ERRATUM TO "THE INTEGER COHOMOLOGY ALGEBRA OF TORIC ARRANGEMENTS"

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Abstract. We point out two errors in the paper "The integer cohomology algebra of toric arrangements", Advances in Mathematics, Vol. 313, pp. 746–802, 2017. The main error concerns Theorem 4.2.17. In particular the Diagram (8) does not commute in general. This invalidates the description for the ring structure of $H^*(M(\mathcal{A});\mathbb{Z})$ given in Theorem A and B. Still, under some restrictive hypotesis on $\mathcal{A}$ the results of Theorem 4.2.17 holds. We show a workaround to provide a description of the cohomology ring $H^*(M(\mathcal{A});\mathbb{Z})$ when $\mathcal{A}$ is a real complexified toric arrangement. The second error concerns the proof Theorem 7.2.1. The claim holds, but the proof is incorrect. We refer to a counterexample for the argument given in the proof and we provide references for a correct proof.

Contents

1. Presentation of the cohomology algebra: the incorrect result

The claim of Theorem 4.2.17 in [CD17] does not hold for all choices of the basis chamber upon which relies the construction of the subcomplexes $\mathcal{S}_L$. In particular, for some such choice the Diagram (8) of [CD17] does not commute. This invalidates the description for the ring structure of $H^*(M(\mathcal{A});\mathbb{Z})$ given in Theorem A and B: in particular, the rings $A(\mathcal{A})$ and $B(\mathcal{A})$ are isomorphic to a graded algebra associated to a filtration of $H^*(M(\mathcal{A});\mathbb{Z})$ induced by the Leray spectral sequence, but in general they are not isomorphic to the ring $H^*(M(\mathcal{A});\mathbb{Z})$ itself.

Remark 1.0.1. The claim Theorem 4.2.17 in [CD17] does hold if the facets F_0 and F_0' that define the complexes $\mathcal{S}_L := \mathcal{S}_{F_0}$ and $\mathcal{S}_{L'} = \mathcal{S}_{F_0'}$ (see [CD17, Def. 4.2.16.]) are adjacent to the same chamber B_0 of $\mathcal{A}_0$.

In particular, under the restrictive hypotesis on $\mathcal{A}$ that there exists a chamber B_0 of the arrangement $\mathcal{A}_0$ such that for every layer L of $\mathcal{A}_0$ the support of the intersection $L \cap \overline{B}_0$ is L, the results of Theorem A and B of [CD17] hold.

In the following Sections 2 and 3 we show a workaround that allows us nevertheless to provide a description of the cohomology ring $H^*(M(\mathcal{A});\mathbb{Z})$ as a subring of the direct sum $\oplus_{L\in\mathcal{C}} H^*(\mathcal{S}_L;\mathbb{Z})$ when $\mathcal{A}$ is a real complexified toric arrangement. Unfortunately the workaround presented here does not apply to non real complexified toric arrangements. Hence in this erratum we will assume that all toric arrangements are real complexified.

For a general toric arrangement $\mathcal{A}$, not necessarily real complexified, a description of the ring $H^*(M(\mathcal{A});\mathbb{Z})$ in the style of the Orlik-Solomon algebra of hyperplane arrangements, obtained using other methods, can be found in [CDD+18].

2. Two classes of subcomplexes

We aim at defining (cellular) representatives for certain homology classes. Intuitively, these classes will be of two types.

The first type of classes, called $\widehat{\lambda}^M_B$ will represent cycles that are parallel to 1-dimensional layers M and lie in certain subcomplexes $\mathcal{S}_L$ with $M \subset L$ (more properly in $\mathcal{S}_{L,F_0}$, with $M \subset L$, under certain conditions on F_0, L and B, see Remark 2.2.9 below). The second type are classes $\widehat{\omega}_H$, representing loops around a codimension-1 layer $H \in \mathcal{A}$.

2.1. Setup.

We start by recalling the setup and some notation from the original article.

Let $\mathcal{A}$ denote a finite toric arrangement in the complex torus T. Recall that a *layer* of $\mathcal{A}$ is any connected component of an intersection of elements of $\mathcal{A}$. We call $\mathcal{C}$ the set of all layers, partially ordered by reverse inclusion. This poset is ranked (by the layer's codimension) and we denote by $\mathcal{C}_i$ the set of elements of $\mathcal{C}$ with rank i. To any $L \in \mathcal{C}$ we can associate the arrangement $\mathcal{A}_L = \{H \in \mathcal{A} \mid L \subseteq H\}$ in T and the arrangement $\mathcal{A}^L = \{H \cap L \mid H \notin \mathcal{A}_L\}$ determined by $\mathcal{A}$ in the torus L.

A central tool is the category $\mathcal{F}(\mathcal{A})$, whose objects are all faces of the induced (polyhedral) cellularization of the compact torus and where morphisms $F \to G$ correspond to the boundary cells of G attached to F (see [dD15, Rmk. 3.3]. Given any $F \in \mathcal{F}(\mathcal{A})$, there is a unique minimal layer containing F, called the support of F and denoted by $\operatorname{supp}(F)$. We will sometimes write $\mathcal{A}^F$ for $\mathcal{A}^{\operatorname{supp}(F)}$ and $\mathcal{A}_F$ for $\mathcal{A}_{\operatorname{supp}(F)}$.

To every face $F \in \operatorname{Ob}\mathcal{F}$ we associate the "local" real hyperplane arrangement $\mathcal{A}[F]$ (this is the real part of the (complexified) hyperplane arrangement defined by $\mathcal{A}$ in the tangent space to T at any point in the relative interior of F). Moreover, associated to $\mathcal{A}$ we consider an "abstract" arrangement of hyperplanes in $\mathbb{R}^d$ that we call $\mathcal{A}_0$, which can be thought of as the union of all $\mathcal{A}[F]$ where F ranges in $\operatorname{Ob}\mathcal{F}$ (omitting repetition of hyperplanes). Key is the fact that, for every $F \in \operatorname{Ob}\mathcal{F}$, $\mathcal{A}[F]$ is a subarrangement of $\mathcal{A}_0$. In particular, for every layer $L \in \mathcal{C}$ there is a subspace $X_L \in \mathscr{L}(\mathcal{A}_0)$ defined as the intersection of the hyperplanes associated to hypertori containing L. Given any $F \in \mathcal{F}(\mathcal{A})$, we let X^F_L be the smallest flat of $\mathcal{A}[F]$ containing X_L.

As is customary for arrangements of hyperplanes in real vector spaces, after choosing a "positive side" of each hyperplane we can associate to every point x

in the ambient space a *sign vector* $\gamma_x \in \{0,+,-\}^{\text{hyperplanes}}$ whose value on any hyperplane H is $0,+,-$ according to whether x lies on H, on the positive side of H or on the negative side of H. A *face* is then the set of all x with a fixed sign vector. The set of all faces is partially ordered by inclusion of topological closures. The top-dimensional faces are called *chambers*.

For every F we will thus consider the poset of faces $\mathcal{F}(\mathcal{A}[F])$ where $G \leqslant K$ if $G \subseteq \overline{K}$, and the set $\mathcal{T}(\mathcal{A}[F])$ of chambers. For every morphism $m : F \to G$ there is a natural inclusion $i_m : \mathcal{F}(\mathcal{A}[G]) \to \mathcal{F}(\mathcal{A}[F])$ and in particular we call F_m the image of the minimal element of $\mathcal{F}(\mathcal{A}[G])$ (see [CD17, §4.1]).

Now for each arrangement $\mathcal{A}[F]$ one can construct the associated Salvetti complex $\mathrm{Sal}(\mathcal{A}[F])$, which models the homotopy type of the complement of the complexification of $\mathcal{A}[F]$. A natural construction of $\mathrm{Sal}(\mathcal{A}[F])$ is as the order complex of the partially ordered set $\mathcal{S}(\mathcal{A}[F])$ of all pairs $[G,C]$ with $C \in \mathcal{T}(\mathcal{A}[F])$ and $G \leqslant C$ in $\mathcal{F}(\mathcal{A}[F])$, partially ordered via $[G,C] \geqslant [G',C']$ if $C_{G'} = C'$ (this means: no hyperplane in $\mathcal{A}[G']$ separates C from C', see [CD17, Def. 3.3.1]) For each chamber C we consider the subposet $\mathcal{S}_C \subseteq \mathcal{S}(\mathcal{A}[F])$ of all pairs $[G,K]$ such that $K = C_G$ where G ranges over $\mathcal{F}(\mathcal{A}[F])$. It will be useful to stratify $\mathcal{S}(\mathcal{A}[F])$ via the subposets $\mathcal{S}^G(\mathcal{A}[F]) := \bigcup_{C \geqslant G} \mathcal{S}_C$. For details on these constructions see [CD17, §3.3].

Returning to the toric arrangement $\mathcal{A}$, a model for the complement of $M(\mathcal{A}) := T \backslash \bigcup \mathcal{A}$ can be obtained from the diagram $\mathscr{D}$ on the index category $\mathcal{F}(\mathcal{A})^{op}$ that associates to every object F the poset $\mathcal{S}(\mathcal{A}[F])$ and to every morphism $m : F \to G$ the order-preserving map $\mathscr{D}(m) : \mathcal{S}(\mathcal{A}[G]) \to \mathcal{S}(\mathcal{A}[F])$, $[G,K] \mapsto [i_m(G), i_m(K)]$. Then the "Grothendieck construction" $\int \mathscr{D}$ gives an acyclic category that is homotopy equivalent to $M(\mathcal{A})$. Crucial to our discussion will be a certain type of subcategories of $\in \mathscr{D}$. For every $Y \in \mathcal{C}$ and every $F_0 \in \mathcal{F}(\mathcal{A}_0)$ whose linear hull $|F_0|$ is X_Y consider the subdiagram $\mathscr{D}_{Y,F_0}$ of $\mathscr{D}$ induced on the subcategory $\mathcal{F}(\mathcal{A}^Y)$ of $\mathcal{F}(\mathcal{A})$ by the subposets $\mathscr{D}_{Y,F_0}(F) := \mathcal{S}^{F_0}(\mathcal{A}[F])$. Then, we set $\mathcal{S}_{Y,F_0} := \int \mathscr{D}_{Y,F_0}$.[1]

Definition 2.1.1. Given any chamber $C \in \mathcal{T}(\mathcal{A}_0)$ and any $F \in \mathcal{F}(\mathcal{A})$ we denote by $C(F)$ the unique chamber of $\mathcal{A}[F]$ containing C.

2.2. **The cycles $\widehat{\lambda}$.**

Definition 2.2.1. Let $\mathcal{A}$ be an essential toric arrangement in a torus T of dimension d. For every $M \in \mathcal{C}_{d-1}$ fix, once and for all, a chamber ${}^M C \in \mathcal{T}(\mathcal{A}_0)$ adjacent to X_M, and choose a minimal gallery in $\mathcal{T}(\mathcal{A}_0)$

$$ {}^M C = C_0, C_1, \ldots, C_{k(M)} = \mathrm{op}_{M_0}(-{}^M C). $$

Here, for every face $F \subseteq \mathbb{R}^d$ of $\mathcal{A}_0$ we write $-F$ for the negative of F viewed as a set of vectors in $\mathbb{R}^d$. Moreover, if C is a chamber adjacent to X_M, with $\mathrm{op}_{X_M}(C)$ we mean the unique chamber such that $\overline{C} \cap X_M = \overline{\mathrm{op}_{X_M}(C)} \cap X_M$ and $S(C, \mathrm{op}_{X_M}(C)) = \mathcal{A}_{X_M}$. In particular, note that $S(C, -\mathrm{op}_{X_M}(C)) = \mathcal{A}_0 \backslash \mathcal{A}_{X_M}$.

Remark 2.2.2. The choice of a different ${}^M C$, say ${}^M C'$, would give a different gallery, say $C'_0, \ldots, C'_{k(M)}$. Let $\rho := R_1, \ldots, R_h$ be a minimal gallery from C_0 to C'_0. Notice that, since $S(C_0, C'_0) \subseteq \mathcal{A}_{X_M}$, we have $S(C_0, C'_0) \cap S(C'_0, C'_{k(M)}) = \varnothing$. Thus the concatenation of ρ with $C'_0, \ldots, C'_{k(M)}$ is a minimal gallery, as is the concatenation of $C_0, \ldots, C_{k(M)}$ with $\tilde{\rho} := \mathrm{op}_{X_M}(-\rho)$.

[1]In the original paper this is Definition 4.2.6 and 4.2.8. Here we added the subscript Y for clarity.

Given any face $F \subseteq M$, let

$$C_0^F, \ldots, C_{k(M,F)}^F$$

be an enumeration of the set $\{C_i(F)\}_{i=0,\ldots,k(M)}$ in increasing index order and call W_i^F the wall separating C_i^F from C_{i+1}^F in $\mathcal{A}[F]$.

Remark 2.2.3. Notice that the sequence $C_0^F, W_1^F, C_1^F, \ldots$ defines a minimal gallery in $\mathcal{A}[F]$, hence it does never cross any hyperplane in $\mathcal{A}[M]$. In terms of sign vectors (see [CD17, §3.2.1]), $\gamma_{C_i^F}(H) = \gamma_{W_i^F}(H) = \gamma_{^M C}(H)$ for all i and all $H \in \mathcal{A}[F] \cap \mathcal{A}[M]$.

