# On the combinatorics of minimal complexes for complexified arrangements

Emanuele Delucchi
SUNY Binghamton

Conference in honor of Peter Orlik
Fields Institute, Toronto
August 19.-22, 2008

# Arrangements of hyperplanes

An arrangement of real hyperplanes is a set

$$\mathcal{A} := \{H_1, \ldots, H_n\}$$

of linear hyperplanes in $\mathbb{R}^d$.

# Arrangements of hyperplanes

An arrangement of real hyperplanes is a set

$$\mathcal{A} := \{H_1, \dots, H_n\}$$

of linear hyperplanes in $\mathbb{R}^d$.

$\mathcal{A}$ defines a stratification of $\mathbb{R}^d$ into open polyhedral strata, called faces.

$\mathcal{F}(\mathcal{A})$ is the set of all faces, partially ordered by reverse inclusion of the topological closures: $F_1 > F_2 \Leftrightarrow \overline{F_1} \subseteq \overline{F_2}$.

# Arrangements of hyperplanes

An arrangement of real hyperplanes is a set

$$\mathcal{A} := \{H_1, \dots, H_n\}$$

of linear hyperplanes in $\mathbb{R}^d$.

$\mathcal{A}$ defines a stratification of $\mathbb{R}^d$ into open polyhedral strata, called faces.

$\mathcal{F}(\mathcal{A})$ is the set of all faces, partially ordered by reverse inclusion of the topological closures: $F_1 > F_2 \Leftrightarrow \overline{F_1} \subseteq \overline{F_2}$.

# Arrangements of hyperplanes

An arrangement of real hyperplanes is a set

$$\mathcal{A} := \{H_1, \ldots, H_n\}$$

of linear hyperplanes in $\mathbb{R}^d$.

$\mathcal{A}$ defines a stratification of $\mathbb{R}^d$ into open polyhedral strata, called faces.

$\mathcal{F}(\mathcal{A})$ is the set of all faces, partially ordered by reverse inclusion of the topological closures: $F_1 > F_2 \Leftrightarrow \overline{F_1} \subseteq \overline{F_2}$.

$\mathcal{C}(\mathcal{A})$ is the set of connected components of $\mathbb{R}^d \setminus \mathcal{A}$, called chambers of $\mathcal{A}$.

# Arrangements of hyperplanes

An arrangement of real hyperplanes is a set

$$\mathcal{A} := \{H_1, \ldots, H_n\}$$

of linear hyperplanes in $\mathbb{R}^d$.

$\mathcal{A}$ defines a stratification of $\mathbb{R}^d$ into open polyhedral strata, called faces.

$\mathcal{F}(\mathcal{A})$ is the set of all faces, partially ordered by reverse inclusion of the topological closures: $F_1 > F_2 \Leftrightarrow \overline{F_1} \subseteq \overline{F_2}$.

$\mathcal{C}(\mathcal{A})$ is the set of connected components of $\mathbb{R}^d \setminus \mathcal{A}$, called chambers of $\mathcal{A}$.

$\mathcal{L}(\mathcal{A})$ is the lattice of all *flats* $X = \bigcap_{H \in \mathcal{K}} H$ for $\mathcal{K} \subseteq \mathcal{A}$, partially ordered by reverse inclusion: $X > Y \Leftrightarrow X \subset Y$.

# Complexified arrangements

Let $\mathcal{A}$ be as before, and for every $i = 1, \ldots n$ choose $\alpha_i \in \mathbb{R}^d$ normal to $H_i$.

The complexification of $\mathcal{A}$ is the set

$$\mathcal{A}_{\mathbb{C}} := \{H_1^{\mathbb{C}}, \ldots, H_n^{\mathbb{C}}\} \text{ where } \; H_i^{\mathbb{C}} := \{z \in \mathbb{C}^d \mid \langle z \mid \alpha_i \rangle = 0\}.$$

# Complexified arrangements

Let $\mathcal{A}$ be as before, and for every $i = 1, \dots n$ choose $\alpha_i \in \mathbb{R}^d$ normal to $H_i$.

The complexification of $\mathcal{A}$ is the set

$$\mathcal{A}_{\mathbb{C}} := \{H_1^{\mathbb{C}}, \dots, H_n^{\mathbb{C}}\} \text{ where } H_i^{\mathbb{C}} := \{z \in \mathbb{C}^d \mid \langle z \mid \alpha_i \rangle = 0\}.$$

We consider its *complement*, that is

$$\mathcal{M}(\mathcal{A}) := \mathbb{C}^d \setminus \bigcup_{i=1}^{n} H_i^{\mathbb{C}}.$$

# The Salvetti complex $Sal(\mathcal{A})$

[Salvetti '87]

Given a complexified arrangement $\mathcal{A}$, consider the oriented graph $\Gamma(\mathcal{A})$ with:

$V(\Gamma(\mathcal{A}))$ : chambers of $\mathcal{A}$

$E(\Gamma(\mathcal{A}))$ contains $(C_1, C_2)$
iff $C_1$ is adjacent to $C_2$.

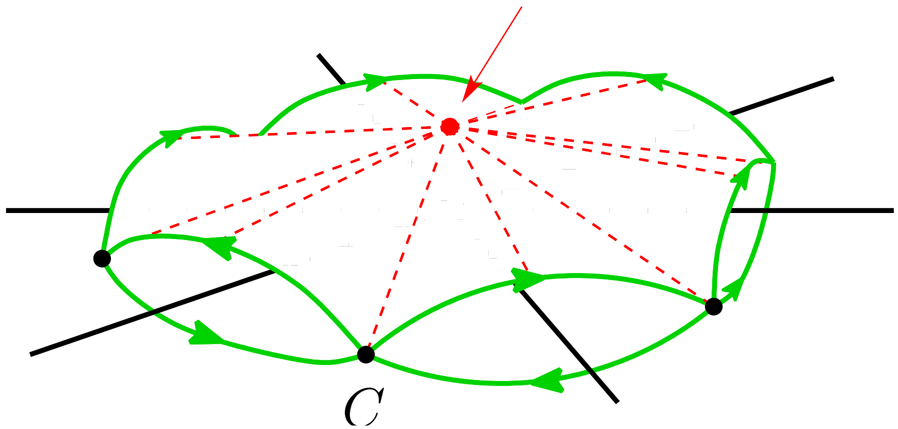
# The Salvetti complex $Sal(\mathcal{A})$

[Salvetti '87]

Given a complexified arrangement $\mathcal{A}$, consider the oriented graph $\Gamma(\mathcal{A})$ with:

$V(\Gamma(\mathcal{A}))$ : chambers of $\mathcal{A}$

$E(\Gamma(\mathcal{A}))$ contains $(C_1, C_2)$ iff $C_1$ is adjacent to $C_2$.



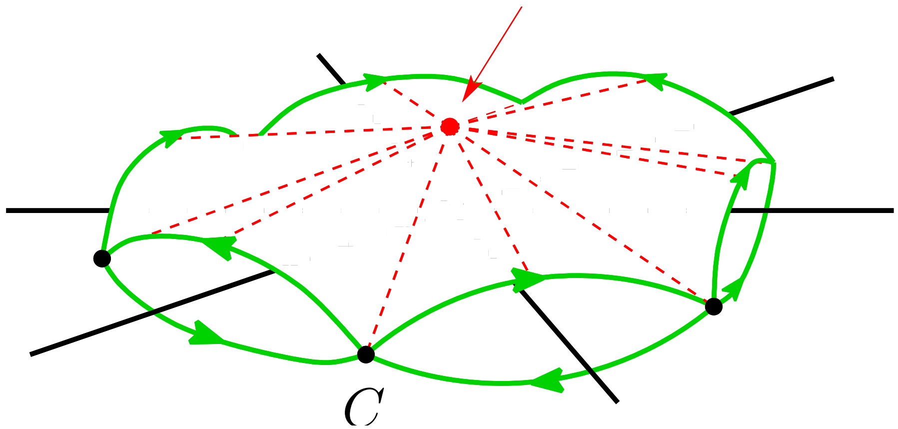
# The Salvetti complex $Sal(\mathcal{A})$

[Salvetti '87]

Given a complexified arrangement $\mathcal{A}$, consider the oriented graph $\Gamma(\mathcal{A})$ with:

$V(\Gamma(\mathcal{A}))$ : chambers of $\mathcal{A}$

$E(\Gamma(\mathcal{A}))$ contains $(C_1, C_2)$ iff $C_1$ is adjacent to $C_2$.

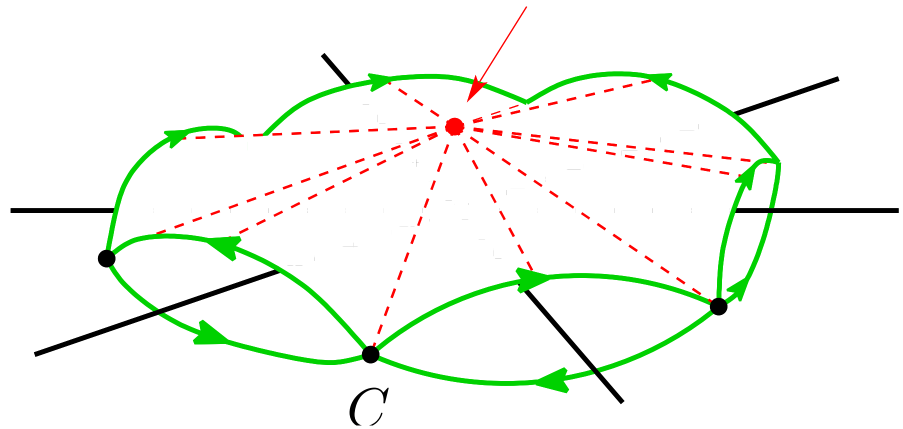



- $Sal(\mathcal{A})^1 = \Gamma(\mathcal{A})$

# The Salvetti complex $Sal(\mathcal{A})$

[Salvetti '87]

Given a complexified arrangement $\mathcal{A}$, consider the oriented graph $\Gamma(\mathcal{A})$ with:




$V(\Gamma(\mathcal{A}))$ : chambers of $\mathcal{A}$

$E(\Gamma(\mathcal{A}))$ contains $(C_1, C_2)$ iff $C_1$ is adjacent to $C_2$.

- $Sal(\mathcal{A})^1 = \Gamma(\mathcal{A})$

- A $k$-cell $[F, C]$ for every codimension-$k$ face $F$ and every chamber $C < F$.

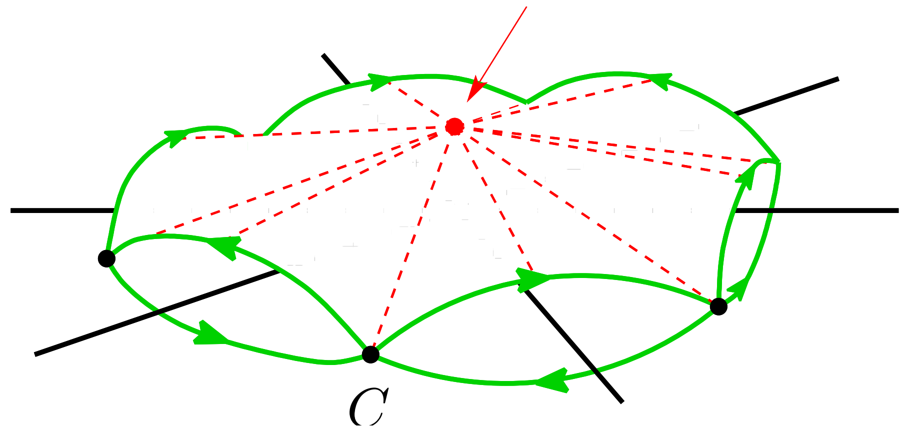
# THE SALVETTI COMPLEX $Sal(\mathcal{A})$

[Salvetti '87]

Given a complexified arrangement $\mathcal{A}$, consider the oriented graph $\Gamma(\mathcal{A})$ with:

$V(\Gamma(\mathcal{A}))$ : chambers of $\mathcal{A}$

$E(\Gamma(\mathcal{A}))$ contains $(C_1, C_2)$ iff $C_1$ is adjacent to $C_2$.




