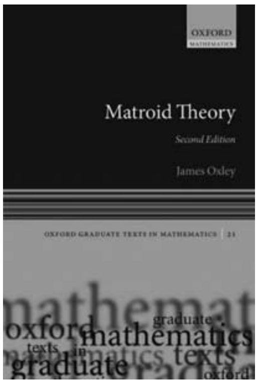

James Oxley
Matroid Theory
Second Edition, Oxford Graduate Texts in Mathematics 21
Oxford University Press 2011
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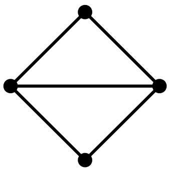
Reviewer: Emanuele Delucchi

Matroid theory is a sprawling field of combinatorics with unique structural features and far-reaching applications. To convey a first impression of the topic let us consider the similarities between the following objects.



$$\begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

Figure 1. The graph G, the matrix A

Both objects consist of 5 'elements' (edges or column-vectors). A closer look shows that there are as many spanning trees in G as there are 'bases' (maximal independent subsets among the columns) of A. An even closer look reveals that a bijection between the set of spanning trees of G and the set of bases of A is induced by any bijection between the edges of G and the columns of A that pairs the 'middle' edge of G to the third column of A and the 'upper' edges with the first two columns. The edges of the planar dual G^* of G are naturally paired with edges of G (Figure 2): under this pairing, a spanning tree of G^* corresponds to the complement of a spanning tree of G.

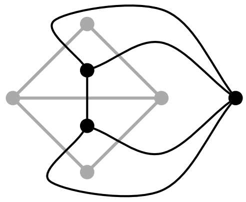


Figure 2. The dual graph G^*

Now consider the set P of planes that have the columns of A as normal vectors (Figure 3). Families of planes containing a common line correspond to cycles of G^*. Moreover, the reader is encouraged to check that P subdivides $\mathbb{R}^3$ into $\chi_G(-1)$ regions, where χ_G is the chromatic polynomial of G (for a solution see the end of the review).

These are not coincidences but shadows of the idea of a matroid on the wall of Plato's cave.

If you are interested in A, you'll say that a (finite) matroid is given by a family of equicardinal subsets of a finite set, satisfying an exchange property inspired by Steinitz' Basis Exchange Theorem. Equivalently, a finite matroid is given by a family of incomparable subsets of a finite set satisfying a characteristic property of cycles in graphs (i.e., that the union of two nondisjoint cycles contains a third cycle); this may be your choice if you are interested in G^* or in the lines of P.

This landscape of different, equivalent definitions is a distinguishing feature of matroid theory, and one that makes the theory powerful. In fact, matroid theory has come to have a wealth of applications in many areas of discrete mathematics and optimization theory – and even beyond, in fields as diverse as the study of Grassmannians or tropical mathematics (see EMS Newsletter No. 83). The usefulness of matroids rests on a solid and lively theory which, of itself, has constituted a fruitful research topic ever since its origins in the 1930s.

The fact that matroid theory has not lost any (and in fact has gained) momentum as a research topic in recent years, coupled with the wide range of applications, makes the task of writing a textbook on the subject particularly challenging. James Oxley, himself a prominent matroid theorist, did not shy from this task and, in 1992, published the first edition of *Matroid Theory*. The book turned out to be a valuable introductory textbook as well as a practical reference work for mathematicians from other fields where matroids are applied. Among the many nice features of the book are the consistently concise yet complete statements and the precise system of internal references, both of which make the book easy to navigate even along paths that do not follow the order of the chapters (the necessity of totally ordering the content of a book being particularly unsuited to matroid theory where – as has been said – the many different, equivalent approaches deserve to be treated as equals).

This review is about the second edition of Oxley's book, which is a major improvement on the first edition.

The whole text has undergone a thorough and detailed revision which has improved many aspects, from the wording of some sentences to the choice of examples and exercises. In particular, the exercises have been thoroughly updated according to the development of the theory: as in the first edition, they are not only numerous but also guide the reader through some important results that are only quickly touched upon in the expository part.