For every $B \in \mathcal{T}(\mathcal{A}_0)$, set

$$\begin{aligned} v_i(M;B,F) &:= [C_i^F, C_i^F] \text{ for } i = 0, \ldots, k(M,F), \\ e_i(M;B,F) &:= [W_{i+1}^F, B(F)_{W_{i+1}^F}] \text{ for } i = 0, \ldots, k(M,F) - 1. \end{aligned}$$

Now consider the subposet of $\mathcal{S}(\mathcal{A}[F])$ induced on

$$\mathrm{Path}(B;M,F) := \{v_i(M;B,F)\}_{i=0,\ldots,k(M,F)} \cup \{e_i(M;B,F)\}_{i=0,\ldots,k(M,F)-1}.$$

This poset has the following form:

$$\begin{tikzcd}[column sep=small] & e_0(B;M,F) & & e_1(B;M,F) & & \cdots \\ v_0(B;M,F) \arrow[ur] & & v_1(B;M,F) \arrow[ul] \arrow[ur] & & v_2(B;M,F) \arrow[ul] & \cdots \end{tikzcd}$$

so that $|\mathrm{Path}(B;M,F)|$ is a topological path from $v_0(B;M,F)$ to $v_{k(M,F)}(B;M,F)$.

Remark 2.2.4. Notice that if $\dim(F) = 1$, then $k(M,F) = 0$ so $\mathrm{Path}(L;M,F)$ is a single vertex which we will denote $v(L;M,F)$.

Moreover, and crucially, for any two $B \neq B'$ we have $v_i(M;B',F) = v_i(M;B,F)$ for all i and $e_i(M;B',F) = e_i(M;B,F)$ if and only if the affine span of W_i does not separate B from B' (hence $B(F)$ from $B'(F)$). If we set

$$\overline{e}_i(M;B,F) := [W_i^F, (-B(F))_{W_i^F}]$$

we can state more precisely

$$e_i(M;B',F) = \begin{cases} e_i(M;B,F) & \text{if } |W_i^F| \notin S(B,B') \\ \overline{e}_i(M;B,F) & \text{otherwise} \end{cases}$$

For every $H \in \mathcal{A}_F \backslash \mathcal{A}_M$ there is a unique i such that $|W_i^F| = H_0$, thus we can define a subcategory $\Xi(H;B,F)$:

$$\begin{tikzcd} e_i(M;B,F) & e_i(M;B',F) = \overline{e}_i(M;B,F) \\ v_i(M;B,F) \arrow[u] \arrow[ur] & v_{i+1}(M;B,F) \arrow[ul] \arrow[u] \end{tikzcd} \tag{1}$$

Definition 2.2.5. Let $\mathcal{A}$ be an essential toric arrangement in a torus T of dimension d. For every $M \in \mathcal{C}_{d-1}$ and every $B \in \mathcal{T}(\mathcal{A}_0)$, define the induced subcategory of $\int \mathscr{D}$ on the vertex set

$$\Lambda_B^M := \bigcup_{F \subseteq M} \{(F,X) \mid X \in \mathrm{Path}(B;M,F)\}$$

In order to understand the structure of the subcategory Λ_B^M let us first consider the category $\mathcal{F}(\mathcal{A}^M)$. Since it is the quotient of $\mathcal{F}((\mathcal{A}^\uparrow)^M)$ by a regular action, Every object P of $\mathcal{F}(\mathcal{A}^M)$ of dimension 0 is the origin of two arrows and every object G of dimension 1 is the target of two arrows. Choose an object P_0 of dimension 0 and consider the two arrows, say m_1, m_2 , originating in P_0. Then $F_{m_1} = -F_{m_2}$ in $\mathcal{F}(\mathcal{A}[P_0])$ and in particular exactly one of these – say, m_2 – is adjacent to $-B(P_0)$. Call G_0 the target of m_2, and call P_1 the origin of the other nontrivial morphism ending in G_0. In this way we can naturally label the objects of $\mathcal{F}(\mathcal{A}^M)$ as

$$P_0 \to G_0 \leftarrow P_1 \to G_1 \leftarrow \ldots \to G_{\ell(M)} \leftarrow P_0.$$

Lemma 2.2.6. *In the category $\int \mathscr{D}$ we have, for all i modulo $\ell(M)$, arrows*

$$(G_i, v(B; M, G_i)) \to (P_j, v_l(B; M, P_j))$$

if and only if either $j = i+1$ and $l = 0$, or else $j = i$ and $l = k(M)$. (The index-less $v(B; M, G_i)$ is the only possible r.h.s. associated to G_i, see Remark 2.2.4.)

Corollary 2.2.7. *The category Λ_B^M is of the form*

$$\begin{tikzcd}[column sep=tiny]
 & (P_0, e_0(B;M,P_0)) & & (P_0, e_{k(M,P_0)}(B;M,P_0)) & & (G_0, v(B;M,G_0)) & & (P_1, e_0(B;M,P_1)) & \\
(P_0, v_0(B;M,P_0)) \arrow[ur] & & \cdots & & (P_0, v_{k(M,P_0)}(B;M,P_0)) \arrow[ul] \arrow[ur] & & (P_1, v_0(B;M,P_1)) \arrow[ul] \arrow[ur] & & \cdots
\end{tikzcd}$$

In particular, it is a poset homeomorphic to S^1.

Lemma 2.2.8. *The homotopy class of the path Λ_B^M does not depend on the choice of the chamber ${}^M C$ in Definition 2.2.1*

Proof. Fix a face $F \subseteq M$. The elements of $\mathrm{Path}(B; M, F)$ are, by definition, cells of the Salvetti complex of $\mathcal{A}[F]$. In this interpretation, they correspond to a minimal, positive path from $v_0(M, B, F)$ to $v_{k(M,F)}(M, B, F)$. Now consider the same construction with a different choice for the chamber ${}^M C$, say ${}^M C'$, as in Remark 2.2.2, and let $\mathrm{Path}(B; M, F)'$ be the obtained minimal path. In the same way, the minimal gallery $\rho = R_1, \ldots, R_h$ and $\tilde{\rho}$ of Remark 2.2.2 defines positive minimal paths ρ^F and $\tilde{\rho}^F$ in $\mathrm{Sal}(\mathcal{A}[F])$ such that the concatenation of ρ with $\mathrm{Path}(B; M, F)'$ is a positive minimal path. In particular, the paths $(\tilde{\rho}^F)\,\mathrm{Path}(B; M, F)(\rho^F)^{-1}$ and $\mathrm{Path}(B; M, F)'$ are homotopic in the Salvetti complex of $\mathcal{A}[F]$. Call h_F this homotopy.

Now consider the entirety of Λ_B^M and $(\Lambda_B^M)'$ constructed choosing ${}^M C$ and ${}^M C'$, respectively. Notice that, if G has dimension one, then $(G, \rho^G) = (G, \tilde{\rho}^G)$. Moreover, the homotopies h_F are carried by cells of $(F, \mathrm{Sal}(\mathcal{A}[F]))$, and thus the union of such cells defines a homotopy between the concatenation of the $\mathrm{Path}(B; M, F)'$ (i.e. $(\Lambda_B^M)'$) and the concatenation of the $(\tilde{\rho}^F)\,\mathrm{Path}(B; M, F)(\rho^F)^{-1}$.

$$\begin{tikzcd}[column sep=small]
 & (P_0,e_0(B;M,P_0)) & & (P_0,e_{k(M,P_0)}(B;M,P_0)) & & (G_0,v(B;M,G_0)) \arrow[dd, "\rho^G=\tilde{\rho}^G"'] & & (P_1,e_0(B;M,P_1)) & \\
(P_0,v_0(B;M,P_0)) \arrow[ur] \arrow[dd, "\rho^{P_0}"'] & & \cdots \arrow[dd, phantom, "h_{P_0}"] & & (P_0,v_{k(M,P_0)}(B;M,P_0)) \arrow[ul] \arrow[ur] \arrow[dd, "\rho^{\tilde{P}_0}"'] & & (P_1,v_0(B;M,P_1)) \arrow[ul] \arrow[ur] \arrow[dd, "\rho^{P_1}"'] & & \cdots \arrow[dd, phantom, "h_{P_1}"] \\
 & (P_0,e_0(B;M,P_0)') & & (P_0,e_{k(M,P_0)}(B;M,P_0)') & & (G_0,v(B;M,G_0))' & & (P_1,e_0(B;M,P_1)') & \\
(P_0,v_0(B;M,P_0)') \arrow[ur] & & \cdots & & (P_0,v_{k(M,P_0)}(B;M,P_0)') \arrow[ul] \arrow[ur] & & (P_1,v_0(B;M,P_1)') \arrow[ul] \arrow[ur] & & \cdots
\end{tikzcd}$$

□

Recall from [CD17, Thm. 4.2.3 and Def. 4.2.6] that for every fixed layer L and every face $F_0 \in \mathcal{F}(\mathcal{A}_0)$ whose support is L there is a subcomplex $\mathcal{S}_{L,F_0}$ of $\mathrm{Sal}(\mathcal{A})$ (notice that such complexes were indexed simply by F_0 in [CD17] – here we need a more refined notation. The homotopy type of $\mathcal{S}_{L,F_0}$ is that of $\mathcal{F}(\mathcal{A}^L) \times \mathcal{S}(\mathcal{A}[L])$.

Remark 2.2.9. If M is a one-dimensional layer contained in L, we have $\Lambda_B^M \subseteq \mathcal{S}_{L,F_0}$ if $F_0 = \overline{B} \cap L_0$

Definition 2.2.10. Let $[\Lambda_B^M] \in C_1(|\int \mathscr{D}|)$ denote the cycle supported on Λ_L^M uniquely determined by setting the coefficient of $(v_0(B;M,P_0) \to e_0(B;M,P_0))$ equal to 1.

For the following "basis-change" formula we need to define, for any $F \in \mathcal{F}(\mathcal{A})$ and any two chambers $B, B' \in \mathcal{T}(\mathcal{A}_0)$, the set

$$S_F(B,B') := \{H \in \mathcal{A}_F \mid H_0 \text{ separates } B \text{ from } B'\}. \tag{2}$$

Proposition 2.2.11. *Let $B, B' \in \mathcal{T}$. Then*

$$[\Lambda_B^M] - [\Lambda_{B'}^M] = \sum_{P \subseteq M} \sum_{H \in S_P(B,B') \setminus \mathcal{A}_M} [\Xi(H;B,P)]$$

where $[\Xi(H;B,P)]$ is the $1-$cycle determined by the subcategory defined in (1) with the orientation given by setting the coefficient of $v_i(M;B,P) \to e_i(M;B,P)$ equal to 1.

2.3. The generators $\widehat{\omega}_H$.

Definition 2.3.1. For every $H \in \mathcal{A}$ choose, once and for all, ${}^H C \in \mathcal{T}(\mathcal{A}_0)$. For every $m : F \to G \in \mathrm{Mor}\,\mathcal{F}(\mathcal{A}^H)$ with $\mathrm{supp}(G) = H$, let $C_1 := {}^H C(F)_{F_m}$, $C_2 := (-{}^H C(F))_{F_m}$ be a numbering of the two chambers of $\mathcal{T}(\mathcal{A}[F])$ adjacent to F_m.

$$\Omega^{(m)} = \begin{tikzcd} (F,[F_m,C_1]) \arrow[d, dash] \arrow[dr, dash] & (F,[F_m,C_2]) \arrow[d, dash] \arrow[dl, dash] \\ (F,[C_1,C_1]) & (F,[C_2,C_2]) \end{tikzcd} \subseteq \mathcal{S}(\mathcal{A}[F])$$

Remark 2.3.2. The case where $m = id_G : G \to G$ is instructive. In this case, $\Omega_H^{(id_G)}$ is the order complex of the subposet

$$\Omega^{(id_G)} = \begin{tikzcd} (G,[G,C_1]) \arrow[d, dash] \arrow[dr, dash] & (G,[G,C_2]) \arrow[d, dash] \arrow[dl, dash] \\ (G,[C_1,C_1]) & (G,[C_2,C_2]) \end{tikzcd} = \mathcal{S}(\mathcal{A}[G])$$

Definition 2.3.3. Let $[\Omega^{(m)}]$ denote the $1-$cycle supported on $\Omega^{(m)}$ uniquely determined by the orientation given by setting the coefficient of $((G, [C_1, C_1]) \to (G, [G, C_1]))$ equal to 1.

Definition 2.3.4. Let $H \in \mathcal{A}$ and $L \in \mathcal{C}$. Define

$$\epsilon(H, B) := \begin{cases} 1 & \text{if } H_0 \notin S(^H C, B) \\ -1 & \text{if } H_0 \in S(^H C, B) \end{cases}$$

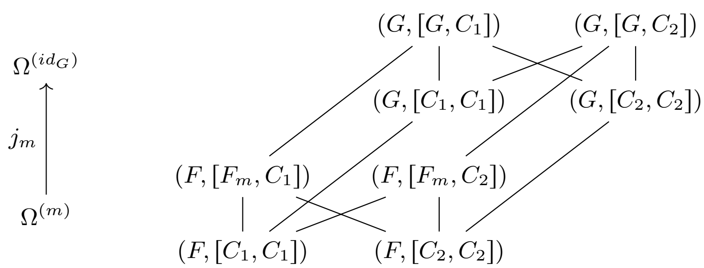
Lemma 2.3.5. *We have the following relations in homology:*

(i) *For* $m : F \to G$, $[\Omega^{(m)}] \simeq [\Omega^{(id_G)}]$
(ii) $[\Omega^{(F \to G)}] \simeq [\Omega^{(F \to G')}]$ *when both* G, G' *of maximal dimension in* $\mathcal{F}(\mathcal{A}^H)$.
(iii) $[\Omega^{(P \to W_i^M(P))}] = \epsilon(H, B)[\Xi(H; B, P)]$ *if* $\mathrm{supp}(W_i^P) = H$.