- $Sal(\mathcal{A})^1 = \Gamma(\mathcal{A})$

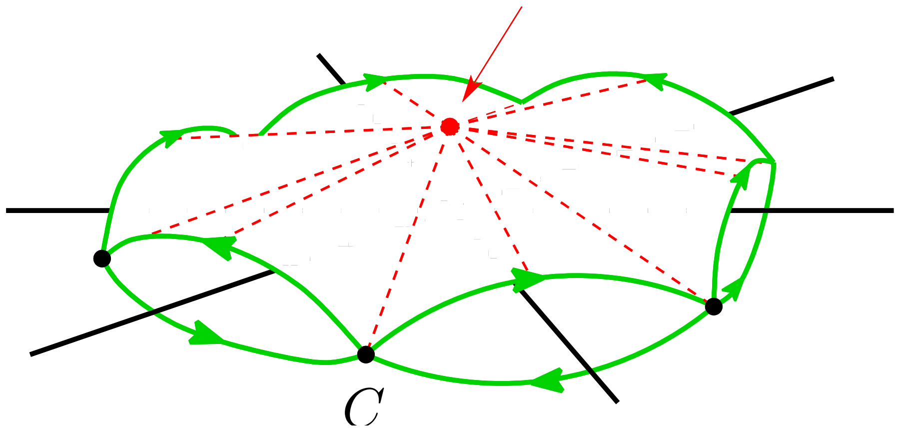
- A $k$-cell $[F, C]$ for every codimension-$k$ face $F$ and every chamber $C < F$.

- $[F, C]^1$: all oriented *(positive)* paths starting at $C$ and passing exactly once *(minimal)* through all $H \supset F$.

# THE SALVETTI COMPLEX $Sal(\mathcal{A})$

[Salvetti '87]

Given a complexified arrangement $\mathcal{A}$, consider the oriented graph $\Gamma(\mathcal{A})$ with:




$V(\Gamma(\mathcal{A}))$ : chambers of $\mathcal{A}$

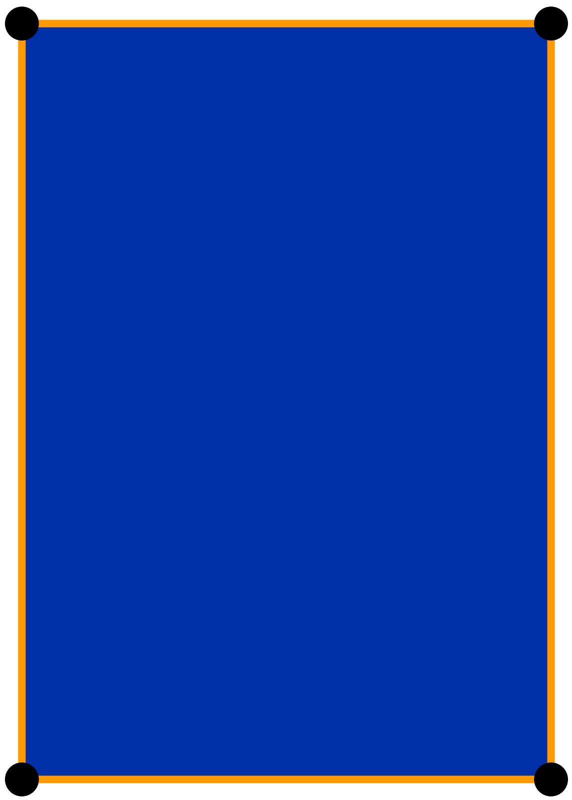
$E(\Gamma(\mathcal{A}))$ contains $(C_1, C_2)$ iff $C_1$ is adjacent to $C_2$.

- $Sal(\mathcal{A})^1 = \Gamma(\mathcal{A})$
- A $k$-cell $[F, C]$ for every codimension-$k$ face $F$ and every chamber $C < F$.
- $[F, C]^1$: all oriented *(positive)* paths starting at $C$ and passing exactly once *(minimal)* through all $H \supset F$.

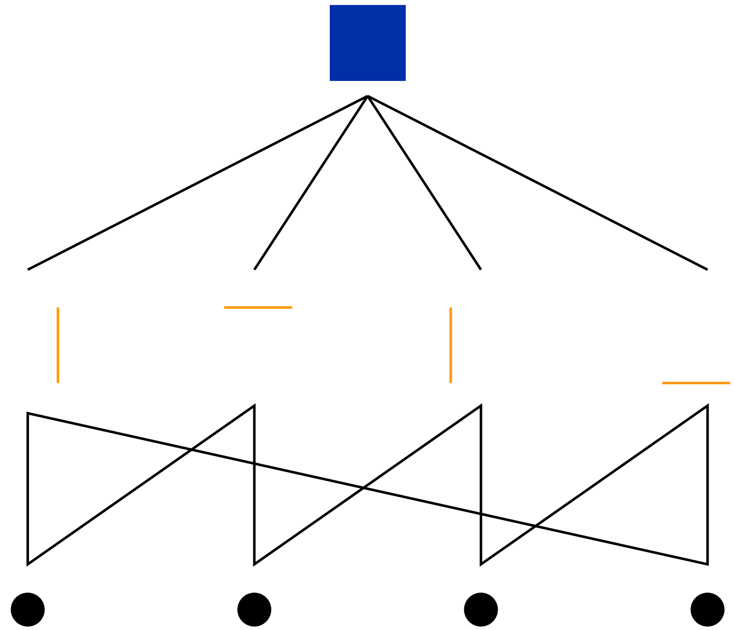
**Theorem. [Salvetti '87]** $Sal(\mathcal{A}) \simeq \mathcal{M}(\mathcal{A})$.

# Discrete Morse theory

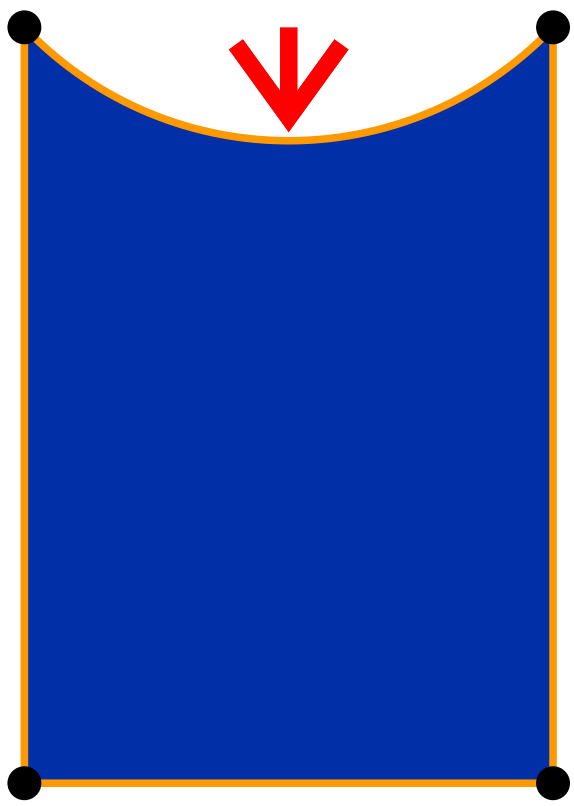
Here is a regular CW complex



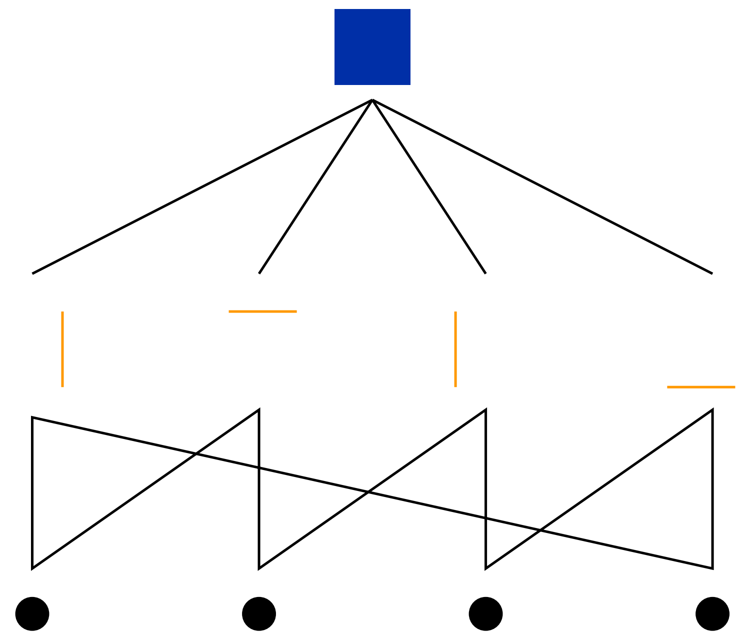
with its poset of cells:



# Elementary collapses...

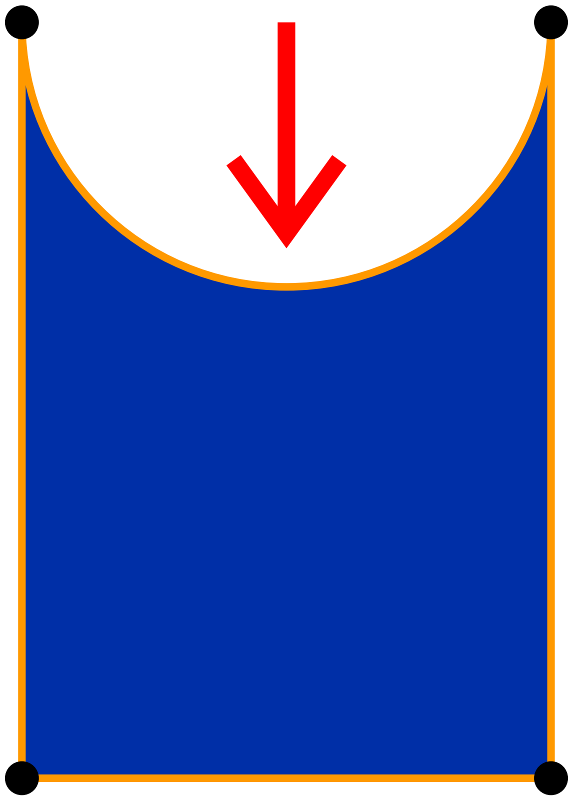


Cells:



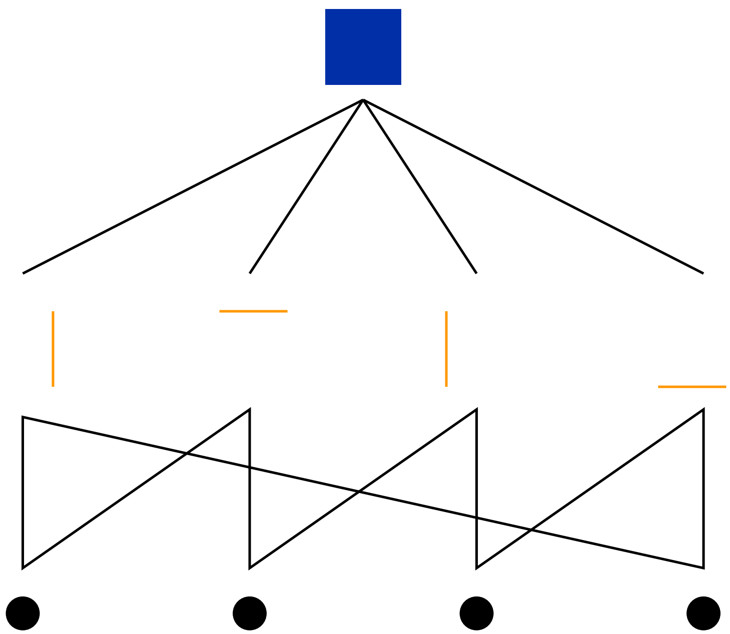
... are homotopy equivalences.

