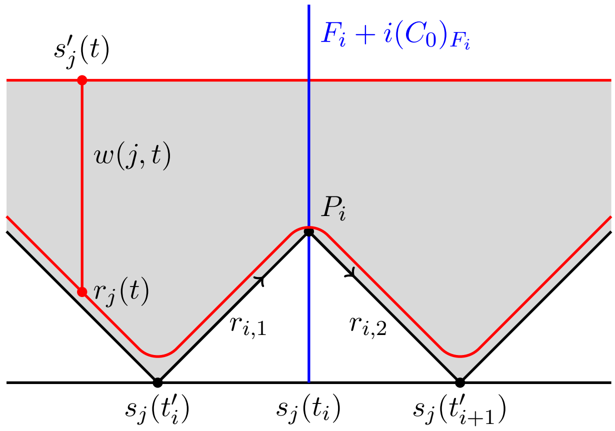


Figure 8.4: Construction for the proof of Lemma 8.4.8

We will next show that for all $j = 1, \ldots, d$ the path

$$s'_j(t) := s_j(t) + iy$$

is homotopic to the positive minimal path $\omega_j \in (C_0 \to u_j C_0)$.

Indeed, write $\omega_j = l_{F_1} \ldots l_{F_k}$ and let $t_1, \ldots t_k$ be such that $s_j(t_i) \in F_i$ for all $i = 1, \ldots, k$. Also, write C_i, C_{i+1} for the source and target chambers of l_{F_i} (note: $C_{k+1} = u_j C_0$) and for $i = 1, \ldots, k-1$ choose $t'_i \in]t_{i-1}, t_i[$, $t'_k := 1$, $t'_0 := 0$. Then $s'_j(t'_i) \in C_i$ for all $i = 1, \ldots, k$.

Recall now that the subset of $M(\mathscr{A}^{\restriction})$ with real part $x \in F$ consists of points with imaginary part belonging to the chambers of $A^{\restriction}_F$. In fact, the edge l_{F_i}, directed from C_i to C_{i+1}, is by construction ([86, p. 608]) the union of two segments, one from a point in $P'_i \in C_i + 0i$ to a point $P_i \in F + i(C_0)_F$, the other from P_i to a point $P'_{i+1} \in C_{i+1} + 0i$. We will parametrize these segments as $r_{i,1}(t)$, $t'_i \leq t \leq t_i$ and $r_{i,2}(t)$, $t_i \leq t \leq t'_{i+1}$. Together, they give a parametrization $r_j(t)$, $0 \leq t \leq 1$ of the positive minimal path ω_j.

The key observation is now that, having chosen $y \in C_0$, we have that

$$s_j(t_h) \in F + i(C_0)_F \text{ for all } h = 1, \ldots, k.$$

Since chambers of arrangements are convex, for all $t \in [0,1]$ there is a straight line segment $w(j,t)$ joining $s_j(t)$ and $r_j(t)$ in $M(\mathscr{A}^{\restriction})$.

The (topological) disk $W_j := \bigcup_{t \in [0,1]} w(j,t)$ defines the desired homotopy between s_j and ω_j.

Now fix $i, j \in \{1, \ldots, n\}$ clearly $s_i\, u_i.(s_j)$ is homotopic to $s_j\, u_j.(s_i)$, and in $\pi_1(M(\mathscr{A}))$ we thus have

$$\tau_i \tau_j = p_*([\omega_i\, u_i.\omega_j]) = p_*([s_i\, u_i.s_j])$$

$$= p_*([s_j\, u_j.s_i]) = p_*([\omega_j\, u_j.\omega_i]) = \tau_j \tau_i.$$

$\square$

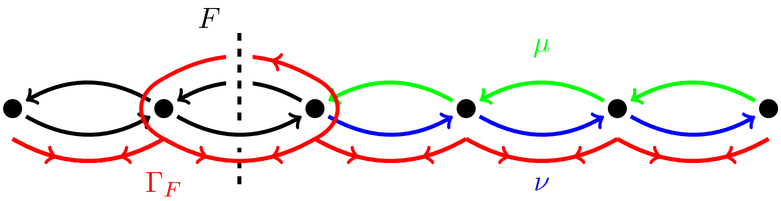


Figure 8.5: Construction for the proof of lemma 8.4.14

Definition 8.4.9. Let $\mathcal{Q}$ be the set of faces that intersect the fundamental region X' of the proof of Lemma 8.4.8. Then $\mathcal{Q}$ contains C_0 and x_0. Let $\mathcal{Q}_i := \mathcal{Q} \cap \mathcal{F}_i^{\upharpoonleft}$. In particular, $\mathcal{Q}_1$ contains the set of faces crossed by s_i, for all i.