Proof. First notice that if $m : F \to G$ with G of codimension 1, then (e.g. by checking Remark 4.1.1 and ff. in [CD17])

$$\Omega^{(m)} = j_m(\Omega^{id_G}).$$

In particular the complex $\mathrm{Sal}(\mathcal{A})$, being obtained as a homotopy colimit, contains the mapping cylinder of $j_m|_{\mathcal{S}(\mathcal{A}[G])}$ in the form of the nerve of the subcategory



which gives a homotopy inside $\mathrm{Sal}(\mathcal{A})$ between Ω^{id_G} and $\Omega^{(m)}$ that sends edges to "corresponding edges". Thus (i) follows.

Part (ii) follows analogously by a homotopy between the two subcomplexes inside the subcomplex $\mathcal{S}^G(\mathcal{A}[F]) = \mathcal{S}^{G'}(\mathcal{A}[F])$ of $\mathrm{Sal}(\mathcal{A})$ (see the discussion around Proposition 3.3.5 in [CD17]).

For part (iii) notice that, for any B, $\Omega^{(P \to W_i^P)}$ and $\Xi(H; B, P)$ are the same subposet. The associated chains differs by a sign depending on whether B is on the same side of H_0 as $^H C$. $\square$

Corollary 2.3.6. *For every* $H \in \mathcal{A}$ *the homology class of any* $[\Omega^{(m)}]$ *does not depend on the choice of* $m : F \to G$ *as long as* $\mathrm{supp}(G) = H$.

Definition 2.3.7. For every $H \in \mathcal{A}$ let us denote by

$$\widehat{\omega}_H \in H_1(\mathrm{Sal}(\mathcal{A}), \mathbb{Z})$$

the homology class of (any) $[\Omega^{(m)}]$ with $m : F \to G$ and $\mathrm{supp}(G) = H$.

2.4. The generators $\widehat{\lambda}_B^M$.

Definition 2.4.1. Let $\widehat{\lambda}_B^M$ be the homology class of $[\Lambda_B^M]$.

Notice that this homology class is well-defined and independent on the choice of ${}^M C$ in Definition 2.2.1 since by Lemma 2.2.8 the paths Λ_B^M obtained with different choices of this chamber are homotopic.

Proposition 2.4.2.

$$\widehat{\lambda}_B^M - \widehat{\lambda}_{B'}^M = \sum_{\substack{F \subseteq M \\ H \in S_F(B,B') \setminus \mathcal{A}_M}} \epsilon(H,B)\widehat{\omega}_H \tag{3}$$

Proof. Lemma 2.3.5, that allows us to eliminate the dependency on P in the right-hand side of the claim of Proposition 2.2.11, and to rewrite it as in this Proposition's claim. ☐

3. Cohomology and recursion

3.1. Quotients of toric arrangements and recursive construction.

Definition 3.1.1. Let $\mathcal{A}$ be a toric arrangement and let $L \in \mathcal{C}$ a layer. Let L_0 be the coset of L that contains the identity of T. Recall that $\mathcal{A}_L$ is the subarrangement of $\mathcal{A}$ given by the hypertori that contains L. Consider the arrangement

$$\overline{\mathcal{A}}_L := \mathcal{A}_L/L_0 := \{H/L_0, H \in \mathcal{A}\} \text{ in } T/L_0.$$

We define the quotient map

$$f_L : M(\mathcal{A}) \to M(\overline{\mathcal{A}}_L)$$

as the composition $\pi_{L_0} \circ i_L$ of the inclusion

$$i_L : M(\mathcal{A}) \to M(\mathcal{A}_L)$$

and the projection

$$\pi_{L_0} : M(\mathcal{A}_L) \to M(\mathcal{A}_L/L_0).$$

Definition 3.1.2. The quotient by L_0 induces order-preserving maps

$$\pi_L : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\overline{\mathcal{A}}_L) \text{ and } \pi_L : \mathcal{F}(\mathcal{A}) \to \mathcal{F}(\overline{\mathcal{A}}_L)$$

and the latter lifts to the natural order-preserving map

$$\mathcal{F}(\mathcal{A}^{\upharpoonright}) \to \mathcal{F}((\mathcal{A}_L)^{\upharpoonright}), \qquad F \mapsto \min_{\subseteq}\{G \in \mathcal{F}((\mathcal{A}_L)^{\upharpoonright}) \mid F \subseteq G\}.$$

Remark 3.1.3. We note two elementary facts about sign vectors that can be gathered directly from Definition 3.2.1 in the original paper.

(1) For all $m \in \operatorname{Mor}\mathcal{F}(\mathcal{A})$ with source object K and every $H \in \mathcal{A}[K]_{X_L^F}$ we have that $\gamma_{F_{\pi_L(m)}}(H/L_0) = \gamma_{F_m}(H)$.
(2) For all $G, K \in \mathcal{A}[F]$ and every $X \in \mathscr{L}(\mathcal{A}[F])$ we have $(G_X)_{(K_X)} = (G_K)_X$.

We see that the linear arrangement $(\overline{\mathcal{A}}_L)_0$ is $(\mathcal{A}_0)_{X_L}/X_L$, and in particular we have a natural map

$$\pi_L : \mathcal{F}(\mathcal{A}_0) \to \mathcal{F}(\mathcal{A}_0)_{X_L} \simeq \mathcal{F}((\overline{\mathcal{A}}_L)_0), F \mapsto F_{X_L}.$$

Similarly, for every $F \in \mathcal{F}(\mathcal{A})$, the arrangement $\overline{\mathcal{A}}_L[\pi_L(F)]$ is the essentialisation [OT92, Lem. 5.30] of the sub-arrangement of $\mathcal{A}[F]_{X_L^F} \subseteq \mathcal{A}[F]$ consisting of all hyperplanes containing X_L^F. Thus the map

$$\pi_L^F : \mathcal{F}(\mathcal{A}[F]) \to \mathcal{F}(\overline{\mathcal{A}}_L[\pi_L(F)]), \quad K \mapsto K/X_L$$

is order preserving and surjective, and restricts to an isomorphism of posets

$$\mathcal{F}(\mathcal{A}[F]_{X_L^F}) \to \mathcal{F}(\overline{\mathcal{A}}_L[\pi_L(F)]).$$

Thus we can identify $\mathcal{S}(\overline{\mathcal{A}}_L[\pi_L(F)])$ with $\mathcal{S}(\mathcal{A}[F]_{X_L^F})$ and [CD17, Def. 3.3.2] gives a natural map

$$b_{X_L^F} : \mathcal{S}(\mathcal{A}[F]) \to \mathcal{S}(\mathcal{A}_L[\pi_L(F)]), \quad [K, C] \mapsto [K_{X_L^F}, C_{X_L^F}]$$

Lemma 3.1.4. *For all layers L,*

$$(\pi_L, b_{X_L^*}) : \mathscr{D}(\mathcal{A}) \Rightarrow \mathscr{D}(\overline{\mathcal{A}}_L)$$

is a natural transformation.

Proof of Lemma 3.1.4. In order to check naturality pick any $m : F \to G$ in $\mathrm{Mor}(\mathcal{F}(\mathcal{A}))$ and $[K, C] \in \mathcal{S}(\mathcal{A}[G])$. With the definitions:

$$\begin{tikzcd}
(F, [i_m(K), i_m(C)]) \arrow[r, "(\pi_L, b_{X_L^F})"] & (\pi_L(F), [(i_m(K))_{X_L^F}, (i_m(C))_{X_L^F}]) \arrow[d, leftrightarrow, "\overset{?}{=}"] \\
 & (\pi_L(F), [i_{\pi_L(m)}(K_{G(L)}), i_{\pi_L(m)}(C_{G(L)})]) \\
(G, [K, C]) \arrow[uu, "(m, j_m)"] \arrow[r, "(\pi_L, b_{G(L)})"] & (\pi_L(G), [K_{G(L)}, C_{G(L)}]) \arrow[u, "(\pi_L(m), j_{\pi_L(m)})"']
\end{tikzcd}$$

and we need to prove equality of the two expressions on the top right-hand-side. It is enough to prove that, for every $K \in \mathcal{F}(\mathcal{A}[G])$,

$$i_{\pi_L(m)}(K_{G(L)}) = (i_m(K))_{X_L^F} \text{ in } \mathcal{A}[F]_{X_L^F}.$$

This we do using the definition [CD17, Rmk. 4.1.1]. First consider the right-hand side: it is defined by

$$\gamma_{(i_m(K))_{X_L^F}}(H) = \gamma_{i_m(K)}(H) = \begin{cases} \gamma_{i_m(K)}(H) & H \notin \mathcal{A}[G] \\ \gamma_K(H) & H \in \mathcal{A}[G] \end{cases} \tag{4}$$

for all $H \in \mathcal{A}[F]_{X_L^F}$.

In the same vein, the left-hand side is determined by

$$\gamma_{i_{\pi_L(m)}(K_{G(L)})}(H) = \begin{cases} \gamma_{F_{\pi_L(m)}}(H) = \gamma_{F_m}(H) & \text{if } H \notin \mathcal{A}[G] \\ \gamma_{K_{G(L)}}(H) = \gamma_K(H) & \text{if } H \in \mathcal{A}[G] \end{cases} \tag{5}$$

where we used Remark 3.1.3.(1). Now, with [CD17, Rmk. 4.1.1] we see that $\gamma_{F_m}(H) = \gamma_{i_m(K)}(H)$ for $H \notin \mathcal{A}[G]$, completing the check of the identity between the two sides of the required equality, as expressed in Equations (4) and (5). $\square$

Definition 3.1.5. Call

$$\Phi_L : \mathrm{Sal}(\mathcal{A}) \to \mathrm{Sal}(\overline{\mathcal{A}}_L)$$

the (cellular) map induced by the natural transformation of Lemma 3.1.4

We collect some properties on the behaviour of this map.

Lemma 3.1.6. *Let $\mathcal{A}$ be an essential toric arrangement in a torus T of dimension d. Fix a layer $L \in \mathcal{C}$.*

(1) For every $M \in \mathcal{C}_{d-1}$ such that $X_M \subseteq X_L$ and every chamber $B \in \mathcal{T}(\mathcal{A}_0)$, $\Phi_L(\Lambda_B^M)$ is a single vertex. In particular the induced homology homomorphism satisfies

$$\Phi_{L*}(\widehat{\lambda}_B^M) = 0.$$

*(2) Consider any $G \in \mathcal{F}(\mathcal{A})$ with $\operatorname{supp}(G) = H \in \mathcal{A}$.
If $H \supseteq L$, then $\Phi_L(\Omega^{(id_G)}) = \Omega^{(id_{\pi_L(G)})}$.
More generally, choosing ${}^{\pi_L(H)}C := ({}^H C)_{G^{(L)}}/L_0$ for every $H \in \mathcal{A}$, we have*

$$\Phi_{L*}(\widehat{\omega}_H) = \begin{cases} \widehat{\omega}_{\pi_L(H)} & \text{if } H \supseteq L \\ 0 & \text{if } H \nsupseteq L \end{cases}$$

(3) For every layer Y and all $F_0 \in \mathcal{F}(\mathcal{A}_0)$ with

$$\Phi_L(\mathcal{S}_{Y,F_0}) \subseteq \mathcal{S}_{\pi_L(Y),\pi_L(F_0)}.$$

Proof. (1): With Remark 2.2.3, for all $F \subseteq M$, we have $(C_i^F)_{X_L^F} = (W_i^F)_{X_L^F} = ({}^M C)_{X_L^F}$ for all faces $F \subseteq M$ and $0 \leqslant i \leqslant k(M,F)$. Therefore, comparing Definition 2.2.5 we see $\Lambda_B^M = \{(\pi_L(F), [{}^M C_{X_L^F}, {}^M C_{X_L^F}])\}$, a singleton.

(2): If $H \supseteq L$, then $G^{(L)} = H$. Direct computations of the image under Φ_L for each of the elements of Ω^{id_G} (Remark 2.3.2) and a glance at Definition 2.3.3 verify the claims in this case. If $H \nsupseteq L$, then $G^{(L)}$ is trivial and $\Phi_L(\Omega^{(id_G)})$ is a single vertex.

(3): Recall that $\mathcal{S}_{Y,F_0}$ is defined as $\int \mathscr{D}_{Y,F_0}$ for a subdiagram $\mathscr{D}_{Y,F_0}$ of $\mathscr{D}$ on the index category $\mathcal{F}(\mathcal{A}^Y)$ [CD17, Def. 4.2.6]. Similarly, if $\overline{\mathscr{D}}$ is the diagram giving $\operatorname{Sal}(\overline{\mathcal{A}}_L)$, then $\mathcal{S}_{\pi_L(Y),\pi_L(F_0)}$ is $\int \overline{\mathscr{D}}_{\pi_L(Y),\pi_L(F_0)}$ for a subdiagram over the index category $\mathcal{F}(\overline{\mathcal{A}}_L^{\pi_L(Y)})$. Since $\Phi_L(\mathcal{S}_{Y,F_0}) = \int \Phi_L(\mathscr{D}_{Y,F_0})$, in order to prove the claim we have to prove that Φ_L restricts to a natural transformation $\mathscr{D}_{Y,F_0} \Rightarrow \overline{\mathscr{D}}_{\pi_L(Y),\pi_L(F_0)}$.

Now obviously $\pi_L(\mathcal{F}(\mathcal{A}^Y)) \subseteq \mathcal{F}((\overline{\mathcal{A}}_L)^{\pi_L(Y)})$, thus we are left proving that, for every face F in $\mathcal{F}(\mathcal{A}^Y)$, $b_{X_L^*}(\mathcal{S}^{F_0}(\mathcal{A}[F])) \subseteq \mathcal{S}^{\pi_L(F_0)}(\overline{\mathcal{A}}_L[\pi_L(F)])$.