# Elementary collapses...

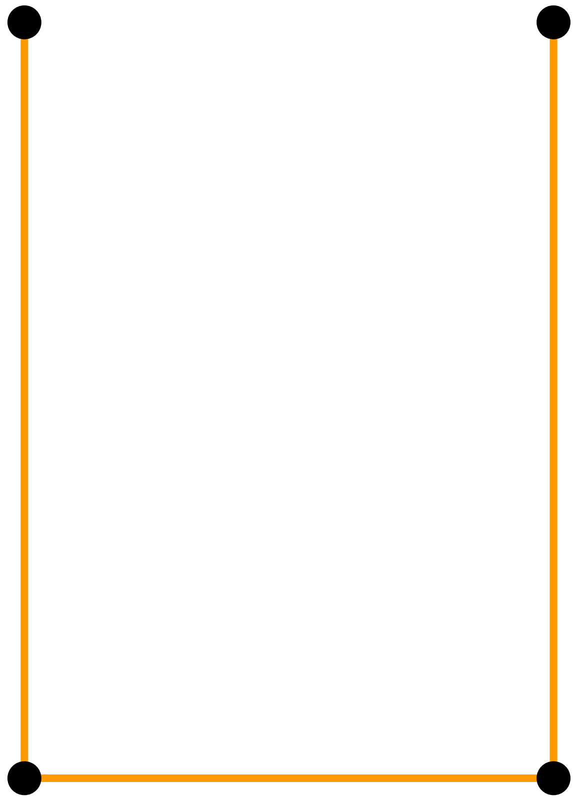


...are homotopy equivalences.

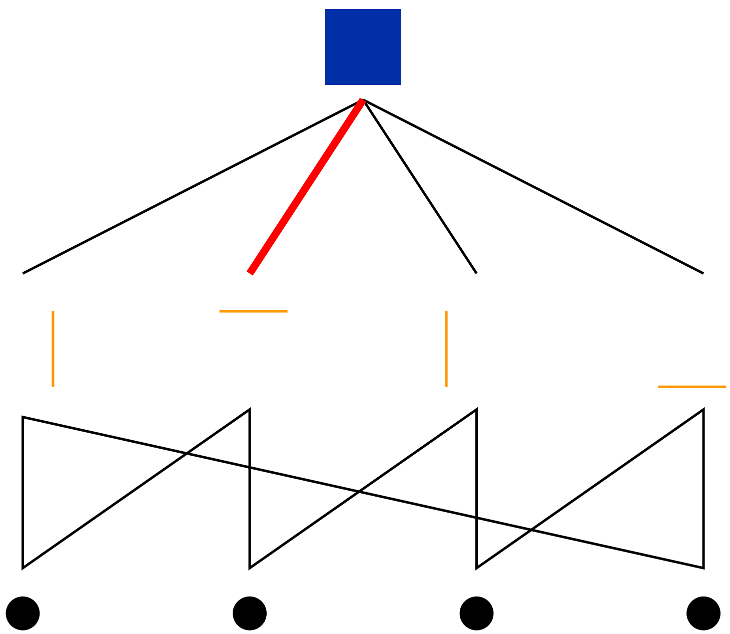
Cells:



# Elementary collapses...

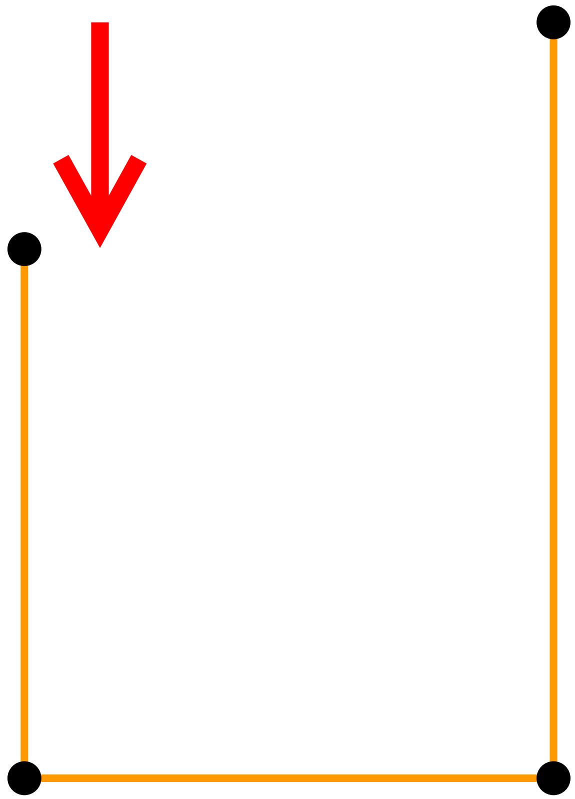


Cells:

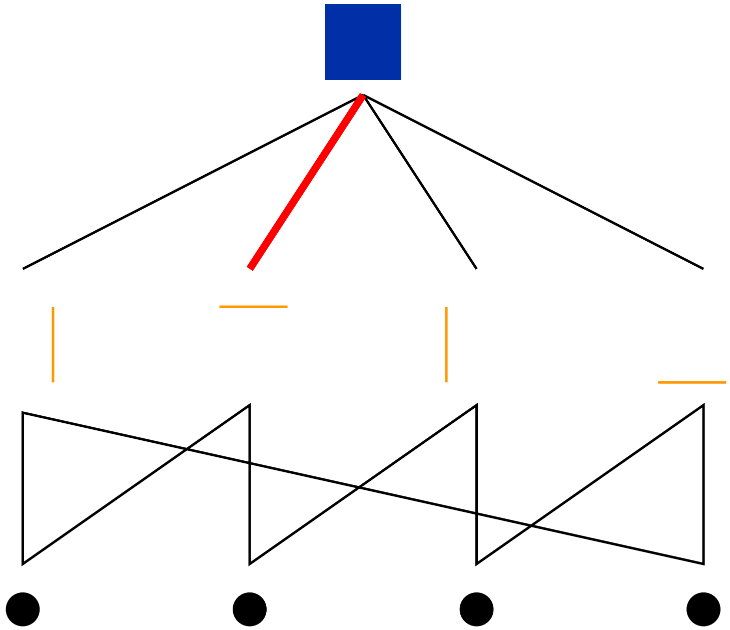


... are homotopy equivalences

# Elementary collapses...



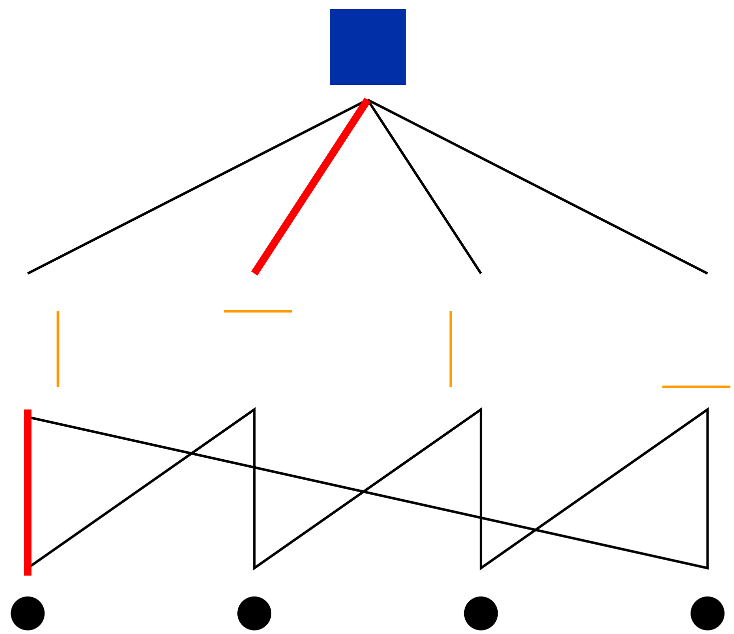
Cells:



... are homotopy equivalences

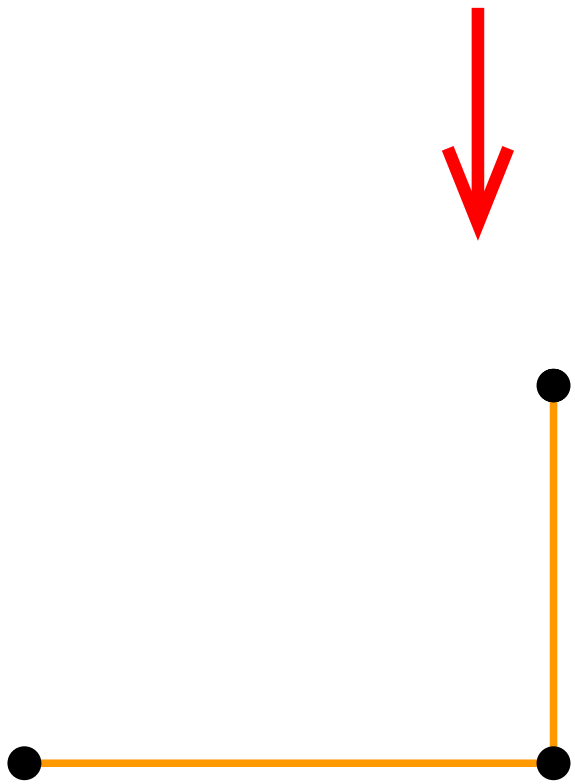
# Elementary collapses...