Recall the parametrization $s_i(t)$ of the segments s_i, and call $\mathcal{B}$ the set of faces of the polyhedron X' which intersect the convex hull of $\{s_i([0,1[) \mid i \in I\}$ for some $I \subseteq \{1, \ldots, d\}$. Notice that every face of X' is a translate of some face in $\mathcal{B}$ by an element $u_1^{m_1} \cdots u_n^{m_n}$ with $m_1, \ldots, m_n \in \{0, 1\}$.

Definition 8.4.10. Let

$$\overline{\mathcal{F}^{\upharpoonleft}} := \{F \in \mathcal{Q} \mid F \cap B = \emptyset \text{ for all } B \notin \mathcal{B}\}$$

Then $\overline{\mathcal{F}^{\upharpoonleft}}$ is a set of representatives for the orbits of the action of Λ on $\mathcal{F}^{\upharpoonleft}$.

Definition 8.4.11. For any given $F \in \mathcal{F}^{\upharpoonleft}$ let $\overline{F}$ be the unique element of $\Lambda F \cap \overline{\mathcal{F}^{\upharpoonleft}}$. Then, call u_F the unique element of Λ such that $F = u_F \overline{F}$.

Define

$$\Gamma_F := \omega_{u_F}(u_F.\beta_{\overline{F}})\omega_{u_F}^{-1}$$

Remark 8.4.12.

(1) For all $F \in \mathcal{F}_1^{\upharpoonleft}$ and all $u \in \Lambda$

$$p_*(\Gamma_{uF}) = \tau_u p_*(\Gamma_F)\tau_u^{-1}.$$

(2) If $F \in \overline{\mathcal{F}^{\upharpoonleft}}_1$, then $\Gamma_F = \beta_F$.

(3) If $F \in \mathcal{Q}$, then u_F has nonnegative coordinates with respect to $u_1, \ldots, u_n$. (Recall the discussion before Definition 8.4.10.)

(4) Since X' is convex, $\mathcal{Q}_0$ contains the vertices of a positive minimal path between any two elements of $\mathcal{Q}_0$.

Definition 8.4.13. For $j = 1, \ldots, d$ let

$$\Omega_j := \{F \in \mathcal{F}_1^{\upharpoonleft} : F \text{ is crossed by } \omega_j^{(k)} \text{ for some } k\},$$

And set $\Omega := \bigcup_j \Omega_j$.

Lemma 8.4.14. *For all $i = 1, \dots, n$, the subgroup of $\pi_1(M(\mathscr{A}^{\upharpoonright}))$ generated by the elements β_F with $F \in \Omega_i$ is contained in the subgroup generated by the Γ_F, $F \in \Omega_i$.*

Proof. Let w.l.o.g. $F \in \Omega_1$, and say that $F = u_1^k \overline{F}$. If $k \geq 0$, by construction we have $\Gamma_F = \beta_F$.

Suppose then $k < 0$, and in this case $C' := (C_0)_F \neq (u_1^k C_0)_F$. Let ν denote the positive minimal path from C' to C_0 that follows the segments s_1. We argue by induction on the length $d(F)$ of ν: if $d(F) = 0$ we have in fact $\Gamma_F = \beta_F$.