Pick any $[G,K] \in \mathcal{S}^{F_0}(\mathcal{A}[F])$. By definition this means that $K = B_G$ for some chamber $B \in \mathcal{T}(\mathcal{A}[F])$ adjacent to F_0. Now, since π_L^F is order preserving, $\pi_L^F(B)$ is adjacent to $\pi_L^F(F_0)$ and $\pi_L^F(B)_{\pi_L^F(G)} = (B_{X_L^F})_{G_{X_L^F}} = (B_G)_{X_L^F} = \pi_L^F(K)$, where the second equality uses Remark 3.1.3.(2). Thus, $b_{X_L^F}([G,K]) \in \mathcal{S}^{\pi_L(F_0)}(\overline{\mathcal{A}}_L[\pi_L(F)])$. □

Corollary 3.1.7. *Fix a chamber $B \in \mathcal{T}(\mathcal{A}_0)$ and a layer $L \in \mathcal{C}(\mathcal{A})$. We have*

$$\Phi_{L*}(H_*(\mathcal{S}_{T,B};\mathbb{Q})) \subset H_*(\mathcal{S}_{\pi_L(T),\pi_L(B)};\mathbb{Q})$$

In particular, for every $M \in \mathcal{C}_{d-1}$ we have

$$\Phi_{L*}(\widehat{\lambda}_B^M) \in H_*(\mathcal{S}_{\pi_L(T),\pi_L(B)};\mathbb{Q}). \tag{6}$$

Proof. The result follows from the following commutative diagram:

$$\begin{tikzcd} \Lambda_B^M \arrow[r, hook] & \mathcal{S}_{T,B} \arrow[r, hook, "\varphi_T"] \arrow[d, "\Phi_L"] & \operatorname{Sal}(\mathcal{A}) \arrow[d, "\Phi_L"] \\ & \mathcal{S}_{\pi_L(T),\pi_L(B)} \arrow[r, hook, "\varphi_{\pi_L(T)}"] & \operatorname{Sal}(\overline{\mathcal{A}}_L) \end{tikzcd}$$

where the existence of the leftmost vertical arrow follows from Lemma 3.1.6, (3). □

Remark 3.1.8. Let $L \in \mathcal{C}$ be a layer and consider the map $\Phi_L : \mathrm{Sal}(\mathcal{A}) \to \mathrm{Sal}(\overline{\mathcal{A}}_L)$. Let $Y \subseteq L$. We have that $\pi_L(L) = \pi_L(Y)$ if and only if $Y \subseteq L$.

Remark 3.1.9. Let $F \in \mathcal{F}(\mathcal{A}_0)$ with $\mathrm{supp}(F) = L_0$. Recall (see [CD17, Lem. 4.2.15]) that the subcomplex $\mathcal{S}_{L,F} \subset \mathrm{Sal}(\mathcal{A})$ is homotopy equivalent to the product $L \times M(\mathcal{A}_{\mathbb{C}}[L])$, where $M(\mathcal{A}_{\mathbb{C}}[L])$ is the complement of the essentialization of the complexified central linear arrangement $\mathcal{A}_{\mathbb{C}}[L] = \mathcal{A}[L] \otimes_{\mathbb{R}} \mathbb{C}$. Hence the cohomology ring of $\mathcal{S}_{L,F}$ is generated in degree 1.

In particular the cohomology ring $H^*(\mathcal{S}_{\pi_L(L),\pi_L(F)}; \mathbb{Q})$ is the Orlik-Solomon algebra generated by the restrictions of the forms $\omega_{\pi_L(H)}$ for $L \subset H$.

Lemma 3.1.10. *Let $\mathcal{A}$ be a toric arrangement invariant by the action of an element $g \in T$. Then the multiplication by g induces a maps $\mu_g : \mathrm{Sal}(\mathcal{A}) \to \mathrm{Sal}(\mathcal{A})$ and $\mu_g : \mathcal{S}_{L,F} \to \mathcal{S}_{L,F}$ such that the following diagram commute*

$$\begin{tikzcd}
M(\mathcal{A}) \arrow[r, "g"] & M(\mathcal{A}) \\
\mathrm{Sal}(\mathcal{A}) \arrow[u, hook] \arrow[r, "\mu_g"] & \mathrm{Sal}(\mathcal{A}) \arrow[u, hook] \\
\mathcal{S}_{L,F} \arrow[u, hook] \arrow[r, "\mu_g"] & \mathcal{S}_{L,F} \arrow[u, hook]
\end{tikzcd}$$

Moreover the map $\mu_g : \mathcal{S}_{L,F} \to \mathcal{S}_{L,F}$ is homotopy equivalent to the identity.

Proof. The multiplication by g on T lifts to a translation τ_g in the universal cover $V \simeq \mathbb{R}^d$ of T, where the periodic arrangement $\mathcal{A}^{\upharpoonright}$ is invariant under τ_g. In particular, τ_g leaves the poset of faces invariant and, hence, induces an automorphism $\mu_g^{\upharpoonright}$ of the Salvetti complex $\mathrm{Sal}(\mathcal{A}^{\upharpoonright})$. Now τ_g commutes with the standard inclusion $\iota : \mathrm{Sal}(\mathcal{A}^{\upharpoonright}) \hookrightarrow M(\mathcal{A}^{\upharpoonright})$ as well as with the translations of $\mathbb{Z}^d \subseteq \mathbb{R}^d$. Hence, so does $\mu_g^{\upharpoonright}$ and, since $\mathrm{Sal}(\mathcal{A}) = \mathrm{Sal}(\mathcal{A}^{\upharpoonright})/\mathbb{Z}^d$ (see [CD17, Thm. 4.1.3]), it induces the required map μ_g and the top half of the diagram commutes. Now, the explicit form of μ_g as a simplicial map on $|\int \mathscr{D}|$ is

$$\mu_g(F, [G, C]) = (gF, [G, C])$$

for all $F \in \mathcal{F}(\mathcal{A})$ and every $[G, C] \in \mathcal{S}(\mathcal{A}[F]) = \mathcal{S}(\mathcal{A}[gF])$. (Recall that, by definition, $\mathcal{A}[F] = \mathcal{A}[L]$ where L is the layer supporting F, and so $\mathcal{A}[F] = \mathcal{A}[F']$.)

In particular, this map restricts to every layer L and to $\mu_g : \mathcal{S}_{L,F} \to \mathcal{S}_{L,F}$. Under the homotopy equivalence $\mathcal{S}_{L,F} \simeq \mathcal{F}(\mathcal{A}^L) \times \mathcal{S}(\mathcal{A}[L])$ of [CD17, Lem. 4.2.15] the map μ_g is the identity on the second component and the cellular map induced by multiplication with g in T in the first component. But the continous map $L \to L$ defined by multiplication with g is homotopic to the identity on $L \simeq \mathcal{F}(\mathcal{A}^L)$ - and any homotopy accomplishing this can be composed with the identity to give a homotopy between $\mu_g : \mathcal{S}_{L,F} \to \mathcal{S}_{L,F}$ and the identity. □

Lemma 3.1.11. *Let $\mathcal{A}'$ be a sub-arrangement of $\mathcal{A}$ of the same rank. The inclusion $M(\mathcal{A}) \hookrightarrow M(\mathcal{A}')$ induces a map*

$$\Psi : \mathrm{Sal}(\mathcal{A}) \to \mathrm{Sal}(\mathcal{A}')$$

that restricts to

$$\Psi : \mathcal{S}_{L,F} \to \mathcal{S}_{\tilde{L},\tilde{F}}$$

where $\tilde{L}$ *(resp.* $\tilde{F}$*) is the smallest layer of* $\mathcal{A}'$ *containing* L *(resp. the smallest face of* $\mathcal{A}'$ *with support* $\tilde{L}$ *containing* F*) such that the following diagram commute*

$$\begin{tikzcd}
M(\mathcal{A}) \arrow[r, hook] & M(\mathcal{A}') \\
\mathrm{Sal}(\mathcal{A}) \arrow[u, hook] \arrow[r, "\Psi"] & \mathrm{Sal}(\mathcal{A}') \arrow[u, hook] \\
\mathcal{S}_{L,F} \arrow[u, hook] \arrow[r, "\Psi"] & \mathcal{S}'_{\tilde{L},\tilde{F}} \arrow[u, hook]
\end{tikzcd}$$

Proof. Consider the arrangements $\mathcal{A}^{\upharpoonleft}$ and $(\mathcal{A}')^{\upharpoonleft}$ in $V = \mathbb{R}^d$. Clearly every open cell F of the polyhedral stratification of V induced by $\mathcal{A}^{\upharpoonleft}$ is contained in a unique cell $s(F)$ of the stratification induced by $(\mathcal{A}')^{\upharpoonleft}$. This defines an order-preserving function $s : \mathcal{F}(\mathcal{A}^{\upharpoonleft}) \to \mathcal{F}((\mathcal{A}')^{\upharpoonleft})$ that induces a poset map $\mathrm{Sal}(\mathcal{A}^{\upharpoonleft}) \to \mathrm{Sal}((\mathcal{A}')^{\upharpoonleft})$, $[G, C] \mapsto [s(G), s(C)]$ (consider any two chambers C, C' of $\mathcal{A}^{\upharpoonleft}$: every hyperplane separating $s(C)$ from $s(C')$ also separates C from C', therefore $[G, C] \geqslant [G'C']$ implies $[s(G), s(C)] \geqslant [s(G'), s(C')]$). The canonical inclusion $\xi : \mathrm{Sal}(\mathcal{A}^{\upharpoonleft}) \hookrightarrow M(\mathcal{A}^{\upharpoonleft})$ sends every $[G, C]$ to a point in the (open and convex) set $G + iC \subseteq V \otimes \mathbb{C}$. This inclusion gives a section of the deformation retraction $r : M(\mathcal{A}^{\upharpoonleft}) \to M((\mathcal{A}')^{\upharpoonleft})$. Therefore, the diagram on the left-hand side commutes, and hence so does, up to homotopy, the one on the left-hand side.

$$\begin{tikzcd}
M(\mathcal{A}^{\upharpoonleft}) \arrow[r, hook] \arrow[d, "r"] & M((\mathcal{A}')^{\upharpoonleft}) \arrow[d, "r"] \\
\mathrm{Sal}(\mathcal{A}^{\upharpoonleft}) \arrow[r, "s"] & \mathrm{Sal}((\mathcal{A}')^{\upharpoonleft})
\end{tikzcd}
\qquad
\begin{tikzcd}
M(\mathcal{A}^{\upharpoonleft}) \arrow[r, hook] & M((\mathcal{A}')^{\upharpoonleft}) \\
\mathrm{Sal}(\mathcal{A}^{\upharpoonleft}) \arrow[u, hook] \arrow[r, "s"] & \mathrm{Sal}((\mathcal{A}')^{\upharpoonleft}) \arrow[u, hook]
\end{tikzcd}$$

Passing to the torus means considering the quotient by the action of the group of translations $\mathbb{Z}^d \subseteq V$ - call $q : V \to T$ this map. Since s commutes with q, it descends to a functor of acyclic categories

$$\Psi : \mathrm{Sal}(\mathcal{A}) \to \mathrm{Sal}(\mathcal{A}').$$

Since the inclusion ξ can be chosen equivariantly, the image under q of the above diagram is a commutative diagram

$$\begin{tikzcd}
M(\mathcal{A}) \arrow[r, hook] & M(\mathcal{A}') \\
\mathrm{Sal}(\mathcal{A}) \arrow[u, hook] \arrow[r, "\Psi"] & \mathrm{Sal}(\mathcal{A}') \arrow[u, hook]
\end{tikzcd}$$

Now notice that for every $F \in \mathcal{F}(\mathcal{A})$ we have $\tilde{F} = q(s(F^{\upharpoonleft}))$ for every choice of $F^{\upharpoonleft}$ in $q^{-1}(F)$. Therefore, as a simplicial map between $|\int \mathscr{D}(\mathcal{A})|$ and $|\int \mathscr{D}(\mathcal{A}')|$, Ψ is given on vertices as

$$\Psi(F, [G, C]) = (\tilde{F}, [\tilde{G}, \tilde{C}]). \tag{7}$$

It follows that Ψ restricts to a map $\mathcal{S}_{L,F} \to \mathcal{S}_{\tilde{L},\tilde{F}}$. $\square$

Remark 3.1.12. From the equality (7) we immediately deduce that for every H such that $H \not\supseteq \widetilde{L}$, the image $\Psi_*(\widehat{\omega}_H)$ vanishes. In order to see this, notice that we can choose a representative of the homology class $\widehat{\omega}_H$ to be $\Omega^{id_G} \subseteq \mathcal{S}_{L,F}$ for some face G of $\mathcal{A}$ supported solely on H. Now, if H is such that $H \not\supseteq \widetilde{L}$, the image $\Psi_*(\widehat{\omega}_H)$ vanishes. In fact, in this case $\mathcal{A}'[\widetilde{G}]$ is the empty arrangement with a unique face K. Now applying Equation (7) to the explicit expression of Ω^{id_G} given in Remark 2.3.2, the image $\Psi(\Omega^{id_G})$ is the single vertex $(\widetilde{G}, [K, K])$.

Let $i : \mathcal{A}' \hookrightarrow \mathcal{A}$ be the inclusion map and assume that for $H \in \mathcal{A}'$ we choose ${}^{i(H)}C \in \mathcal{T}(\mathcal{A}_0)$ as the smallest chamber in $\mathcal{T}(\mathcal{A}_0)$ that contains ${}^{H}C \in \mathcal{T}(\mathcal{A}'_0)$. Then if $H \supseteq \widetilde{L}$ with Equation (7) one checks that $\Psi(\Omega^{id_G}) = \Omega^{id_{\widetilde{G}}}$, thus $\Psi_*(\widehat{\omega}_H) = \widehat{\omega}_H$.

3.2. **Choices for a presentation.** Let $\mathcal{A}$ be an essential toric arrangement in a torus T of dimension r. In order to provide a presentation of the cohomology ring $H^*(M(\mathcal{A}); \mathbb{Q})$ from the combinatorial data we need to make some choices.