Cells:

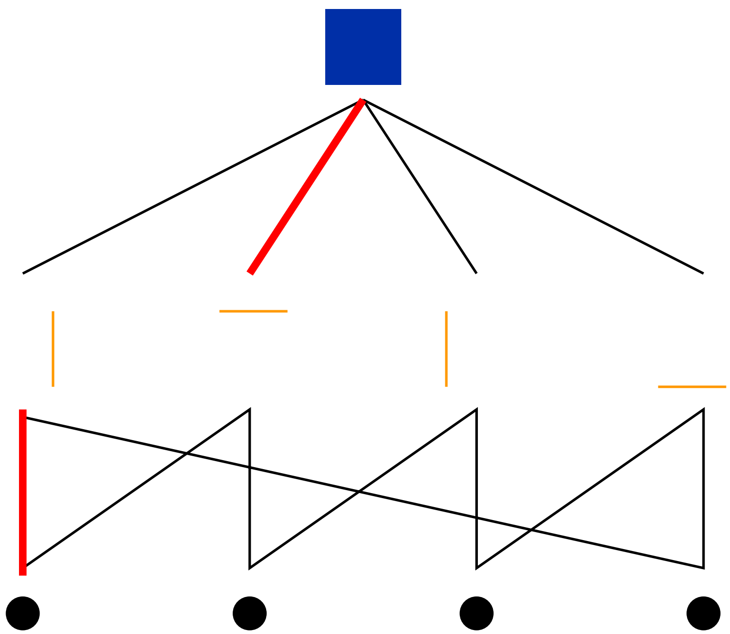


... are homotopy equivalences

# Elementary collapses...



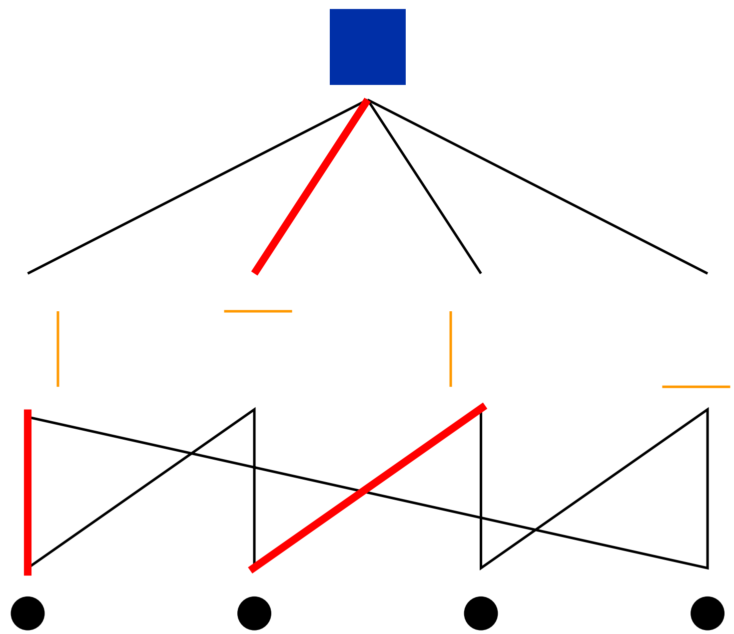
Cells:



... are homotopy equivalences

# ELEMENTARY COLLAPSES...

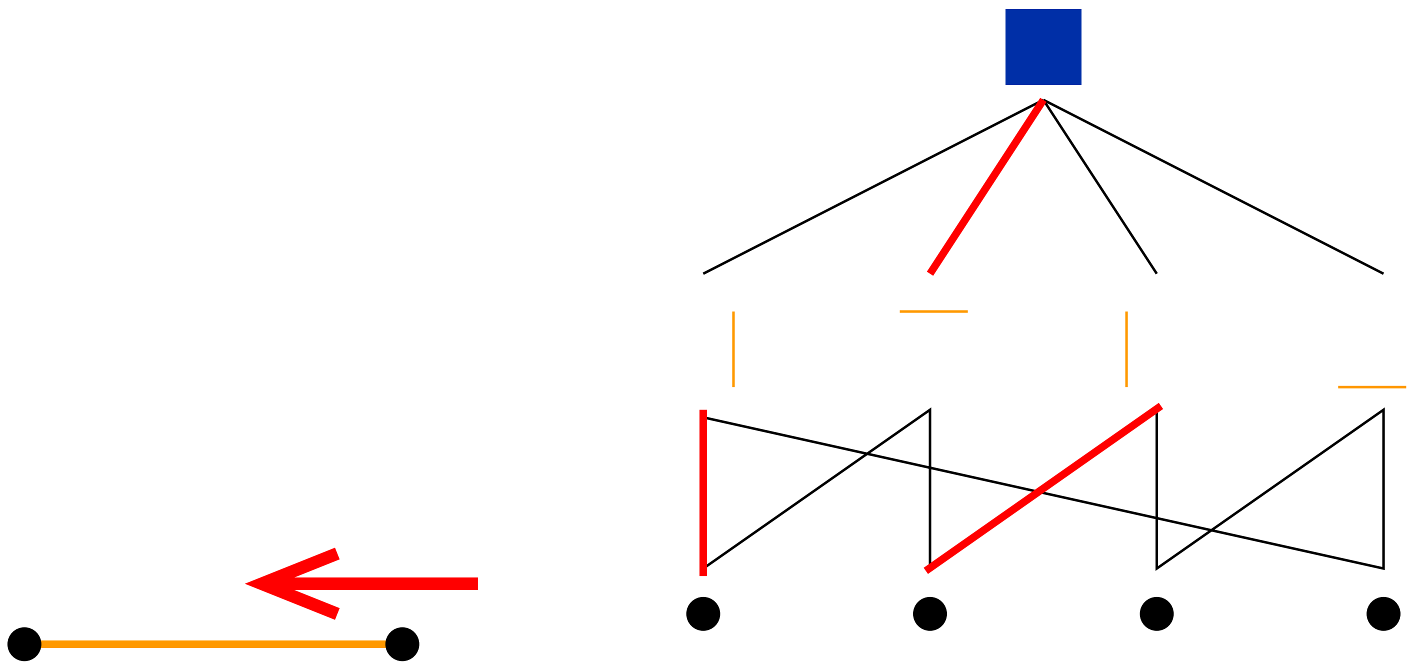
Cells:



... are homotopy equivalences

# Elementary collapses...

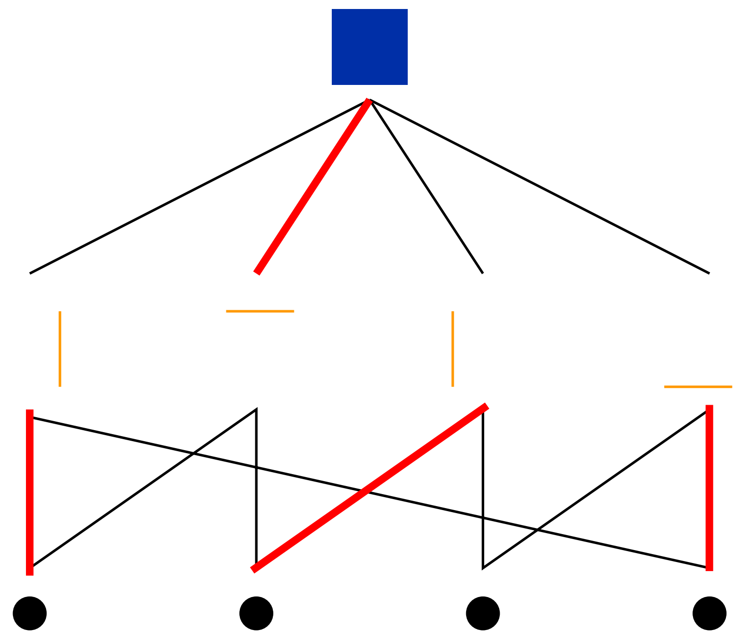
Cells:



... are homotopy equivalences

# ELEMENTARY COLLAPSES...

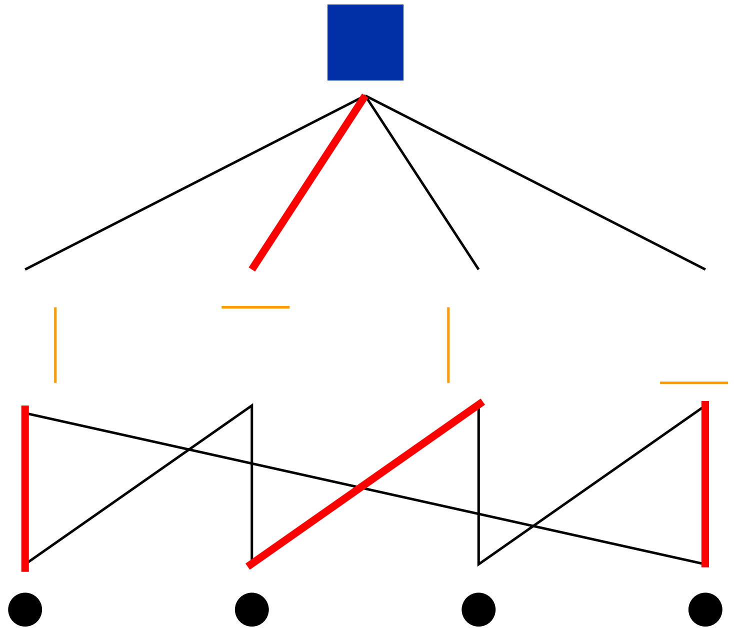
Cells:



... are homotopy equivalences

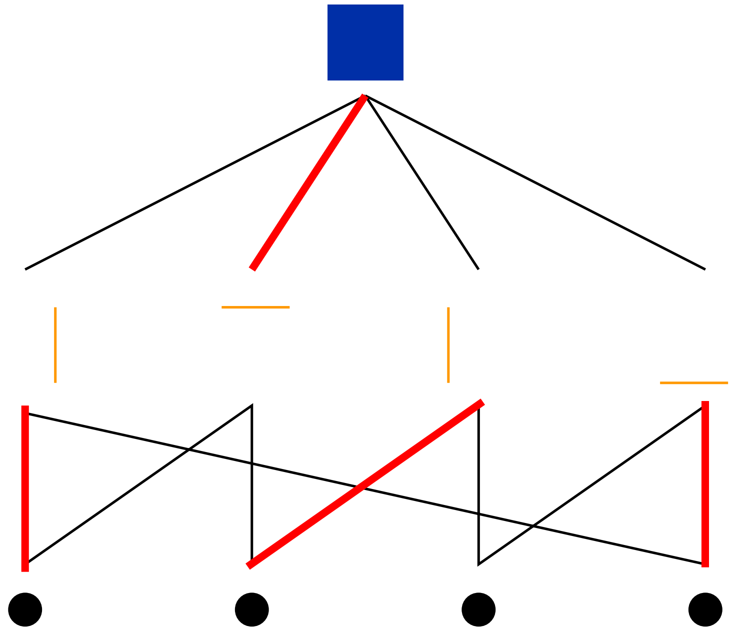
# Acyclic matchings

The sequence of collapses is encoded in a matching of the Hasse diagram of the poset of cells.



# Acyclic matchings

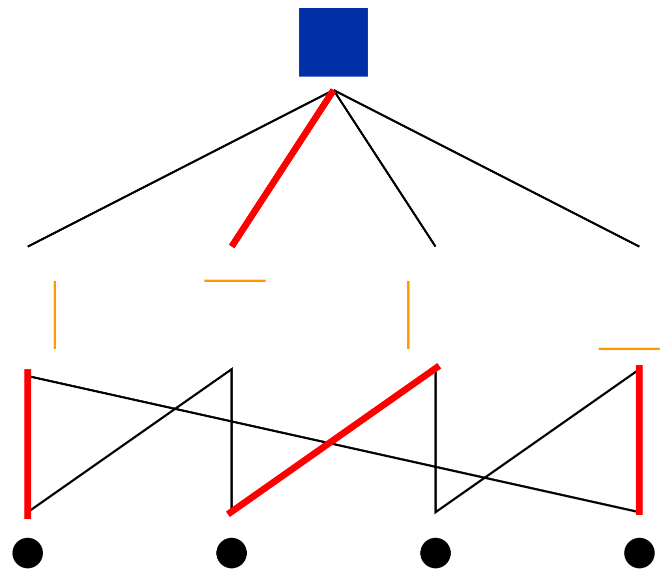
The sequence of collapses is encoded in a matching of the Hasse diagram of the poset of cells.