The overall structure of the first seven chapters has been mostly retained. After a preliminary introduction of some background and motivation from linear algebra and graph theory, the first chapter presents some of the most well known axiomatizations of matroids and explains a widely used geo-

Figure 3. The set of planes P

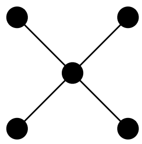


Figure 4. The geometric representation of our example matroid

metric representation of matroids with small rank (according to which our example matroid would be depicted as in Figure 4).

Chapters 2–4 deal with basic structural properties (duality, minors and connectivity), before two important classes of matroids are introduced in Chapters 5 and 6 (about graphical matroids and representable matroids). In Chapter 6, a new section on Dowling matroids, with a view on the more general theory of bias matroids (to which a substantial number of the exercises are devoted), allows the author to state the complete classification of universal models for matroid varieties obtained by Kahn and Kung in 1989. Chapter 7 presents some basic operations between matroids (connections, 2-sums, extensions and quotients) and features a new section about the free product of matroids introduced by Crapo and Schmitt in 2005 in order to study the number of nonisomorphic matroids on a given number of elements.

Starting with Chapter 8, the structure of the book has undergone a major reorganisation showing a shift of focus from the analysis of particular classes of matroids to the description of general theoretical structures. Thus, chapters about ternary matroids and regular matroids have given way to Chapter 10 about excluded-minor theorems, characterizing some matroid classes through 'forbidden submatroids', and to Chapter 13 about Seymour's theorem, a deep structural result with applications in computational matrix theory, which is treated far more extensively than in the first edition and is proved through a previously unpublished argument (Oxley credits much of it to private conversations with Jim Geleen). Chapters 8, 9, 11, 12 about higher connectivity, binary matroids, submodularity and matroid unions and about the 'Splitter theorem' have been kept, albeit in a substantially revised and expanded form. Without going into too much detail let us mention the dramatic expansion of Chapter 8 (on higher connectivity) to include a matroid version of Menger's theorem and Tutte's 'whirls and wheels' theorem characterizing the 'minimal' 3-connected matroids.

Both editions feature a 'window on research'. The first edition's last chapter about unsolved problems has been complemented here by Chapter 14 on research in representability and structure. This reflects Oxley's own research interests and the author himself states in the preface that this has been his criterion in deciding which topics – other than those that, in Oxley's words, "virtually select themselves" – had to be included in the book. To have mathematical topics explained with the words of someone who feels strongly (and positively) about them is helpful and motivating. On the other hand, some of the main gateways between matroid theory and other fields of mathematics get surprisingly little or no mention, a feat that may disorient some potential readers who come to the book from other areas of mathematics seeking to understand how matroid theory relates to their research. Oxley addresses this problem in the beginning of Chapter 15 with a survey of some alternative textbooks and introductory texts. But even there, one does not find any reference of, for example, polynomial invariants of matroids – a topic that is absent from the book except for a passing citation in Section 15.3. As another example, I'll mention my unsuccessful search through the book for any reference to matroid polytopes (even the otherwise very practical and thorough index did not help), an increasingly relevant topic for which, to my knowledge, a comprehensive dedicated introductory text has yet to be published. Of course, it would be a Herculean (if not Sisyphean) task to try to write an all-encompassing textbook on matroids; and so my comment is less a negative point to Oxley's book than it is a call on the community to fill this gap.

All things considered, the improvements in the second edition will ensure that, as matroid theory continues to develop and to broaden the scope of its applications, Oxley's book will remain a valuable companion, both as a reference and as an introductory work, for specialists and neophytes alike.

Solution to the quiz: The chromatic polynomial of G is $\chi_G(t) = t^4 - 5t^3 + 8t^2 - 4t$ and indeed the number of regions is $18 = \chi_G(-1)$.

Emanuele Delucchi [delucchi@math.uni-bremen.de] obtained his PhD in mathematics from ETH Zurich in 2006. He was a postdoctoral researcher at the University of Pisa and at the MSRI in Berkeley. From 2008 to 2010 he has been a visiting assistant professor at SUNY Binghamton and, since then, a senior lecturer at the University of Bremen where he obtained his Habilitation in 2011. His research is in combinatorics and topology, with a special focus on the theory of hyperplane arrangements.