Choice 3.2.1. For every layer $L \in \mathcal{C}$ we choose a chamber $B(L) \in \mathcal{T}(\mathcal{A}_0)$ such that the intersection $\overline{B}(L) \cap X_L$ has maximal dimension and we set

$$F(L) := \overline{B}(L) \cap X_L \in \mathcal{F}(\mathcal{A}_0).$$

Hence we will simply write $\mathcal{S}_L$ for $\mathcal{S}_{L,F(L)}$.

Notice that the face $F(T)$ is actually a chamber in $\mathcal{T}(\mathcal{A}_0)$. When the setting of the toric arrangement $\mathcal{A}$ is understood we will write Λ^M for $\Lambda^M_{F(T)}$ and $\widehat{\lambda}^M$ for $\widehat{\lambda}^M_{F(T)}$. It follows from Remark 2.2.9 that we have $\Lambda^M \subset \mathcal{S}_T = \mathcal{S}_{T,F(T)}$. Hence $\widehat{\lambda}^M$ is a homology class in $H_1(\mathcal{S}_T; \mathbb{Q})$.

Choice 3.2.2. Once for all we choose elements $M_1, \ldots, M_r \in \mathcal{C}_{r-1}$ such that

$$\widehat{\mathcal{B}}_T(\mathcal{A}) := \{\widehat{\lambda}^{M_1}, \ldots, \widehat{\lambda}^{M_r}\}$$

is a basis of $H_1(\mathcal{S}_T; \mathbb{Q})$.

This can be done since the arrangement is essential and hence the $\mathbb{Q}$-span of the defining characters is $\mathrm{Hom}(T, \mathbb{C}^*) \otimes \mathbb{Q} \simeq H^1(\mathcal{S}_T; \mathbb{Q})$. By duality the set of 1-dimensional layers $\mathcal{C}_{r-1}$ generate $H_1(T; \mathbb{Q})$. Moreover we have that the projection $\mathcal{S}_T \to T_c$, which is a homeomorphism, maps $\widehat{\lambda}^M \mapsto M$. Hence the set $\{\widehat{\lambda}^M \mid M \in \mathcal{C}_{r-1}\}$ generates $H_1(\mathcal{S}_T; \mathbb{Q})$.

Definition 3.2.3. We define the following set:

$$\widehat{\mathcal{B}}(\mathcal{A}) := \{\widehat{\lambda}^{M_1}, \ldots, \widehat{\lambda}^{M_r}\} \cup \{\widehat{\omega}_H \mid H \in \mathcal{A}\}$$

Clearly we also have that the set $\widehat{\mathcal{B}}(\mathcal{A})$ is a basis of $H_1(\mathrm{Sal}(\mathcal{A}); \mathbb{Q})$. In fact, under the natural map $\mathrm{Sal}(\mathcal{A}) \to T_c$ (see [CD17, Rmk. 4.1.6]) each of the classes $\widehat{\omega}_H$ maps to a trivial homology class in T_c (in fact, all vertices of $\Omega_H^{(id_G)}$, see Remark 2.3.2, map to the same vertex of the celluarization of T_c) Moreover, since for each H we have that $\Phi_{H*}(\widehat{\omega}_H)$ is a non-trivial class in $H_1(\mathrm{Sal}(\overline{\mathcal{A}}_H); \mathbb{Q})$, while the cycle $\Phi_{H'*}(\widehat{\omega}_H)$ is trivial in $H_1(\mathrm{Sal}(\overline{\mathcal{A}}_{H'}); \mathbb{Q})$ for $H' \neq H$ (see Lemma 3.1.6) we have that the classes $\widehat{\omega}_H$ are linearly independent. Finally, since we have the equality

$$\mathrm{rk} H_1(\mathrm{Sal}(\mathcal{A}); \mathbb{Q}) = \mathrm{rk} T + |\mathcal{A}|$$

the set $\widehat{\mathcal{B}}(\mathcal{A})$ is a basis of $H_1(\mathrm{Sal}(\mathcal{A}); \mathbb{Q})$.

Definition 3.2.4. Now we can define the set of classes

$$\mathcal{B}_T(\mathcal{A}) := \{\lambda^{M_1}(T), \ldots, \lambda^{M_r}(T)\}$$

as the basis of $H^1(\mathcal{S}_T;\mathbb{Q})$ that is dual to $\widehat{\mathcal{B}}_T(\mathcal{A})$.

Remark 3.2.5. Recall that there is a natural projection on the compact torus $\mathrm{Sal}(\mathcal{A}) \to T_c$ that induces by restrictions maps $\mathcal{S}_L \to T_c$ (see [CD17, Rmk. 4.1.6 and Thm. 4.2.3]). In particular there is is a natural isomorphism between $H^*(\mathcal{S}_T;\mathbb{Q})$ and $H^*(T_c;\mathbb{Q})$ induced by the maps $\mathcal{S}_T \to T_c$. Using the inclusions

$$\mathcal{S}_T \hookrightarrow \mathrm{Sal}(\mathcal{A}) \to T_c$$

we can identify $H^*(T_c;\mathbb{Q}) \simeq H^*(\mathcal{S}_T;\mathbb{Q})$ with a sub-algebra of $H^*(\mathrm{Sal}(\mathcal{A});\mathbb{Q})$.

Definition 3.2.6. For $i = 1, \ldots, r$ we will write λ^{M_i} for the cohomology class in $H^1(\mathrm{Sal}(\mathcal{A});\mathbb{Q})$ that is the restriction of the class $\lambda^{M_i}(T) \in H^1(\mathcal{S}_T;\mathbb{Q}) \simeq H^1(T_c;\mathbb{Q})$.

Hence we can define the classes ω_H, for $H \in \mathcal{A}$, such that the set

$$\mathcal{B}(\mathcal{A}) := \{\lambda^{M_1}, \ldots, \lambda^{M_r}\} \cup \{\omega_H \mid H \in \mathcal{A}\}$$

is the basis of $H^1(M(\mathcal{A});\mathbb{Q})$ that is dual to $\widehat{\mathcal{B}}(\mathcal{A})$.

For every layer L we will write $\omega_H(L)$ (resp. $\lambda^{M_i}(L)$) for the restriction of the cohomology class ω_H (resp. λ^{M_i}) to $\mathcal{S}_{L,F(L)}$.

We will write $\mathcal{I} \in H^*(M(\mathcal{A});\mathbb{Q})$ for the ideal of $H^*(M(\mathcal{A});\mathbb{Q})$ generated by the classes $\lambda^{M_1}, \ldots, \lambda^{M_r}$. Notice that the definition of the ideal $\mathcal{I}$ depends on the choice of the chamber $B(T)$, but not on the choice of the layers $M_1, \ldots, M_r$.

Choice 3.2.7. For every layer $L \in \mathcal{C}$ of rank $k > 0$ we choose elements $N_1(L), \ldots,$ $N_k(L) \in \mathcal{C}_{r-1}$ contained in L and such that the cycles

$$\{\widehat{\lambda}^{N_1(L)}_{B(L)}, \ldots, \widehat{\lambda}^{N_k(L)}_{B(L)}\}$$

are linearly independent in $H_1(\mathcal{S}_{L,F(L)};\mathbb{Q})$.

Moreover we will consider the set of hypertori

$$H_{i_1}, \ldots, H_{i_s} \in \mathcal{A}$$

such that $L \subset H_{i_j}$, that is the set $\mathcal{A}_L$. Hence we can define the set

Definition 3.2.8.

$$\widehat{\mathcal{B}}_L(\mathcal{A}) := \{\widehat{\lambda}^{N_1(L)}_{B(L)}, \ldots, \widehat{\lambda}^{N_k(L)}_{B(L)}\} \cup \{\widehat{\omega}_{H_{i_1}}, \ldots, \widehat{\omega}_{H_{i_s}}\}$$

Clearly the set $\widehat{\mathcal{B}}_L(\mathcal{A})$ is a basis of $H_1(\mathcal{S}_{L,F(L)};\mathbb{Q})$.

Definition 3.2.9. We write

$$\mathcal{B}_L(\mathcal{A}) = \{\lambda^{N_1(L)}_{B(L)}, \ldots, \lambda^{N_k(L)}_{B(L)}\} \cup \{\omega_{H_{i_1}}(L), \ldots, \omega_{H_{i_s}}(L)\}$$

for the basis of $H^1(\mathcal{S}_{L,F(L)};\mathbb{Q})$ that is dual to $\widehat{\mathcal{B}}_L(\mathcal{A})$.

3.3. Ideals and cohomology maps.

Definition 3.3.1. We define $\mathcal{I}(L) \subset H^*(\mathcal{S}_{L,F(L)};\mathbb{Q})$ as the ideal of $H^*(\mathcal{S}_{L,F(L)};\mathbb{Q})$ generated by the classes $\lambda_{B(L)}^{N_1(L)}, \dots, \lambda_{B(L)}^{N_k(L)}$.

Lemma 3.3.2. *Let $\varphi_L : \mathcal{S}_{L,F(L)} \to \mathrm{Sal}(\mathcal{A})$ be the inclusion map. The induced restriction homomorphism in cohomology $\varphi_L^* : H^1(\mathrm{Sal}(\mathcal{A});\mathbb{Q}) \to H^1(\mathcal{S}_{L,F(L)};\mathbb{Q})$ maps as follows:*

$$\varphi_L^* : \lambda^{M_i} \mapsto \sum a_{hi} \lambda_{B(L)}^{N_h(L)}$$

where the coefficients a_{hi} are given by the relation $\widehat{\lambda}_{F(T)}^{N_h} = \sum a_{hi} \widehat{\lambda}_{F(T)}^{M_i}$ and if $L \subset H$ we have

$$\varphi_L^* : \omega_H \mapsto \omega_H(L) + \sum_{h=1}^{k} \sum_{\substack{F \subseteq N_h(L) \\ H \in S_F(B(L),F(T)) \setminus \mathcal{A}_{N_h(L)}}} \epsilon(H,B) \lambda_{B(L)}^{N_h(L)},$$

while if $L \not\subset H$ we have

$$\varphi_L^* : \omega_H \mapsto \sum_{h=1}^{k} \sum_{\substack{F \subseteq N_h(L) \\ H \in S_F(B(L),F(T)) \setminus \mathcal{A}_{N_h(L)}}} \epsilon(H,B) \lambda_{B(L)}^{N_h(L)}.$$

Proof. Using the homological basis of $H_1(\mathcal{S}_{L,F(L)};\mathbb{Q})$ we can consider the pairings

$$\begin{aligned}
\int_{\widehat{\lambda}_{B(L)}^{N_h}} \varphi_L^*(\lambda^{M_i}) &= \int_{\widehat{\lambda}_{B(L)}^{N_h}} \lambda^{M_i} = \\
&= \int_{\widehat{\lambda}_{F(T)}^{N_h}} \lambda^{M_i} + \sum_{\cdots} \epsilon_H \int_{\widehat{\omega}_H} \lambda^{M_i} = \\
&= \int_{\widehat{\lambda}_{F(T)}^{N_h}} \lambda^{M_i} = \\
&= \int_{\sum a_{hj} \widehat{\lambda}_{F(T)}^{M_j}} \lambda^{M_i} = a_{hi}.
\end{aligned}$$

where the second equality follows from Equation (3) and we don't need to specify the range of the sum and the coefficients of the terms $\widehat{\omega}_H$ since they are not significant in the computation. In the same way we have:

$$\begin{aligned}
\int_{\widehat{\omega}_H} \varphi_L^*(\lambda^{M_i}) &= \int_{\widehat{\omega}_H} \lambda^{M_i} = \\
&= \int_{\widehat{\lambda}_{F(T)}^{N_h}} \lambda^{M_i} + \sum_{\cdots} \epsilon_H \int_{\widehat{\omega}_H} \lambda^{M_i} = 0.
\end{aligned}$$

The analogous computation for ω_H gives:

$$\begin{aligned}\int_{\hat{\lambda}^{N_h}_{B(L)}} \varphi_L^*(\omega_H) &= \int_{\hat{\lambda}^{N_h}_{B(L)}} \omega_H = \\ &= \int_{\hat{\lambda}^{N_h}_{F(T)}} \omega_H + \sum_{\substack{F \subseteq N_k \\ H' \in S_F(B(L),F(T)) \setminus \mathcal{A}_{N_h}}} \epsilon(H, B(L)) \int_{\hat{\omega}_{H'}} \omega_H = \\ &= \sum_{\substack{F \subseteq N_k \\ H' \in S_F(B(L),F(T)) \setminus \mathcal{A}_{N_h}}} \epsilon(H, B(L)) \delta_{H,H'}.\end{aligned}$$

and

$$\int_{\hat{\omega}_{H'}} \varphi_L^*(\omega_H) = \int_{\hat{\omega}_{H'}} \hat{\omega}_H = \delta_{H,H'}.$$

where $H' \in \mathcal{A}_L$ and hence there is a non-zero pairing if and only if $L \subset H$. $\square$

The following result is a strightforward consequence of the formulas of Lemma 3.3.2.

Corollary 3.3.3. *The homomorphism φ_L^* maps $\varphi_L^*(\mathcal{I}) \subset \mathcal{I}(L)$.*

Corollary 3.3.4. *Let $\mu \in H^s(\mathrm{Sal}(\mathcal{A}); \mathbb{Q})$ be the restriction of the class $\mu_T \in H^s(T_c; \mathbb{Q})$. Then $\varphi_L^*(\mu) = 0$ if $\mathrm{rk}(L) > r - s$.*

Proof. This follows immediately from Lemma 3.3.2 since a product of more than $r - \mathrm{rk}(L)$ classes of the type $\lambda^M_{B(L)}$ is zero in $H^*(\mathcal{S}_{L,F(L)}; \mathbb{Q})$. $\square$

The result of Lemma 3.1.6 and Corollary 3.1.7 has the following consequence in cohomology.