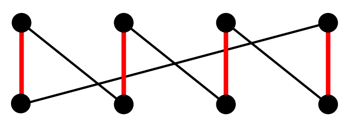
**Question:** Does any matching encode such a sequence?

# Acyclic matchings

The sequence of collapses is encoded in a matching of the Hasse diagram of the poset of cells.



**Question:** Which matchings encode such a sequence?

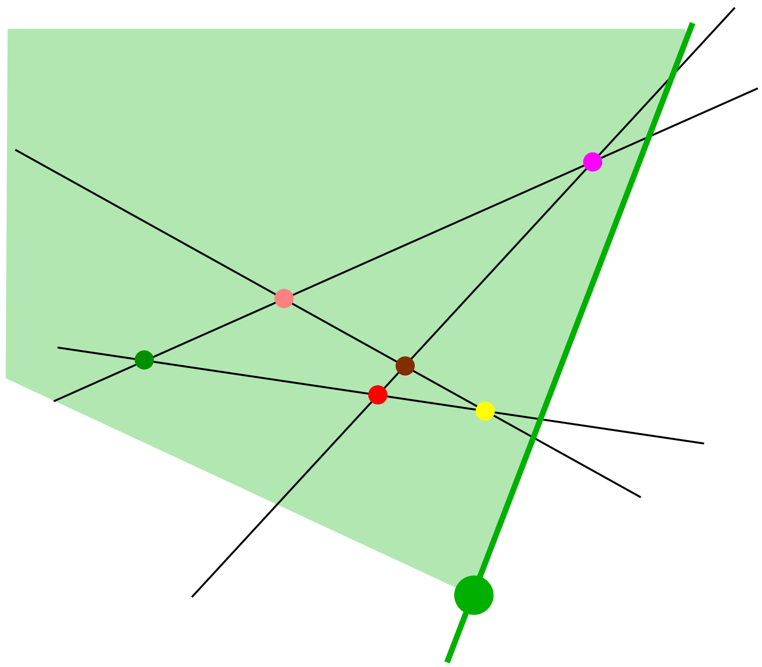
**Answer:** Those whitout "cycles" like .

**Acyclic matchings $\leftrightarrow$ discrete Morse functions.**

# Recall: polar ordering

[Salvetti and Settepanella '07]

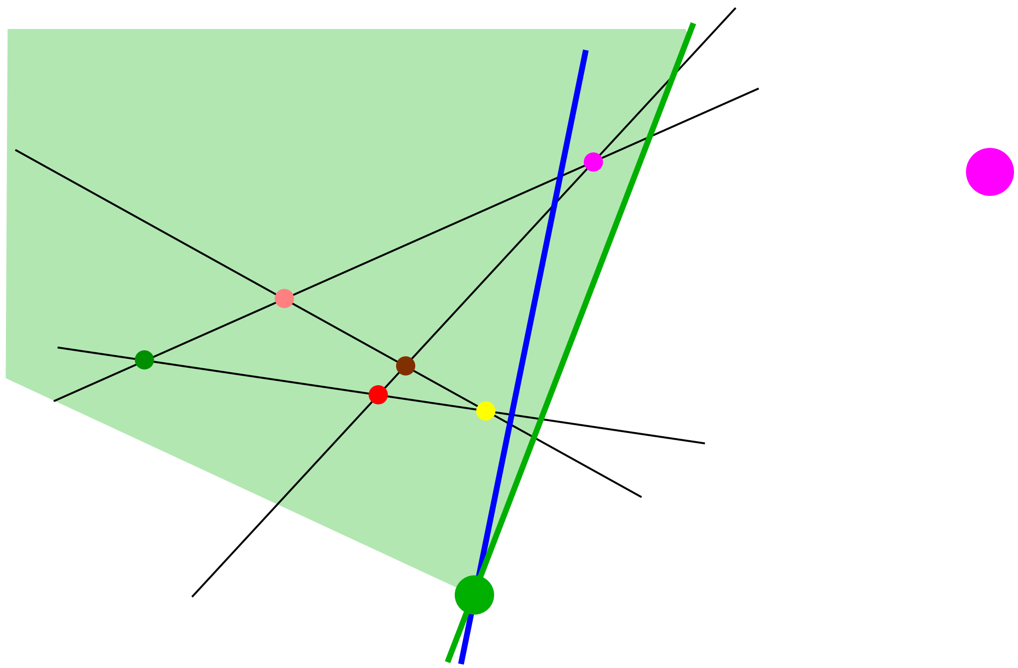
Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a *"generic"* flag of affine subspaces of $\mathbb{R}^d$.



# Recall: polar ordering

[Salvetti and Settepanella '07]

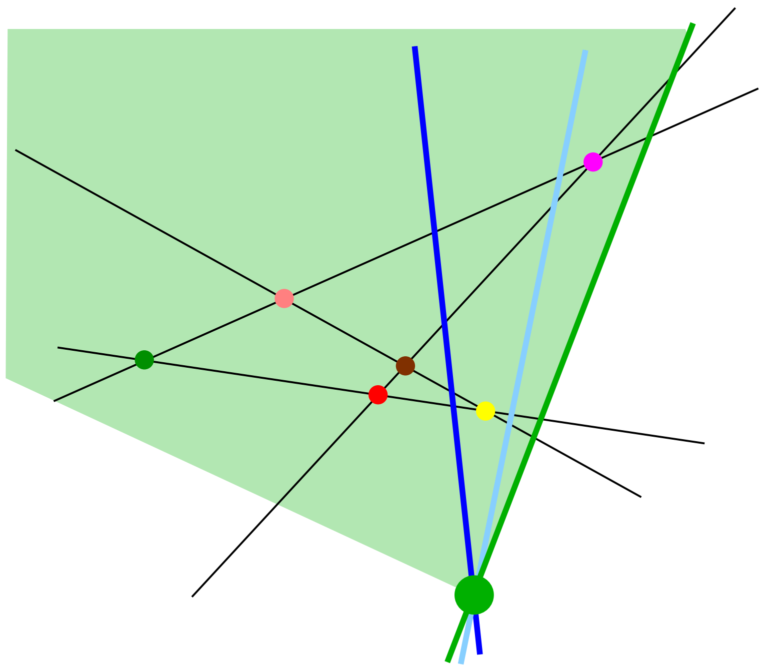
Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a *"generic"* flag of affine subspaces of $\mathbb{R}^d$.

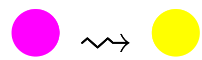


# Recall: polar ordering

[Salvetti and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a *"generic"* flag of affine subspaces of $\mathbb{R}^d$.

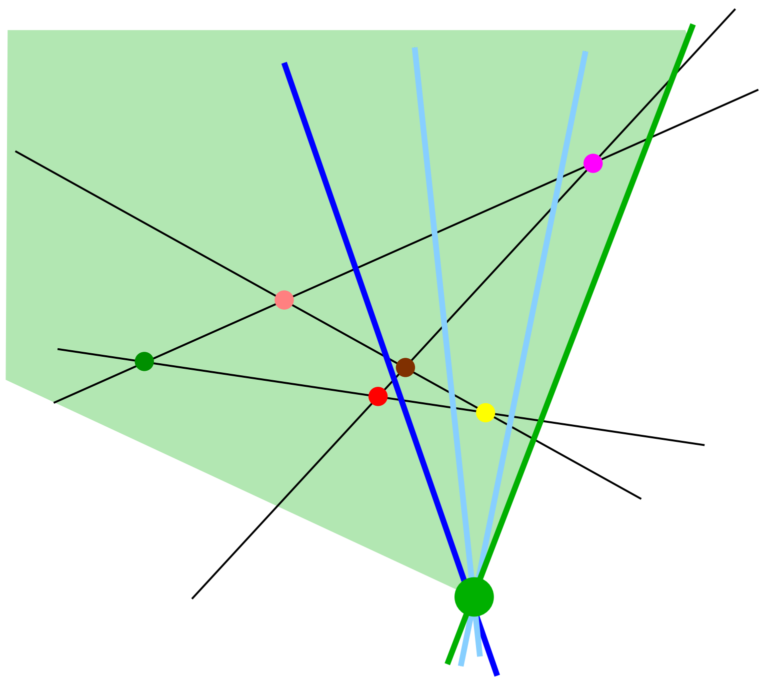


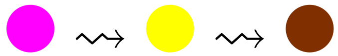


# RECALL: POLAR ORDERING

[Salvetti and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a *"generic"* flag of affine subspaces of $\mathbb{R}^d$.

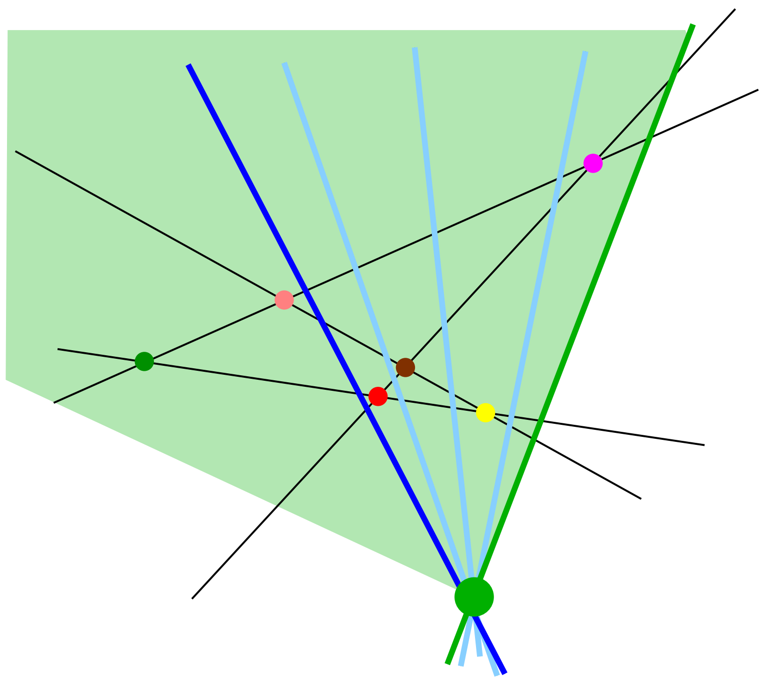


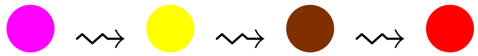


# Recall: polar ordering

[Salvetti and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a *"generic"* flag of affine subspaces of $\mathbb{R}^d$.