Now let $d(F) > 0$. Then

$$\Gamma_F \simeq \nu^{-1} l_F^2 \nu; \quad \beta_F = \mu l_F^2 \mu^{-1}$$

where μ is the positive minimal path from C_0 to C' following s_1. Thus

$$\beta_F = \mu\nu\nu^{-1} l_F^2 \nu(\mu\nu)^{-1} = (\mu\nu)\Gamma_F(\mu\nu)^{-1}$$

where $\mu\nu$ is the product of all $\beta_{F'}$ with F' crossed by μ - therefore, with $F' \in \Omega_1$ and $d(F') < d(F)$. By induction, the claim follows. □

Lemma 8.4.15. *The set $\{\Gamma_F \mid F \in \Omega\}$ generates $\pi_1(\mathcal{M}(\mathscr{A}^{\upharpoonright}))$.*

Proof. Let $F \in \mathcal{F}_1^{\upharpoonright}$, and let H the affine hyperplane supporting F.

By construction, there is $i \in \{1, \dots, d\}$ and $k \in \mathbb{Z}$ such that H is crossed by $\omega_i^{(k)}$ in, say, the face G ('every hyperplane is cut by the coordinate axes').

By Lemma 8.4.6, β_F is then product of β_G and other $\beta_{G'}^{\pm}$ with $G' \in \Omega$. These can be written in terms of the Γ_F by Lemma 8.4.14. □

Relations

We now turn to the study of the relations.

Lemma 8.4.16. *Let $F \in \mathcal{Q}_1$. Then there is a sequence $F_1, \dots, F_k$ of elements of $\mathcal{Q}_1$ such that β_F is homotopic to*

$$(\prod_{i=1}^{k} \Gamma_{F_i})^{-1} \Gamma_F (\prod_{i=1}^{k} \Gamma_{F_i}).$$

Moreover, $(F_1, \dots, F_k) = \Sigma(\omega_{u_F}\, \underline{(u_F C_0 \to (u_F C_0)_F)})$ as in Equation 8.4.3. In particular, the F_i are translates of elements of $\Omega \cap \overline{\mathcal{F}^{\upharpoonright}}$.

Proof. By definition $\Gamma_F = \omega_{u_F} u_F . \beta_{\overline{F}} \omega_{u_F}^{-1}$. Writing μ for a positive minimal path $(u_F C_0 \to (u_F C_0)_F)$ we decompose this into

$$\Gamma_F = \omega_{u_F} \mu (l_F)^2 (\omega_{u_F} \mu)^{-1}.$$

With Remark 8.4.12.(3) we have that $\omega_{u_F}\mu$ is a positive path, and with Lemma 8.4.5 we write it as a product $\prod_j \beta_{G_j}^{\omega_{u_F}\mu}(C_0 \to (C_0)_F)$ where since μ is positive miminal, the G_j are crossed by ω_{u_F} and thus are translates of faces intersecting the segments s_i.

Now, by construction

$$\beta_{G_j}^{\omega_{u_F}\mu} = \Gamma_{G_j}.$$

Then, set

$$\Delta_F := \prod_j \Gamma_{G_j}.$$

Therefore if $(C_0)_F = (u_F C_0)_F$ we are done with

$$\Gamma_F \simeq \Delta_F \beta_F \Delta_F^{-1}, \text{ and thus } \beta_F \simeq \Delta_F^{-1}\Gamma_F\Delta_F.$$

If $(C_0)_F \neq (u_F C_0)_F$, then we may choose a representant of $(C_0 \to (u_F C_0)_F)$ that ends with l_F, so its inverse begins with l_F^{-1} and we have the same relation as above.

□

Keeping the notations of the Lemma we define, for every $F \in \mathcal{Q}_1$,

$$\Delta_F := \prod_{G \in \Sigma(\omega_{u_F}(u_F C_0 \to (u_F C_0)_F))} \Gamma_G; \qquad \Gamma_F^\Delta := \Delta_F^{-1}\Gamma_F\Delta_F \tag{8.4.7}$$

Recall from 8.4.II that to every face $G \in \mathcal{F}_2^{\upharpoonright}$ we have an ordered set $h(G) = (F_1, \ldots, F_k)$ of incident codimension 1 faces, one for every hyperplane containing G. The relations associated with G assert the equality of

$$\beta_{F_{\sigma(1)}} \ldots \beta_{F_{\sigma(k)}} \tag{8.4.8}$$

where σ is a cyclic permutation, and we write β_i for β_{F_i}.