Lemma 3.3.5. *Let $\mathcal{A}$ be a toric arrangement in the torus T. Let $L \in \mathcal{C}$ be a layer of $\mathcal{A}$. Consider the quotient arrangement $\overline{\mathcal{A}}_L$ in $\overline{T} = T/L_0$ and the cellular map $\Phi_L : \mathrm{Sal}(\mathcal{A}) \to \mathrm{Sal}(\overline{\mathcal{A}}_L)$. Moreover assume that $\pi_L(F(T)) = F(T/L_0)$. Then for any hypertorus $H \in \mathcal{A}_L$ the cohomology homomorphism*

$$\Phi_L^* : H^1(\mathrm{Sal}(\overline{\mathcal{A}}_L); \mathbb{Q}) \to H^1(\mathrm{Sal}(\mathcal{A}); \mathbb{Q})$$

induced by Φ_L maps as follows:

$$\Phi_L^* : \omega_{\pi_L(H)} \mapsto \omega_H.$$

Corollary 3.3.6. *Let $\mathcal{A}$ be a toric arrangement in the torus T. Let $L \in \mathcal{C}$ be a layer of $\mathcal{A}$. Consider the quotient arrangement $\mathrm{Sal}(\overline{\mathcal{A}}_L)$ in $\overline{T} = T/L_0$ and the cellular map $\Phi_L : \mathrm{Sal}(\mathcal{A}) \to \mathrm{Sal}(\overline{\mathcal{A}}_L)$. Moreover assume that $\pi_L(F(T)) = F(\overline{T})$. Then for any layer $Y \in \mathcal{C}$, if we consider the cohomology map induced by the restriction*

$$\Phi_{L|} : \mathcal{S}_{Y,F(Y)} \mapsto \mathcal{S}_{\pi_L(Y),\pi_L(F(Y))}$$

the following holds: for any hypertorus $H \in \mathcal{A}$ such that $Y \subset H$ and $L \subset H$ we have

$$\Phi_{L|}{}^*(\omega_{\pi_L(H)}(\pi_L(Y)) = \omega_H(Y).$$

Proof. This follows immediately from the previous lemma using the commutativity of the diagram

$$\begin{tikzcd} \mathcal{S}_{Y,F(Y)} \arrow[r, hook, "\varphi_Y"] \arrow[d, "\Phi_L"] & \mathrm{Sal}(\mathcal{A}) \arrow[d, "\Phi_L"] \\ \mathcal{S}_{\pi_L(Y),\pi_L(F(Y))} \arrow[r, hook, "\varphi_{\pi_L(Y)}"] & \mathrm{Sal}(\overline{\mathcal{A}}_L). \end{tikzcd}$$

□

Recall that we can assume the arrangement $\mathcal{A}$ to be totally ordered and we actually fix such an ordering.

Given any $d-r$-dimensional layer $L \in \mathcal{C}_r$ in the torus T of dimension d, we can consider the linear arrangement $\mathcal{A}[L]$.

The ordering of $\mathcal{A}$ induces an ordering of $\mathcal{A}[L]$. For every element of the nbc-basis associated to the arrangement $\mathcal{A}[L]$ we can consider the corresponding ordered subset $S = (H_{i_1}, \dots, H_{i_r}) \subset \mathcal{A}$.

We will write ω_S for the product $\omega_{H_{i_1}} \cdots \omega_{H_{i_r}} \in H^r(M(\mathcal{A}); \mathbb{Q})$.

Corollary 3.3.7. *Let $\mathcal{A}$ be a toric arrangement in the torus T of dimension d. Let $L \in \mathcal{C}$ be a layer of $\mathcal{A}$ of rank r. Consider the quotient arrangement $\mathrm{Sal}(\overline{\mathcal{A}}_L)$ in $\overline{T} = T/L_0$ and the cellular map $\Phi_L : \mathrm{Sal}(\mathcal{A}) \to \mathrm{Sal}(\overline{\mathcal{A}}_L)$. Assume that $\pi_L(F(T)) = F(\overline{T})$.*

Let $S = (H_{i_1}, \dots, H_{i_r})$ be an element of the nbc-basis of $\mathcal{A}[L]$.

Let $\alpha \in H^*(\mathrm{Sal}(\overline{\mathcal{A}}_L); \mathbb{Q})$ be a class such that $\alpha - \omega_{\pi_L(S)}$ restricts to zero in $H^*(\mathcal{S}_{\pi_L(L)}; \mathbb{Q})$ and α restricts to a class in $\mathcal{I}(\overline{L}') \subset H^*(\mathcal{S}_{\overline{L}'}; \mathbb{Q})$ for $\overline{L}' \neq \pi_L(L)$.

Then we have that the class ${\Phi_L}^*(\alpha) - \omega_S$ restricts to 0 in $H^*(\mathcal{S}_{L'}; \mathbb{Q})$ if $\pi_L(L') = \pi_L(L)$, that is if $L' \subseteq L$, and ${\Phi_L}^*(\alpha)$ restricts to a class of $\mathcal{I}(L')$ in $H^*(\mathcal{S}_{L'}; \mathbb{Q})$ if $\pi_L(L') \neq \pi_L(L)$.

Proof. The result follows from Corollary 3.3.6 by multiplicativity. □

Remark 3.3.8. The cohomology ring $H^*(\mathrm{Sal}(\mathcal{A}); \mathbb{Z})$ is a free $\mathbb{Z}$ module (this has been proved in [dD15], but follows also from [CD17, Rmk. 6.1.1]).

Lemma 3.3.9. *The cohomology ring $H^*(\mathcal{S}_L; \mathbb{Z})$ is a module over $H^*(T; \mathbb{Z})$ generated by the restriction of the classes ω_S for $S \in \mathrm{nbc}(\mathcal{A}[L])$.*

Proof. The lemma follows from [CD17, Thm. 4.2.3], where the homotopy equivalence

$$\Theta_{F_0} : \mathcal{S}_{L,F} \to |F(\mathcal{A}^L)| \times \mathrm{Sal}(\mathcal{A}[L])$$

is given. Since the projection on the first component of Θ_{F_0} is the projection $\mathcal{S}_{L,F} \to L_c$ induced by $\mathrm{Sal}(\mathcal{A}) \to T_c$, we have that the cohomology of $\mathcal{S}_{L,F}$ is a $H^*(T; \mathbb{Z})$-module generated by generators of the Orlik-Solomon algebra $H^*(\mathrm{Sal}(\mathcal{A}[L]); \mathbb{Z})$. Finally the following hold:

a) The homology classes $\widehat{\omega}_H$ are non-trivial (in fact their image in $\mathrm{Sal}(\overline{\mathcal{A}}_H)$ is non-zero).
b) For $H \supset L$ the homology classes $\widehat{\omega}_H$ have a representative in $\mathcal{S}_{L,F}$.
   In fact, for every $G \in \mathcal{F}(A^L)$ the arrangement $\mathcal{A}[G]$ contains H and hence in particular a face W supported on H. Then we can consider the morphism m of $\mathcal{F}(\mathcal{A})$ that arises from the order relation $G \leqslant W$ in $\mathcal{F}(\mathcal{A}[G])$ (ensuring that

$F_m = W$). Thus we see that the complex $\mathcal{S}^F(\mathcal{A}[G])$ contains the subposet Ω^m of Remark 2.3.2.

c) The classes $\widehat{\omega}_H$ project to trivial classes in $T_c = \mathcal{S}_T$.
d) If $\mathcal{B}$ is the essentialization of $\mathcal{A}\backslash\{H\}$, the classes $\widehat{\omega}_H$ maps to trivial classes in $\mathrm{Sal}(\mathcal{B})$ (this claim follows directly from Lemma 3.1.11 and Remark 3.1.11 when $\mathcal{A}$ and $\mathcal{A}\backslash\{H\}$ have the same rank, otherwise from Lemma 3.1.6).

Hence we have that the classes $\omega_H(L)$ for $H \supset L$, that are the restrictions of the corresponding classes $\widehat{\omega}_H$ to $\mathcal{S}_L$, are the standard generators of the Orlik-Solomon algebra $H^*(\mathrm{Sal}(\mathcal{A}[L]); \mathbb{Z})$. □

3.4. **Group action and the cohomology ring.**

Definition 3.4.1. Let $\mathcal{A}$ be a toric arrangement in a torus T. Let $L \in \mathcal{C}_r(\mathcal{A})$ a layer and let $S \in \mathrm{nbc}(\mathcal{A}[L])$. We define the subarrangement $\mathcal{A}_S \subset \mathcal{A}$ as the set of hypertori of $\mathcal{A}$ associated to the elements of S.

Definition 3.4.2. Let $\mathcal{A}$ be a toric arrangement in a torus T. We define the *stabilizer* of $\mathcal{A}$ as the group $G \subset T$ given by

$$G := \{g \in T | \forall H \in \mathcal{A}, gH = H\}.$$

and the *essential stabilizer* of $\mathcal{A}$ as the group $\mathcal{G}$ of the connected components of G. Hence $\mathcal{G} := G/G_0$, where G_0 is the connected component of the identity of G.

Given a layer $L \in \mathcal{C}_r(\mathcal{A})$ and an element $S \in \mathrm{nbc}(\mathcal{A}[L])$ we write G_S (resp. $\mathcal{G}_S$) for the stabilizer (resp. essential stabilizer) of $\mathcal{A}_S$.

Remark 3.4.3. We notice that if $\mathcal{A}$ is essential then G is discrete and we have $\mathcal{G} \simeq G$. In general, by choosing a direct summand of G_0 in T we get a lifting $\mathcal{G} \to G$. Hence we can always identify $\mathcal{G}$ with a subgroup of T that acts by multiplication on the layers in $\mathcal{C}(\mathcal{A})$.

Proposition 3.4.4. *Let $\mathcal{A}$ be an essential arrangement in a torus T of dimension d. Let $L \in \mathcal{C}_r$ a layer of rank $r \geqslant 0$ and let $S \in \mathrm{nbc}(\mathcal{A}[L])$ of length r. For a layer of $Y \in \mathcal{C}(\mathcal{A})$ let $\overline{Y}$ be the smallest layer in $\mathcal{C}(\mathcal{A}_S)$ containing Y. There exists a unique cohomology class $\omega_{S,L} \in H^r(\mathrm{Sal}(\mathcal{A}); \mathbb{Q})$ such that for every layer of $Y \in \mathcal{C}(\mathcal{A})$ and for every face $F \in \mathcal{F}(\mathcal{A}_0)$ with $\mathrm{supp}(F) = Y$ we have:*

i) If $L \subset \overline{Y}$ then $\omega_{S,L} - \frac{\omega_S}{|\mathrm{Stab}_{\mathcal{G}_S}(\overline{Y})|}$ restricts to 0 in $H^r(\mathcal{S}_{Y,F}; \mathbb{Q})$;
ii) If $L \nsubseteq \overline{Y}$ then $\omega_{S,L}$ restricts to 0 in $H^r(\mathcal{S}_{Y,F}; \mathbb{Q})$.

Proof. We prove our statement by induction on d and r, considering the following cases.

a) $[d = 0]$ For $L = T$ and $S = \varnothing$ we can just set $\omega_{S,L}$ as the constant class $1 \in H^0(\mathrm{Sal}(\mathcal{A}); \mathbb{Q})$.
b) $[d = 1, r = 1]$ In this case the result is trivial as we can set $\omega_{S,L} = \omega_H$.
c) $[d > 1, r < d]$ Suppose now that $\dim T = d > 1$. We can assume that the statement is true for any essential arrangement in a torus T' with $\dim T' < d$. Then if $\mathrm{rk}(L) = r < d$ the statement follows also for $\dim T = d$. In fact, given $S = (H_{i_1}, \dots, H_{i_r}) \in \mathrm{nbc}(\mathcal{A}[L])$ of length r, we can consider the quotient arrangement $\overline{\mathcal{A}}_S := \mathcal{A}_S/L_0$, the layer $\overline{L} := L/L_0 \in \mathcal{C}_r(\overline{\mathcal{A}}_S)$ and the element $\overline{S} := \pi_L(S) \in \mathrm{nbc}(\overline{\mathcal{A}}_S[\overline{L}])$. By induction the class $\omega_{\overline{S},\overline{L}}$ exists and we can set

$$\omega_{S,L} := \Phi_L^*(\omega_{\overline{S},\overline{L}}).$$

In fact by induction, $\omega_{\overline{S},\overline{L}} - \frac{\omega_{\overline{S}}}{|\operatorname{Stab}_{\mathcal{G}_{\overline{S}}}(\pi_L(Y))|}$ restricts to 0 in $\mathcal{S}_{\pi_L(Y),R}$ if $\overline{L} \subset \pi_L(Y)$, that is, if $L \subset \overline{Y}$, and $\omega_{\overline{S},\overline{L}}$ restricts to 0 in $\mathcal{S}_{\pi_L(Y),R}$ if $\overline{L} \nsubseteq \pi_L(Y)$, that is, if $L \nsubseteq \overline{Y}$. Hence, since $\mathcal{G}_S \simeq \mathcal{G}_{\overline{S}}$, according to Corollary 3.3.7 we have that

$$\omega_{S,L} - \frac{\omega_S}{|\operatorname{Stab}_{\mathcal{G}_S}(\overline{Y})|}$$

restricts to 0 in $H^*(\mathcal{S}_{Y,F};\mathbb{Q})$ for all Y, F such that $L \subseteq \overline{Y}$ and $\omega_{S,L}$ restricts to 0 in $H^*(\mathcal{S}_{Y',F'};\mathbb{Q})$ if $L \nsubseteq \overline{Y'}$.

d) $[d > 1, r = d]$ First we consider the case when $\mathcal{A} = \mathcal{A}_S$. Since in this case the linear arrangement $\mathcal{A}_0$ is isomorphic to a boolean arrangement, it is enough to consider a single chamber $B_0 \in \mathcal{T}(\mathcal{A}_0)$ and to set $F(Y) = \overline{B_0} \cap Y_0$, because B_0 is adjacent to every hyperplane in $\mathcal{A}_0$. Hence the results stated in [CD17, Thm. A and B] hold in this special setting. In particular from [CD17, Thm. A] we can choose a class $\omega_{S,L} \in H^r(\operatorname{Sal}(\mathcal{A});\mathbb{Q})$ such that $\omega_{S,L} - \omega_S$ restricts to zero in $\mathscr{L}_L^{\operatorname{rk}(L)}$ (and hence in $H^r(\mathcal{S}_{L,F(L)};\mathbb{Q})$) and $\omega_{S,L}$ restricts to zero in $\mathscr{L}_Y^{\operatorname{rk}(Y)}$ for $Y \neq L$. From the definition of coherent elements (see [CD17, Def. 2.3.4]) and from [CD17, Thm. B] we have that $\omega_{S,L}$ restricts to zero in $H^r(\mathcal{S}_{Y,F(Y)};\mathbb{Q})$ for $Y \neq L$. Hence $\omega_{S,L}$ satisfies the required conditions i) and ii) when $F = F(Y)$.