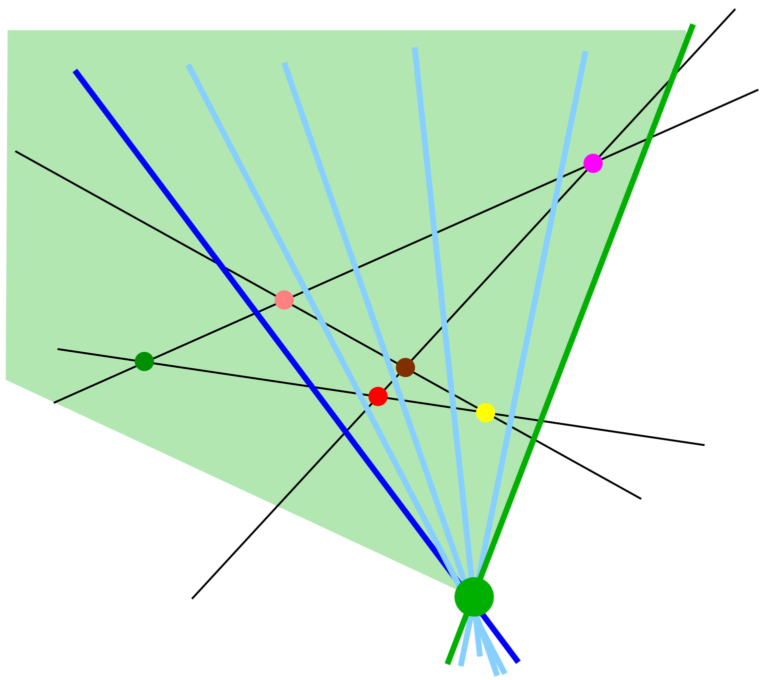


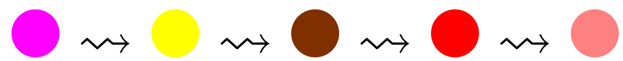


# RECALL: POLAR ORDERING

[Salvetti and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a *"generic"* flag of affine subspaces of $\mathbb{R}^d$.

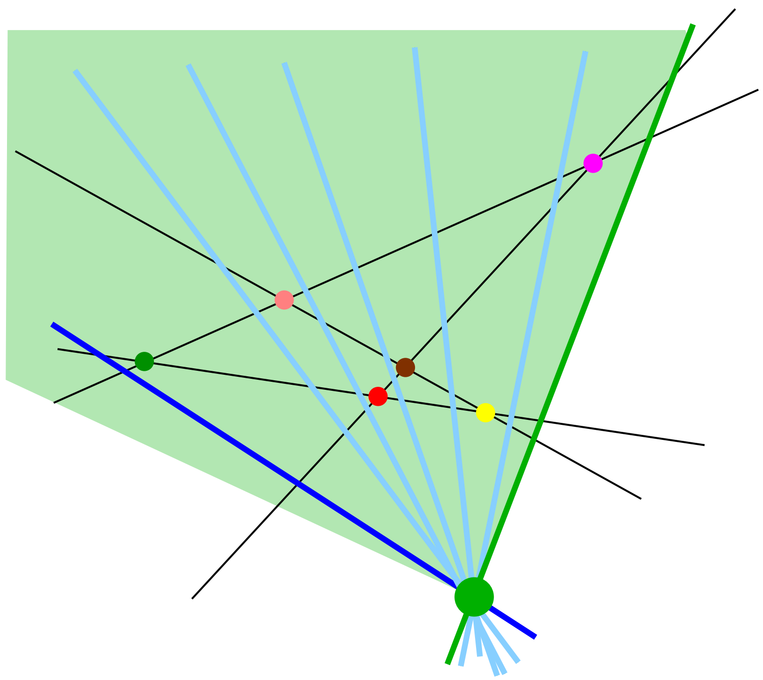


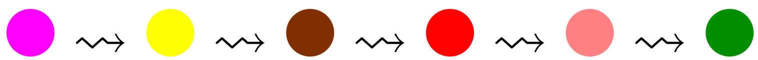


# Recall: polar ordering

[Salvetti and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a *"generic"* flag of affine subspaces of $\mathbb{R}^d$.



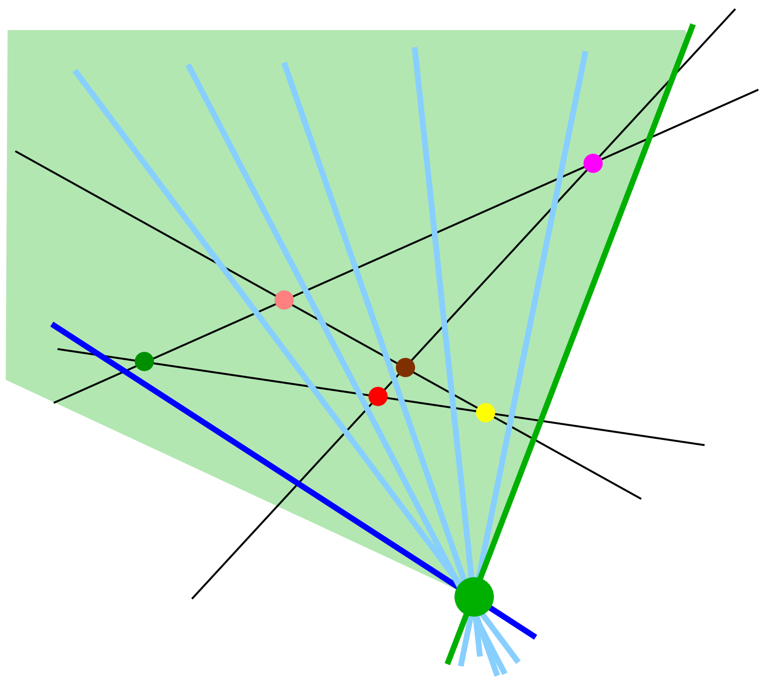


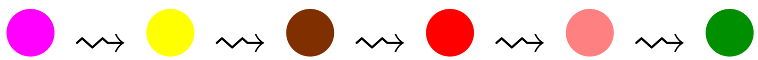
These orderings of the points of the $V_k \cap \mathcal{A}$'s can be extended to a total ordering of $\mathcal{F}$, called polar ordering.

# RECALL: POLAR ORDERING

[Salvetti and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a *"generic"* flag of affine subspaces of $\mathbb{R}^d$.





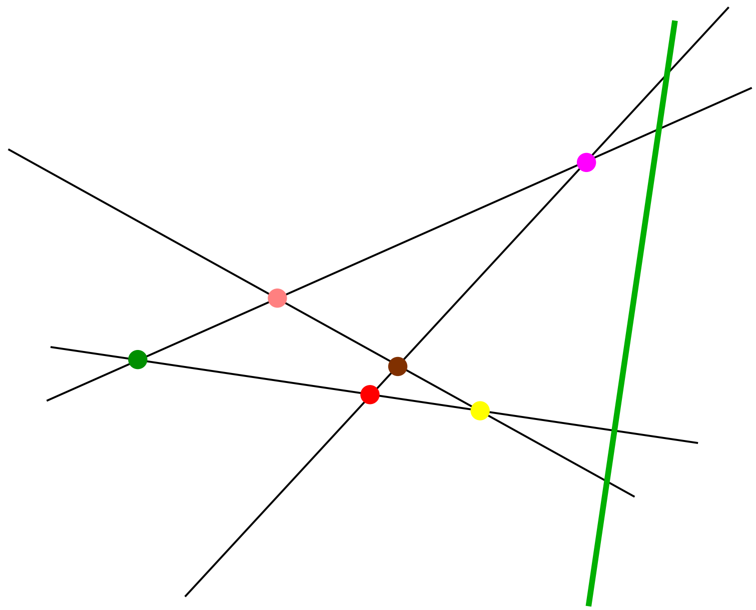
These orderings of the points of the $V_k \cap \mathcal{A}$'s can be extended to a total ordering of $\mathcal{F}$, called polar ordering.

Salvetti and Settepanella give an algorithm that starts from these data to construct a discrete Morse vector field for $Sal(\mathcal{A})$ and the boundary maps of the corresponding minimal complex.

# Combinatorial polar ordering

[D. and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a flag of affine subspaces of $\mathbb{R}^d$ in general position with respect to $\mathcal{A}$.

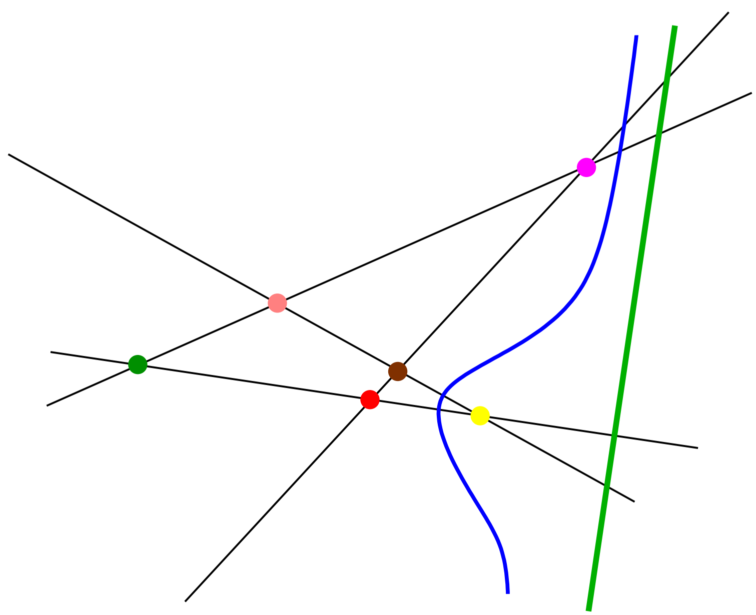


For all $k = 1, \ldots, d$ let the pseudohyperplane $V_{k-1}$ sweep through the arrangement $V_k \cap \mathcal{A}$.

# Combinatorial polar ordering

[D. and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a flag of affine subspaces of $\mathbb{R}^d$ in general position with respect to $\mathcal{A}$.

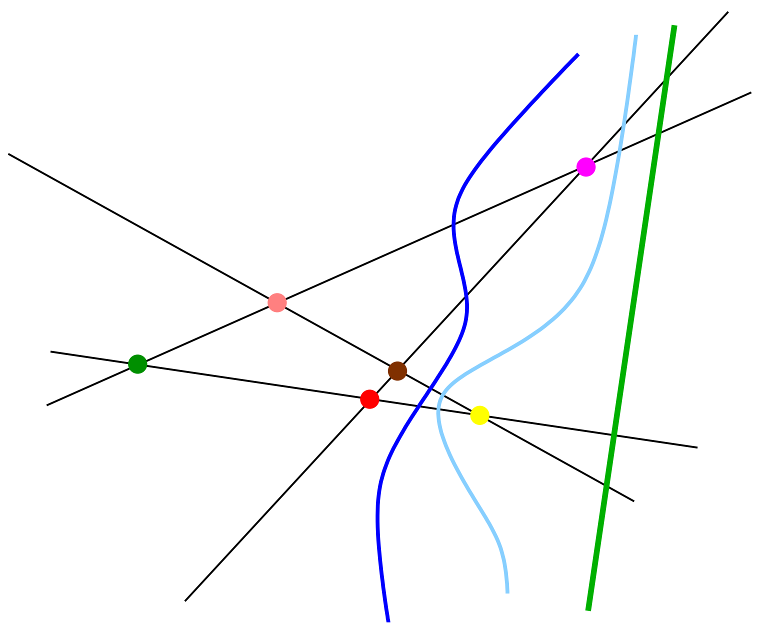


For all $k = 1, \ldots, d$ let the pseudohyperplane $V_{k-1}$ sweep through the arrangement $V_k \cap \mathcal{A}$.