Lemma 8.4.17. *Given $G \in \mathcal{F}_2^{\upharpoonright}$ there is Δ_G such that, for all cyclic permutations σ, we have a homotopy of paths*

$$\beta_{F_\sigma(1)} \ldots \beta_{F_\sigma(k)} \simeq \Delta_G \omega_{u_G} u_G.(\Gamma^\Delta_{u_G^{-1}F_{\sigma(1)}} \ldots \Gamma^\Delta_{u_G^{-1}F_{\sigma(k)}})\omega_{u_G}^{-1}\Delta_G^{-1}.$$

Proof. Let us fix some notation and let $C' := (C_0)_G$, $C'' := (u_G.C_0)_G$, $\mu := (u_G C_0 \to C'')$, $\nu := (C'' \to C')$. By equation (8.4.5) we have the homotopy

$$\beta_{\sigma(1)} \ldots \beta_{\sigma(k)} \simeq (C_0 \to C')\alpha_G(C')(C_0 \to C')^{-1}$$

moreover, with Equation (8.4.4) we see

$$\alpha_G(C') \simeq \nu^{-1}\alpha_G(C'')\nu \simeq \nu^{-1}\mu^{-1}\omega_{u_G}^{-1}\omega_{u_G}\mu\alpha_G(C'')\mu^{-1}\omega_{u_G}^{-1}\omega_{u_G}\mu\nu$$

expanding $\mu\alpha_G(C'')\mu^{-1}$ according to Equation (8.4.5) and defining $\Delta_G := (C_0 \to C')\nu^{-1}\mu^{-1}\omega_{u_G}^{-1}$ we have the homotopy

$$\beta_{\sigma(1)} \ldots \beta_{\sigma(k)} \simeq \Delta_G \omega_{u_G}(u_G.\beta_{u_G^{-1}F_{\sigma(1)}}) \ldots (u_G.\beta_{u_G^{-1}F_{\sigma(k)}})\omega_{u_G}^{-1}\Delta_G^{-1} \tag{8.4.9}$$

From which the claim follows by use of Lemma 8.4.16.

□

Definition 8.4.18. For $F \in \mathcal{F}_1^{\upharpoonright}$ let

$$\gamma_F := p(\Gamma_F).$$

Moreover, for $F \in \mathcal{Q}_1$ let

$$\delta_F := p(\Delta_F); \qquad \gamma_F^\delta := \delta_F^{-1}\gamma_F\delta_F$$

Given $G \in \mathcal{F}_2^\upharpoonleft$ with $h(G) = (F_1, \ldots, F_k)$, let $R_G^\downharpoonleft$ define the relation stating the equality of all words

$$\gamma^\delta_{F_{\sigma(1)}} \cdots \gamma^\delta_{F_{\sigma(k)}}$$

where σ ranges over all cyclic permutations.

Lemma 8.4.19. *If $G \in \mathcal{F}_2^\upharpoonleft$ is a face of codimension 2, then $R_G^\downharpoonleft$ is equivalent to $R_{\overline{G}}^\downharpoonleft$*

Proof. Let $G \in \mathcal{F}_2^\upharpoonleft$. With Lemma 8.4.17 (and the notation thereof) we know that every relation $R_G^\downharpoonleft$ states the equality of all

$$p_*(\Delta_G)p_*(\Gamma^\Delta_{F_{\sigma(1)}} \ldots \Gamma^\Delta_{F_{\sigma(k)}})p_*(\Delta_G)^{-1},$$

where σ runs over all cyclic permutations. The middle term by Equation (8.4.9) is represented by the path

$$\omega_{u_G}(u_G.\beta_{u_G^{-1}F_{\sigma(1)}}) \ldots (u_G.\beta_{u_G^{-1}F_{\sigma(k)}})\omega_{u_G}^{-1}$$

and thus its image under p_* is represented by the same path as

$$p_*(\omega_{u_G})p_*(\beta_{u_G^{-1}F_{\sigma(1)}} \ldots \beta_{u_G^{-1}F_{\sigma(k)}})p_*(\omega_{u_G})^{-1}$$