Recall that for $g \in \operatorname{Stab}_{\mathcal{G}_S}(Y)$ we have a multiplication map $g : Y \to Y$ that from Lemma 3.1.10 is homotopic to the identity map and hence induces the identity map on $H^*(\mathcal{S}_{Y,F})$.

We need to prove that i) and ii) are satisfied also for every complex $\mathcal{S}_{Y,F}$, with $F \neq F(Y)$. This is trivially true for $Y = L$, since in this case $F = F(Y)$ is the only possible face in $\mathcal{F}(\mathcal{A}_0)$ with $\operatorname{supp}(F) = Y_0$. When $Y \neq L$ we can consider the class

$$\omega_{S,L}^Y := \sum_{g \in \operatorname{Stab}_{\mathcal{G}_S}(Y)} g^* \omega_{S,L} \in H^*(\operatorname{Sal}(\mathcal{A});\mathbb{Q}).$$

Let $S = S_1 \sqcup S_2$, where the elements of S_1 are the elements of S that contains Y. Let $\epsilon \in \{\pm 1\}$ be the sign such that $\omega_S = \epsilon \omega_{S_1} \omega_{S_2}$. Consider the map $\Phi_{S_1} : \operatorname{Sal}(\mathcal{A}) \to \operatorname{Sal}(\overline{\mathcal{A}}_{S_1})$. We claim that

$$\omega_{S,L}^Y = \epsilon \Phi_{S_1}^* (\omega_{\overline{S_1}, \pi_Y(Y)}) \omega_{S_2}. \tag{8}$$

The equality follows checking that the two terms agree when restricted to $\mathcal{S}_{W,F(W)}$ for every $W \in \mathcal{C}(A_S)$. In fact we have:

$$\varphi_W^*(\omega_{S,L}^Y) = \frac{\omega_S}{|\operatorname{Stab}_{\mathcal{G}_S}(W)|} \cdot |\operatorname{Stab}_{\mathcal{G}_S}(Y) \cap \operatorname{Stab}_{\mathcal{G}_S}(W)|$$

if $L \subseteq \operatorname{Stab}_{\mathcal{G}_S}(Y).W$ and

$$\varphi_W^*(\omega_{S,L}^Y) = 0$$

otherwise. On the other side we have

$$\varphi_W^*(\epsilon \Phi_{S_1}^*(\omega_{\overline{S_1},\pi_Y(Y)})\omega_{S_2}) = \epsilon \frac{\omega_{S_1}}{|\operatorname{Stab}_{\mathcal{G}_{\overline{S_1}}}(\pi_Y(W))|} \omega_{S_2} = \frac{\omega_S}{|\operatorname{Stab}_{\mathcal{G}_{\overline{S_1}}}(\pi_Y(W))|}$$

if $\pi_Y(Y) \subseteq \pi_Y(W)$ and

$$\varphi_W^*(\epsilon \Phi_{S_1}^*(\omega_{\overline{S_1},\pi_Y(Y)})\omega_{S_2}) = 0$$

otherwise. The equality

$$\frac{|\operatorname{Stab}_{\mathcal{G}_S}(Y) \cap \operatorname{Stab}_{\mathcal{G}_S}(W)|}{|\operatorname{Stab}_{\mathcal{G}_S}(W)|} = \frac{1}{|\operatorname{Stab}_{\mathcal{G}_{\overline{S_1}}}(\pi_Y(W))|}$$

follows since the kernel of the surjective homomorphism

$$\operatorname{Stab}_{\mathcal{G}_S}(W) \to \operatorname{Stab}_{\mathcal{G}_{\overline{S_1}}}(\pi_Y(W))$$

is exactly the subgroup $\operatorname{Stab}_{\mathcal{G}_S}(Y) \cap \operatorname{Stab}_{\mathcal{G}_S}(W)$. Moreover the two conditions $L \subseteq \operatorname{Stab}_{\mathcal{G}_S}(Y).W$ and $\pi_Y(Y) \subseteq \pi_Y(W)$ are equivalent. In fact $\pi_Y(L) = \pi_Y(Y)$ and $\pi_Y(\operatorname{Stab}_{\mathcal{G}_S}(Y).W) = \pi_Y(W)$ and this proves that the first condition implies the second one. Since $\pi_Y(W)$ is the quotient by Y_0 of the smallest layer $\overline{W}$ of $\mathcal{A}_{S_1}$ containing W, the second condition means that $Y \subset \overline{W}$ and then L, which is a point of Y, is contained in $\operatorname{Stab}_{\mathcal{G}_S}(Y).W$, hence the second condition implies the first. Then we have proved Equation (8). In particular we have by induction that for every face $F \in \mathcal{F}(\mathcal{A}_0)$ such that $\operatorname{supp}(F) = Y_0$ the class $\omega_{S,L}$ restricts in $H^*(\mathcal{S}_{Y,F})$ as $\frac{\omega_S}{|\operatorname{Stab}_{\mathcal{G}_S}(Y)|}$ and this proves the proposition if $\mathcal{A} = \mathcal{A}_S$.

If $\mathcal{A} \neq \mathcal{A}_S$ we can consider the map $\operatorname{Sal}(\mathcal{A}) \to \operatorname{Sal}(\mathcal{A}_S)$ induced by the inclusion $M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_S)$ and the result follows applying Lemma 3.1.11 and Remark 3.1.12. □

Theorem 3.4.5. *Let $\mathcal{A}$ be an essential toric arrangement in T. The homomorphism of algebras*

$$\bigoplus_{L \in \mathcal{C}(\mathcal{A})} \varphi_L : H^*(\operatorname{Sal}(\mathcal{A}); \mathbb{Z}) \to \bigoplus_{L \in \mathcal{C}(\mathcal{A})} H^*(\mathcal{S}_L; \mathbb{Z})$$

is injective.

Proof. Let R be a ring and let $\mathcal{I}_R$ (resp. $\mathcal{J}_R$) be the ideal of $A_R^* := H^*(\operatorname{Sal}(\mathcal{A}); R)$ (resp. $B_R^* := \bigoplus_{L \in \mathcal{C}(\mathcal{A})} H^*(\mathcal{S}_L; R)$) generated by the restriction of $H^1(T; R) \simeq H^1(\mathcal{S}_T; \mathbb{R})$. Note that $\mathcal{I}_R$ and $\mathcal{J}_R$ are graded ideals with respect to the cohomological graduation and we will write $(\mathcal{I}_R)_j$ (resp. $(\mathcal{J}_R)_j$) for the graded component of $\mathcal{I}_R$ (rsp. $\mathcal{J}_R$) in A_R^j (resp. B_R^j). Let $\operatorname{Gr}(A_R)$ and $\operatorname{Gr}(B_R)$ be the associated bi-graded groups, where we write $\operatorname{Gr}_i(A_R^j)$ (resp. $\operatorname{Gr}_i(B_R^j)$) for $(\mathcal{I}_R^i/\mathcal{I}_R^{i+i})_j$ (resp. $(\mathcal{J}_R^i/\mathcal{J}_R^{i+i})_j$).

The map $\Phi := \oplus_{L \in \mathcal{C}(\mathcal{A})} \varphi_L$ induces an homomorphism of bi-graded groups

$$\overline{\Phi} : \operatorname{Gr}_i(A_R^j) \to \operatorname{Gr}_i(B_R^j).$$

As recalled in Remark 3.3.8, the cohomology groups $H^*(\operatorname{Sal}(\mathcal{A}); \mathbb{Z})$ and $H^*(\mathcal{S}_L; \mathbb{Z})$ are torsion free and hence $A_\mathbb{Z}$ (resp. $B_\mathbb{Z}$) includes in $A_\mathbb{Q}$ (resp. $B_\mathbb{Q}$). Moreover the injectivity of $\overline{\Phi}$ implies the injectivity of the map Φ. As a consequence of these two facts, in order to prove that Φ is injective for $R = \mathbb{Z}$ it will be enough to prove that $\overline{\Phi}$ is injective when $R = \mathbb{Q}$.

This can be seen showing that $\operatorname{Gr}_i(A_\mathbb{Q}^j)$ and $\overline{\Phi}(\operatorname{Gr}_i(A_\mathbb{Q}^j))$ have the same dimension. In fact if we fix $L \in \mathcal{C}(\mathcal{A})$ with $\operatorname{rk}(L) = l$ we have that $\operatorname{Gr}_i(H^{l+i}(\mathcal{S}_L; \mathbb{Q})) \simeq H^l(M(\mathcal{A}[L]); \mathbb{Q}) \otimes H^i(L; \mathbb{Q})$. For a given $S \in \operatorname{nbc}(\mathcal{A}[L])$ with $|S| = l$ and $\lambda \in H^i(T; \mathbb{Q})$, the class $\omega_{S,L} \cdot \lambda$ belongs to $\operatorname{Gr}_i(A_\mathbb{Q}^{l+i})$.

It follows from Proposition 3.4.4 that $\omega_{S,L}$ maps to $\omega_S(L) \otimes \lambda$ in the graded piece $\operatorname{Gr}_i(H^{l+i}(\mathcal{S}_L; \mathbb{Q}))$.

Moreover for $L' \neq L$ with $\operatorname{rk}(L') = l'$ we have that $\omega_{S,L}$ maps to 0 in the graded piece $\operatorname{Gr}_i(H^{l+i}(\mathcal{S}_{L'}; \mathbb{Q}))$. Again this follows from Lemma 3.3.2 and Proposition

3.4.4 since: either at least one of the hypertori H_s for $s \in S$ does not contains L' and then $\omega_{S,L}$ maps to $\mathcal{J}_R$, either $L' \subsetneq L$ and hence $\omega_S(L)$ has dimension less then l' in $H^*(\mathcal{S}_{L'};\mathbb{Q})$.

As a consequence the images of the classes $\omega_{S,L} \cdot \lambda$ for $L \in \mathcal{C}(\mathcal{A})$, $S \in \mathrm{nbc}(\mathcal{A}[L])$ with $|S| = \mathrm{rk}(L)$ and λ in a basis of $H^i(L;\mathbb{Q})$ in $\mathrm{Gr}(B_\mathbb{Q})$ are linearly independent and the rank of the image of $\mathrm{Gr}(A_\mathbb{Q})$ in $\mathrm{Gr}(B_\mathbb{Q})$ is greater or equal to

$$\sum_{L \in \mathcal{C}(\mathcal{A})} 2^{d-\mathrm{rk}(L)} \dim H^{\mathrm{rk}(L)}(M(\mathcal{A}[L]);\mathbb{Q})$$

that is the dimension of $A_\mathbb{Q}$ (see [Loo93, DCP05]). Hence $\overline{\Phi}$ is an injective homomorphism. □

Proposition 3.4.6. *The classes $\omega_{S,L}$ defined in Proposition 3.4.4 are integer classes.*

Proof. Following the same pattern of the proof of Proposition 3.4.4, we can prove the result by induction. The claim follows immediately for the cases a) (since 1 is an integer class) b) (since the classes ω_H are integer classes) and c) since the pull back of an integer class is an integer class.

Concerning case d), since the restriction of an integer class is an integer class, we can assume that $\mathcal{A} = \mathcal{A}_S$. We will show that for a given layer Y such that $L \subset Y$ and $|\operatorname{Stab}_{\mathcal{G}_S}(Y)| \neq 1$ the restriction of the class $\omega_{S,L}$ in $H^*(\mathcal{S}_{Y,F};\mathbb{Q})$ is an integer class. We consider the group $\mathcal{N} := \operatorname{Stab}_{\mathcal{G}_S}(Y)$ and the quotient $\mathrm{Sal}(\mathcal{A}) \to \mathrm{Sal}(\mathcal{A})/\mathcal{N}$. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Sal}(\mathcal{A}) \arrow[r] & \mathrm{Sal}(\mathcal{A})/\mathcal{N} \\ \mathcal{S}_{Y,F} \arrow[u, hook] \arrow[r, "\pi_\mathcal{N}"] & \mathcal{S}_{Y,F}/\mathcal{N} \arrow[u, hook] \end{tikzcd}$$

We have an homotopy equivalence

$$\begin{tikzcd} \mathcal{S}_{Y,F} \arrow[r, "\pi_\mathcal{N}"] \arrow[d, "\simeq"] & \mathcal{S}_{Y,F}/\mathcal{N} \arrow[d, "\simeq"] \\ \mathrm{Sal}(\mathcal{A}[Y]) \times Y \arrow[r] & \mathrm{Sal}(\mathcal{A}[Y]) \times (Y/\mathcal{N}) \end{tikzcd}$$

Let $m = |\operatorname{Stab}_{\mathcal{G}_S}(Y)|$. Notice that when $i = \dim(Y)$ the cohomology map

$$H^i(Y/\mathcal{N};\mathbb{Q}) \to H^i(Y;\mathbb{Q})$$

induced by the m-fold covering $Y \to Y/\mathcal{N}$ is the multiplication by m.