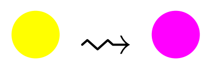
# Combinatorial polar ordering

[D. and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a flag of affine subspaces of $\mathbb{R}^d$ in general position with respect to $\mathcal{A}$.



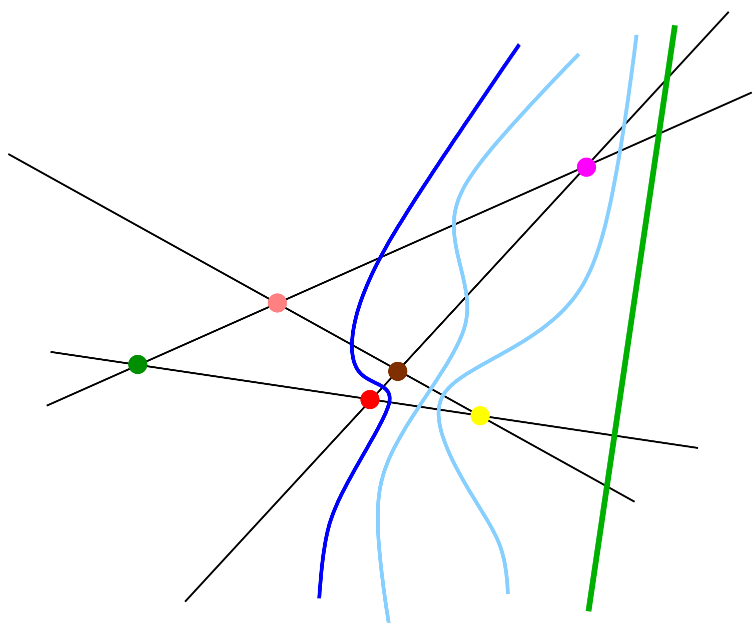
For all $k = 1, \ldots, d$ let the pseudohyperplane $V_{k-1}$ sweep through the arrangement $V_k \cap \mathcal{A}$.



# Combinatorial polar ordering

[D. and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a flag of affine subspaces of $\mathbb{R}^d$ in general position with respect to $\mathcal{A}$.

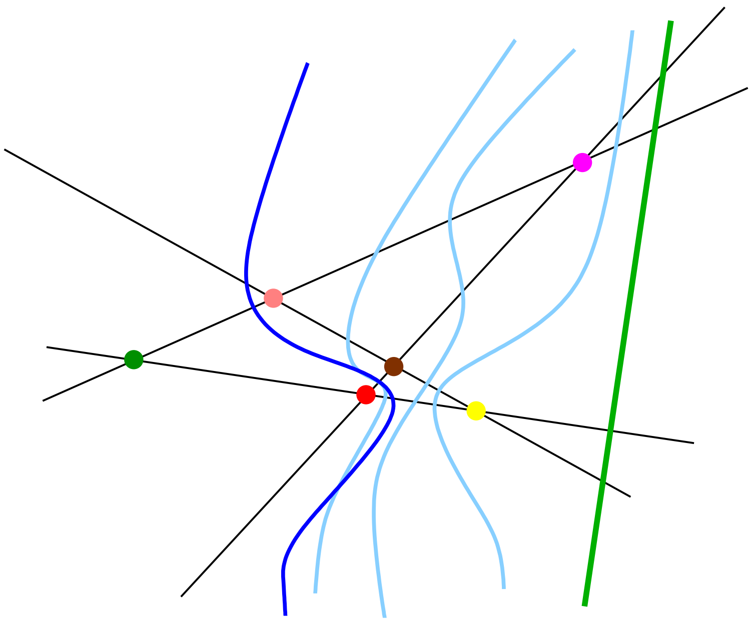


For all $k = 1, \ldots, d$ let the pseudohyperplane $V_{k-1}$ sweep through the arrangement $V_k \cap \mathcal{A}$.

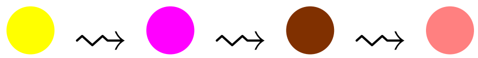
# Combinatorial polar ordering

[D. and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a flag of affine subspaces of $\mathbb{R}^d$ in general position with respect to $\mathcal{A}$.



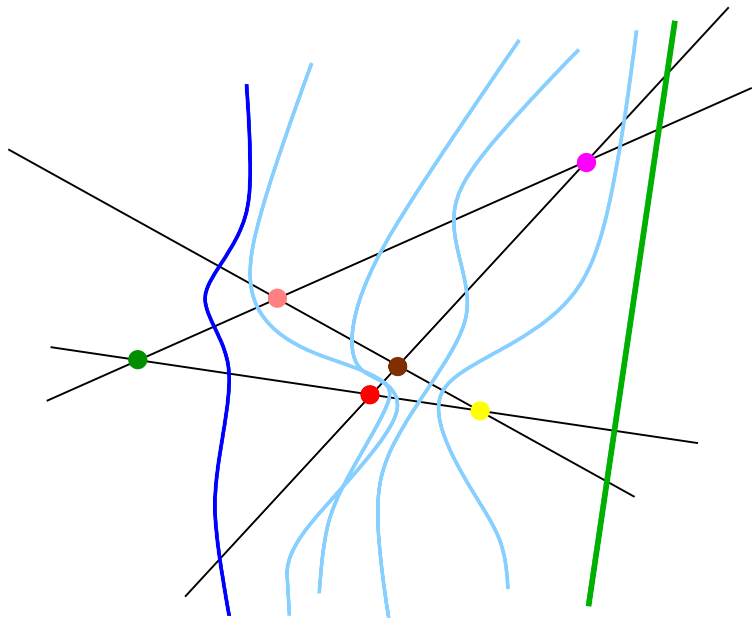
For all $k = 1, \ldots, d$ let the pseudohyperplane $V_{k-1}$ sweep through the arrangement $V_k \cap \mathcal{A}$.



# Combinatorial polar ordering

[D. and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a flag of affine subspaces of $\mathbb{R}^d$ in general position with respect to $\mathcal{A}$.

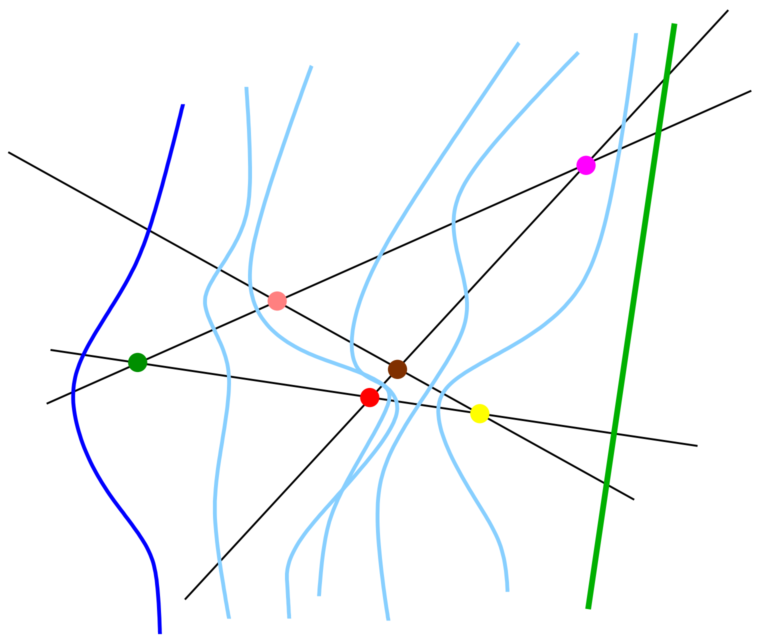


For all $k = 1, \ldots, d$ let the pseudohyperplane $V_{k-1}$ sweep through the arrangement $V_k \cap \mathcal{A}$.

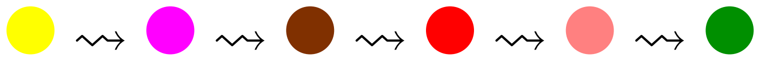
# Combinatorial polar ordering

[D. and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a flag of affine subspaces of $\mathbb{R}^d$ in general position with respect to $\mathcal{A}$.



For all $k = 1, \ldots, d$ let the pseudohyperplane $V_{k-1}$ sweep through the arrangement $V_k \cap \mathcal{A}$.

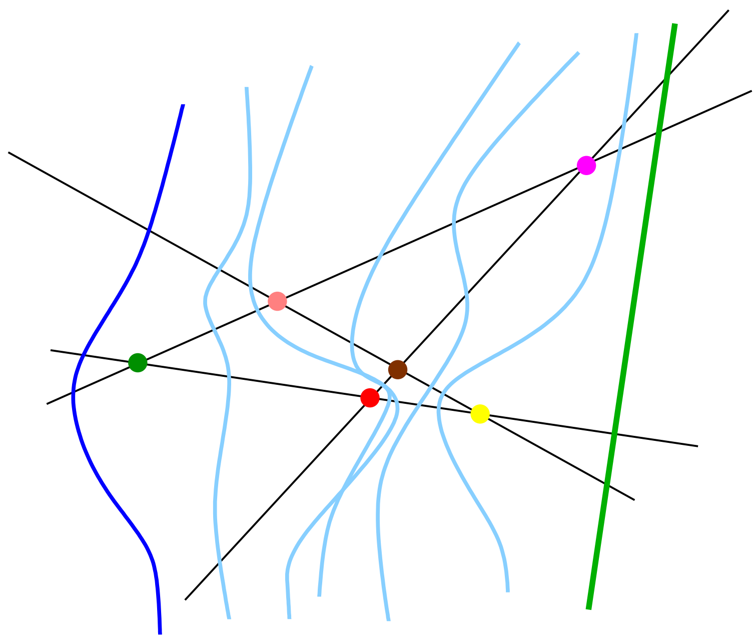


These orderings of the points of the $V_k \cap \mathcal{A}$'s can be extended to a total ordering of $\mathcal{F}$, called combinatorial polar ordering.

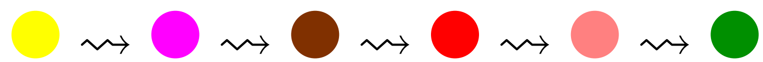
# Combinatorial polar ordering

[D. and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a flag of affine subspaces of $\mathbb{R}^d$ in general position with respect to $\mathcal{A}$.



For all $k = 1, \ldots, d$ let the pseudohyperplane $V_{k-1}$ sweep through the arrangement $V_k \cap \mathcal{A}$.



These orderings of the points of the $V_k \cap \mathcal{A}$'s can be extended to a total ordering of $\mathcal{F}$, called combinatorial polar ordering.

**Theorem. [D., Settepanella '07]** *Every combinatorial polar ordering induces a discrete Morse vector field with a minimal number of critical cells. The field depends only of the combinatorial type of the flag.*

# *Follow-up* arrangements

[Delucchi and Settepanella '07]

# *Follow-up* ARRANGEMENTS

[Delucchi and Settepanella '07]

Notice that the intersection with $V_1$ orders the "lines" of $\mathcal{A} \cap V_2$, and we may order the points of this arrangement so that:

- points on "earlier" lines come first,
- among points on the same line, those nearer to $V_1$ come first.

# *Follow-up* arrangements

[Delucchi and Settepanella '07]

Notice that the intersection with $V_1$ orders the "lines" of $\mathcal{A} \cap V_2$, and we may order the points of this arrangement so that:

- points on "earlier" lines come first,
- among points on the same line, those nearer to $V_1$ come first.