Where $u_G^{-1}F_{\sigma(i)} \in \mathcal{Q}_1$ for all i. Now we apply Lemma 8.4.16. The element $\mu := p_*(\omega_{u_G}) \in \pi_1(T_\Lambda)$ is such that, for every cyclic permutation σ,

$$p_*(\Gamma^\Delta_{F_{\sigma(1)}} \ldots \Gamma^\Delta_{F_{\sigma(k)}}) = \mu\, p_*(\Gamma^\Delta_{\overline{F}_{\sigma(1)}} \ldots \Gamma^\Delta_{\overline{F}_{\sigma(k)}})\mu^{-1}$$

and therefore relation $R_G^\downharpoonleft$ is equivalent to relation $R_{\overline{G}}^\downharpoonleft$. □

Presentation

In this closing section we discuss presentations for $\pi_1(M(\mathscr{A}))$.

Lemma 8.4.20. *For all $F \in \mathcal{Q}_1$ let $(F_1, \ldots F_k) = \Sigma(\omega_{u_F}(u_F C_0 \to (u_F C_0)_F))$. We have*

$$\delta_F = \prod_{i=1}^k \tau_{u_{F_i}} \gamma_{\overline{F}_i} \tau_{u_{F_i}}^{-1}$$

and, in particular, γ_F^δ can be written as a word in the $\tau_1, \ldots, \tau_n$ and γ_F with $F \in \overline{\mathcal{F}^\upharpoonleft}_1$.

Proof. This is an easy computation using Remark 8.4.12.(1). □

In Particular, the relations $R^\downharpoonleft$ can be written in terms of the τ_i and the γ_F with $F \in \overline{\mathcal{F}^\upharpoonleft}_1$. We have immediately

Theorem 8.4.21. *The group $\pi_1(\mathcal{M}(\mathscr{A}))$ is presented as*

$$\langle \tau_1, \ldots, \tau_n;\ \gamma_F,\ F \in \mathcal{F}_1 \mid \tau_i\tau_j = \tau_j\tau_i \text{ for } i,j = 1,\ldots,n;\ R_G^{\downharpoonleft},\ G \in \mathcal{F}_2\rangle,$$

where we identify $\mathcal{F}_1$ with $\overline{\mathcal{F}^{\upharpoonleft}}$ and $\mathcal{F}_2$ with $\overline{\mathcal{F}^{\upharpoonleft}}_2$.

This presentation, while not very economical in terms of generators, has the advantage that the relations can be described with an acceptable amount of complexity.

Using Lemma 8.4.15 and Remark 8.4.12.(1) we can let, for all $G \in \overline{\mathcal{F}^{\upharpoonleft}}_2$, $\widetilde{R}_G^{\downharpoonleft}$ denote the relations obtained from $R_G^{\downharpoonleft}$ by substituting every γ_F with the corresponding expression in terms of the generators $\tau_1, \ldots, \tau_d$ and $\gamma_{F'}$ with $F' \in \overline{\mathcal{F}^{\upharpoonleft}} \cap \Omega$. Under the identification of $\mathcal{F}_1$ with $\overline{\mathcal{F}^{\upharpoonleft}}$, these are the faces on the compact torus that are crossed by some fixed chosen reppresentants of the generators $\tau_1, \ldots, \tau_d$.

Theorem 8.4.22. *The group $\pi_1(\mathcal{M}(\mathscr{A}))$ is presented as*

$$\langle \tau_1, \ldots, \tau_n;\ \gamma_F,\ F \in p(\Omega) \cap \mathcal{F}_1 \mid \tau_i\tau_j = \tau_j\tau_i \text{ for } i,j = 1,\ldots,n;\ \widetilde{R}_G^{\downharpoonleft},\ G \in \mathcal{F}_2\rangle.$$

Remark 8.4.23. The number of generators (and relations) can in principle be reduced further, by adequate choice of the coordinates of T_Λ. The computations, however, become quite more involved and untransparent. We thus omit them here, leaving the question open for a presentation with generators and relations corresponding to layers instead of faces (which exists in the case of complexified hyperplane arrangements, as shown by Salvetti in [86] by simplifying the presentation given above in 8.4.3).

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