Recall that we are assuming that $r = \mathrm{rk}(T) = \mathrm{rk}(\mathcal{A}) = |S|$ and hence $i = \dim Y = r = \mathrm{rk}(Y)$.

The class $\omega_S \in H^r(\mathrm{Sal}(\mathcal{A});\mathbb{Z})$ is $\mathcal{N}$-invariant and hence it is the pullback of an integer class $\overline{\omega}_S \in H^r(\mathrm{Sal}(\mathcal{A})/\mathcal{N};\mathbb{Z})$. Moreover the class $\overline{\omega}_S$ restricts to a class $\beta \in H^r(\mathcal{S}_{Y,F}/\mathcal{N};\mathbb{Z}) \simeq H^{\mathrm{rk}(Y)}(\mathrm{Sal}(\mathcal{A}[Y]);\mathbb{Z}) \otimes H^i(Y;\mathbb{Z})$. This implies that $\varphi^*_{Y,F}(\omega_S) = \pi^*_\mathcal{N}(\beta)$ is m-times an integer class in $H^r(\mathcal{S}_{Y,F};\mathbb{Z})$ and hence $\omega_{S,L}$ restricts to an integer class in $\mathcal{S}_{Y,F}$. From the injectivity of the map

$$H^*(\mathrm{Sal}(\mathcal{A});\mathbb{Z}) \to \bigoplus_{Y \in \mathcal{C}} H^*(\mathcal{S}_Y;\mathbb{Z})$$

it follows that the class $\omega_{S,L}$ is an integer class. □

Theorem 3.4.7. *Let $\mathcal{A}$ be an essential toric arrangement in T. The integer cohomology ring $H^*(\mathrm{Sal}(\mathcal{A});\mathbb{Z})$ is generated as a module over $H^*(T;\mathbb{Z})$ by the classes $\omega_{S,L}$ for $L \in \mathcal{C}(\mathcal{A})$ and $S \in \mathrm{nbc}(\mathcal{A}[L])$.*

Proof. Since we have proved in Theorem 3.4.5 that the map $\Phi = \oplus_{L\in\mathcal{C}(\mathcal{A})}\varphi_L$ is injective over $\mathbb{Z}$, it will be enough to show that the image of the homomorphism of algebras

$$\bigoplus_{L\in\mathcal{C}(\mathcal{A})} \varphi_L : H^*(\mathrm{Sal}(\mathcal{A});\mathbb{Z}) \to \bigoplus_{L\in\mathcal{C}(\mathcal{A})} H^*(\mathcal{S}_L;\mathbb{Z})$$

is the $H^*(T;\mathbb{Z})$-module generated by the restrictions of the classes $\omega_{S,L}$ for $L \in \mathcal{C}(\mathcal{A})$ and $S \in \mathrm{nbc}(\mathcal{A}[L])$.

We keep the notation of the proof of Theorem 3.4.5. Let A'_R the sub-$H^*(T;R)$-module of A_R generated by the classes $\omega_{S,L}$ for $L \in \mathcal{C}(\mathcal{A})$ and $S \in \mathrm{nbc}(\mathcal{A}[L])$. Using the injectivity of $\Phi : A^*_{\mathbb{Z}} \to B^*_{\mathbb{Z}}$ and the fact that the surjectivity of $\overline{\Phi} : \mathrm{Gr}(A'_R) \to \overline{\Phi}(\mathrm{Gr}(A_R))$ implies the surjectivity of $\Phi : A'_R \to \Phi(A_R)$, in order to prove that $A'_{\mathbb{Z}} = A_{\mathbb{Z}}$ we will show that

$$\overline{\Phi}(\mathrm{Gr}(A'_{\mathbb{Z}})) = \overline{\Phi}(\mathrm{Gr}(A_{\mathbb{Q}})) \cap \mathrm{Gr}(B_{\mathbb{Z}}).$$

In fact, since we have the inclusions $\overline{\Phi}(\mathrm{Gr}(A'_{\mathbb{Z}})) \subset \overline{\Phi}(\mathrm{Gr}(A_{\mathbb{Z}})) \subset \overline{\Phi}(\mathrm{Gr}(A_{\mathbb{Q}})) \cap \mathrm{Gr}(B_{\mathbb{Z}})$, the equality between the first and the last term implies the equality between the first and the second one. As we have seen in the proof of Theorem 3.4.5, $\overline{\Phi}$ maps the class of $\omega_{S,L}$ in $\mathrm{Gr}_0(H^{\mathrm{rk}(L)}(\mathrm{Sal}(\mathcal{A});\mathbb{Z}))$ to the class $\omega_S(L)$ in $\mathrm{Gr}_0(H^{\mathrm{rk}(L)}(\mathcal{S}_L;\mathbb{Z})$. Since $H^*(\mathcal{S}_L;\mathbb{Z}) \simeq H^*(M(\mathcal{A}[L]);\mathbb{Z}) \otimes H^*(L)$ we have that the set of classes $\omega_S(L)$ for $S \in \mathrm{nbc}(\mathcal{A}[L])$ is a set of generators of

$$\bigoplus_i \mathrm{Gr}_i(H^{\mathrm{rk}(L)+i}(\mathcal{S}_L;\mathbb{Z})) \simeq \bigoplus_i H^{\mathrm{rk}(L)}(M(\mathcal{A}[L]);\mathbb{Z}) \otimes H^i(L)$$

as a $H^*(T;\mathbb{Z})$-module. The sum of these modules, for $L \in \mathcal{C}(\mathcal{A})$, is the intersection $\overline{\Phi}(\mathrm{Gr}(A_{\mathbb{Q}})) \cap \mathrm{Gr}(B_{\mathbb{Z}})$. Hence the claim follows. □

Example 3.4.8. As an example of our result we provide an explicit description of the cohomology of the complement of the toric arrangement $\mathcal{A} = \{H_0, H_1, H_2\}$ in $T = (\mathbb{C}^*)^2$ given by:

$$H_0 = \{z \in T | z_0 = 1\}; \quad H_1 = \{z \in T | z_0 z_1^2 = 1\}; \quad H_2 = \{z \in T | z_1 = 1\}.$$

The associated hyperplane arrangement $\mathcal{A}_0$ in $V = \mathbb{R}^2$ is given by the corresponding hyperplanes

$$W_0 = \{x \in V | x_0 = 0\}; \quad W_1 = \{x \in V | x_0 + 2x_1 = 0\}; \quad W_2 = \{x \in V | x_1 = 0\}.$$

We consider in $\mathcal{T}(\mathcal{A}_0)$ the chambers $B_0 = \{x \in V | x_0 < 0, x_0 + 2x_1 > 0\}$ and $B_1 = \{x \in V | x_1 > 0, x_0 + 2x_1 < 0\}$ (see Figure 1). The poset of layers $\mathcal{C}(\mathcal{A})$ is given by the elements T, H_0, H_1, H_2 and the points $P = \{(1, 1)\}$, $Q = \{(1, -1)\}$. In order to define the subcomplexes S_L for $L \in \mathcal{C}(\mathcal{A})$ we need to choose the chamber $B(L)$. We can do this as follows: $B(H_2) = B_1$, $B(L) = B_0$ for $L \neq H_2$. Moreover for $H \in \mathcal{A}$ choose ${}^HC : B(H)$. As a basis $\widehat{\mathcal{B}}_T(\mathcal{A})$ we can choose the set $\{\widehat{\lambda}^{H_0}_{B_0}, \widehat{\lambda}^{H_2}_{B_0}\}$. All the other choices of basis are natural. In Table 1 we describe the restriction of each generator of the cohomology of $\mathrm{Sal}(\mathcal{A})$ to each one of the subcomplexes $S_L = S_{L,F(L)}$, for $L \in \mathcal{C}(\mathcal{A})$. Cells are empty when a class restricts to zero. The multiplicative structure of the cohomology of $\mathrm{Sal}(\mathcal{A})$ is induced by the multiplcative structure on each subcompex.

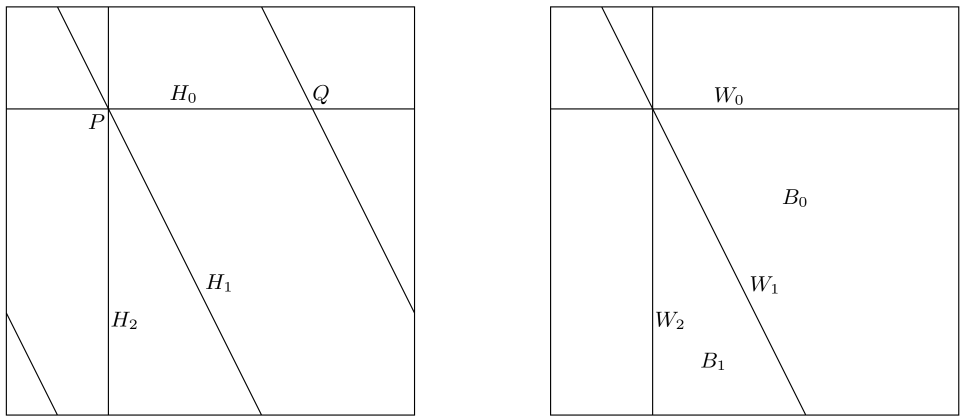


FIGURE 1. The real picture of the toric arrangement $\mathcal{A}$ of Example 3.4.8 (on the left) and of the corresponding central arrangement $\mathcal{A}_0$ (on the right).

$\mathrm{Sal}(\mathcal{A})$	$\mathcal{S}_T$	$\mathcal{S}_{H_0}$	$\mathcal{S}_{H_1}$	$\mathcal{S}_{H_2}$	$\mathcal{S}_P$	$\mathcal{S}_Q$
$\lambda_{B_0}^{H_0}$	$\lambda_{B_0}^{H_0}$	$\lambda_{B_0}^{H_0}$	$\lambda_{B_0}^{H_1}$			
$\lambda_{B_0}^{H_2}$	$\lambda_{B_0}^{H_2}$		$2\lambda_{B_0}^{H_1}$	$\lambda_{B_1}^{H_2}$		
ω_{H_0}		ω_{H_0}			ω_{H_0}	ω_{H_0}
ω_{H_1}			ω_{H_1}	$2\lambda_{B_1}^{H_2}$	ω_{H_1}	ω_{H_1}
ω_{H_2}				ω_{H_2}	ω_{H_2}	
$\omega_{\{0,2\},P}$					$\omega_{H_0}\omega_{H_2}$	
$\omega_{\{1,2\},P}$				$\lambda_{B_1}^{H_2}\omega_{H_2}$	$\omega_{H_1}\omega_{H_2}$	
$\omega_{\{1,2\},Q}$						$\omega_{H_1}\omega_{H_2}$

TABLE 1

4. Representations of arithmetic matroids

In Section 7 of [CD17] we investigate the dependency of our presentation of from the combinatorial data. There, we claim the following result (where the last qualifier was implicit in the paragraphs preceding this theorem in [CD17]).

Theorem 4.0.1 ([CD17, Thm. 7.2.1])**.** *If an arithmetic matroid with a basis of multiplicity* 1 *is representable by a matrix A, then, if we fix such a basis, the matrix A is unique up to sign reversal of the column vectors and up to a unimodular transformation from the left.*

The proof given in the paper is not correct. As explained in [Len19], the argument of the proof in case b) fails for example for the matrix

$$X = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{pmatrix}$$

since it is not possible to make the bottom right entry positive preserving all other signs while in the proof this is assumed to be possible.

However, the claim of Theorem 7.2.1 of [CD17] is true: a correct proof follows from the results by Lenz [Len19, Thm. 1.1] and in more generality from Pagaria [Pag19, Thm. 3.5].

5. Summary of corrections

We now summarize the modifications in the body of [CD17] that are required by the corrections in this Erratum. The details for the corrections of §2–6 are given in Sections 1–3. The details for the correction of §7 are given in Section 4.

§2 The rings $A(\mathcal{A})$ and $B(\mathcal{A})$ are isomorphic to a graded algebra associated to a filtration of $H^*(M(\mathcal{A});\mathbb{Z})$ induced by the Leray spectral sequence, but in general they are not isomorphic to the ring $H^*(M(\mathcal{A});\mathbb{Z})$ itself. The statements of Thm. A and B should be replaced by Prop. 3.4.4, Thm. 3.4.5, Thm. 3.4.7 of this Erratum.

§3 The results in this sections holds without any change.

§4 The statements previous to Thm. 4.2.17 holds. Thm. 4.2.17 holds only if the subcomplexes $\mathcal{S}_L$ and $\mathcal{S}_{L'}$ are defined choosing $F_0 = \overline{B} \cap L_0$ and $F_0' = \overline{B} \cap L_0'$ for a common chamber B. Notice that Def. 4.2.16 require to choose a face F_0 and this is not explicit in the notation of $\mathcal{S}_L$. Lem. 4.2.18 and Sch. 4.2.19 holds.

§5 Lem. 5.1.2, Thm. 5.1.3 and Thm. 5.1.5 hold. Cor. 5.1.6 and the following statements are false.

§6 Lem. 6.1.2 and Thm. 6.1.3 hold. Thm. 6.2.4 is false. We refer to [CDD+18] for a description of the cohomology of the complement in the general case.

§7 Thm. 7.2.1 holds, but the proof given in [CD17] is wrong. Example 7.3 gives only a description of the graded ring associated to the filtration of the cohomology of the complement of the toric arrangement.

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