Similarly one gets, for any $k$, a total ordering of the points of $\mathcal{A} \cap V_k$.

## *Follow-up* arrangements

[Delucchi and Settepanella '07]

Notice that the intersection with $V_1$ orders the "lines" of $\mathcal{A} \cap V_2$, and we may order the points of this arrangement so that:

- points on "earlier" lines come first,
- among points on the same line, those nearer to $V_1$ come first.

Similarly one gets, for any $k$, a total ordering of the points of $\mathcal{A} \cap V_k$.

The follow-up ordering associated to the flag $V_k$ is the ordering of the points obtained by concatenating the above orderings for increasing $k$.

# *Follow-up* ARRANGEMENTS

[Delucchi and Settepanella '07]

Notice that the intersection with $V_1$ orders the "lines" of $\mathcal{A} \cap V_2$, and we may order the points of this arrangement so that:

- points on "earlier" lines come first,
- among points on the same line, those nearer to $V_1$ come first.

Similarly one gets, for any $k$, a total ordering of the points of $\mathcal{A} \cap V_k$.

The follow-up ordering associated to the flag $V_k$ is the ordering of the points obtained by concatenating the above orderings for increasing $k$.

**Definition.** *A real arrangement is called* follow-up *if there is a flag* $(V_k)_k$ *such that the corresponding follow-up ordering is a combinatorial polar ordering.*

# *Follow-up* ARRANGEMENTS

[Delucchi and Settepanella '07]

Notice that the intersection with $V_1$ orders the "lines" of $\mathcal{A} \cap V_2$, and we may order the points of this arrangement so that:

- points on "earlier" lines come first,
- among points on the same line, those nearer to $V_1$ come first.

Similarly one gets, for any $k$, a total ordering of the points of $\mathcal{A} \cap V_k$.

The follow-up ordering associated to the flag $V_k$ is the ordering of the points obtained by concatenating the above orderings for increasing $k$.

**Definition.** *A real arrangement is called* follow-up *if there is a flag* $(V_k)_k$ *such that the corresponding follow-up ordering is a combinatorial polar ordering.*

Fact: For such arrangements, the Salvetti-Settepanella minimal complex can be computed in a particularly handy way.

# *Follow-up* arrangements

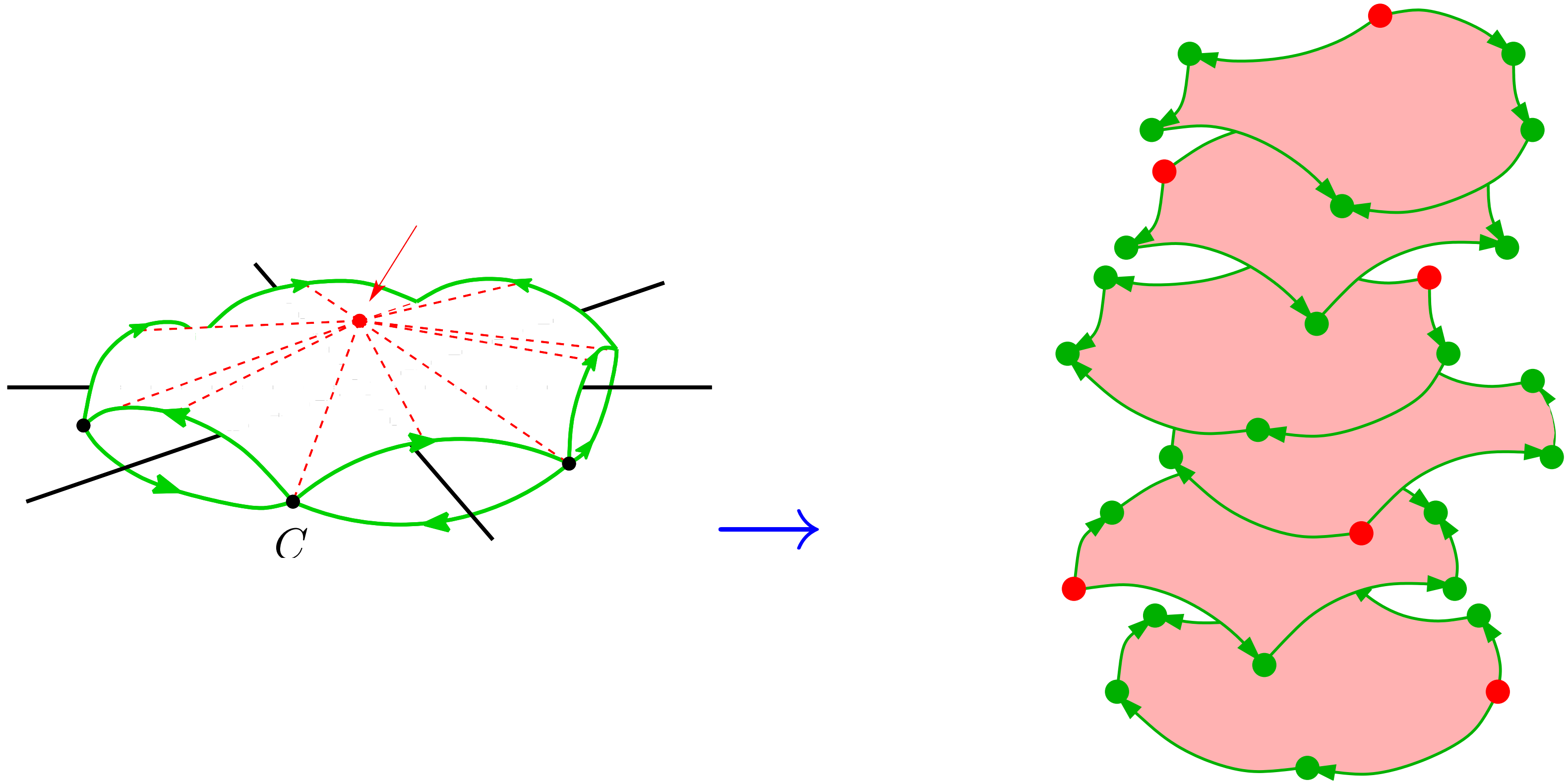
[Delucchi and Settepanella '07]

**Theorem. [D., Settepanella '07]** *The 2-dimenional follow-up arrangements can be completely characterized. In general:*

- *supersolvable arrangements are follow-up*
- *not every Coxeter arrangement is follow-up ($H_3$ is not)*

# A NEW STRATIFICATION OF $Sal(\mathcal{A})$

Let $\mathcal{A}$ from now be a central arrangement.




$Sal(\mathcal{A})$ is the union of combinatorially isomorphic maximal cells, one for every chamber.

# A NEW STRATIFICATION OF $Sal(\mathcal{A})$

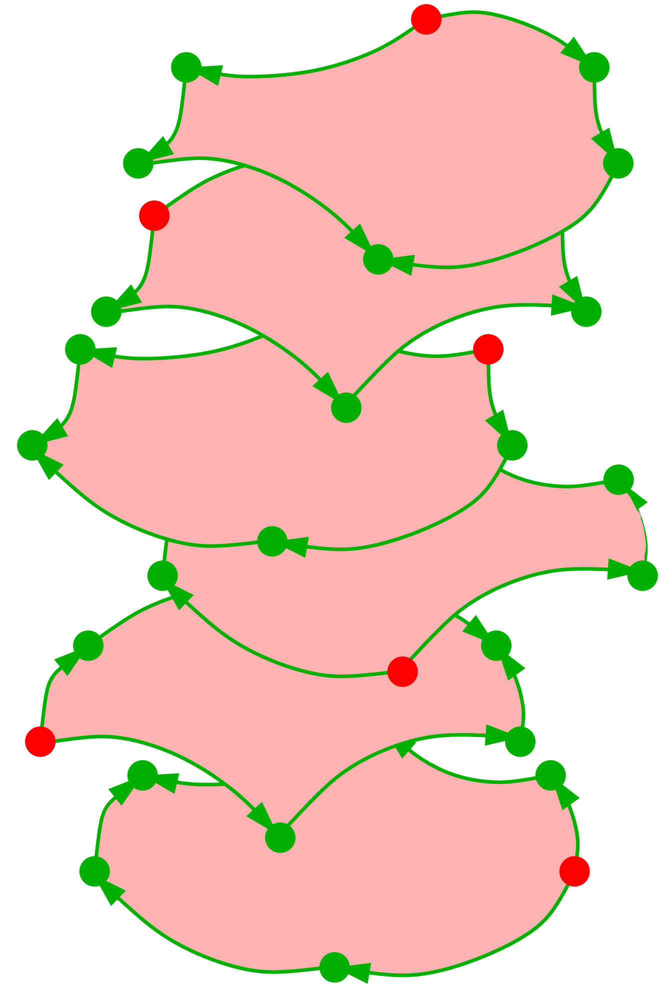
Given a chamber $C$, let $\mathcal{S}_C$ be the poset of cells of the subcomplex given by the maximal cell associated to $C$.

By abuse of notation, write

$$Sal(\mathcal{A}) = \bigcup_{C \in \mathcal{C}} \mathcal{S}_C.$$

**Question:**

What is $\mathcal{S}_{C_1} \cap \mathcal{S}_{C_2}$ for $C_1 \neq C_2$?

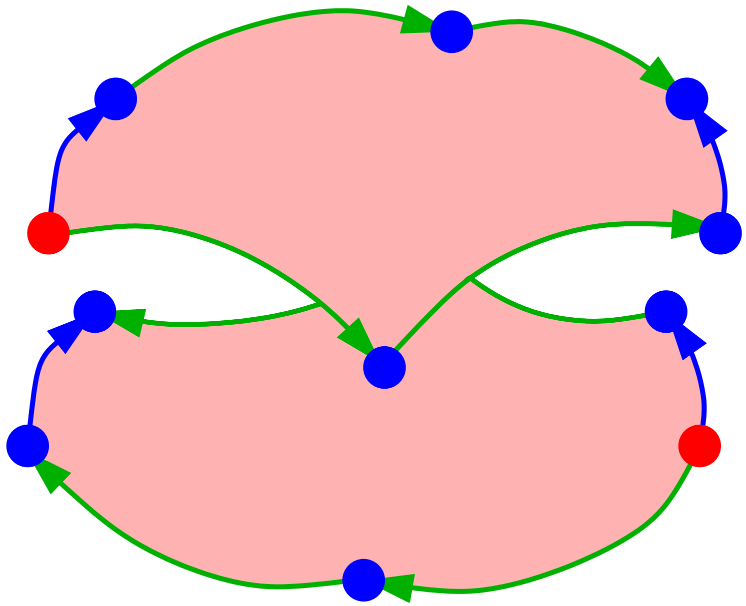


# Order on chambers

**Definition.** *Let $C_1$, $C_2$ be chambers of $\mathcal{A}$, $F$ any face.*
$S(C_1, C_2) \subset \mathcal{A}$*: the set of hyperplanes separating $C_1$ from $C_2$,*
$\mathrm{supp}(F)$*: the set of hyperplanes containing $F$.*

# Order on chambers

**Definition.** *Let $C_1$, $C_2$ be chambers of $\mathcal{A}$, $F$ any face.*
$S(C_1, C_2) \subset \mathcal{A}$*: the set of hyperplanes separating $C_1$ from $C_2$,*
supp$(F)$*: the set of hyperplanes containing $F$.*

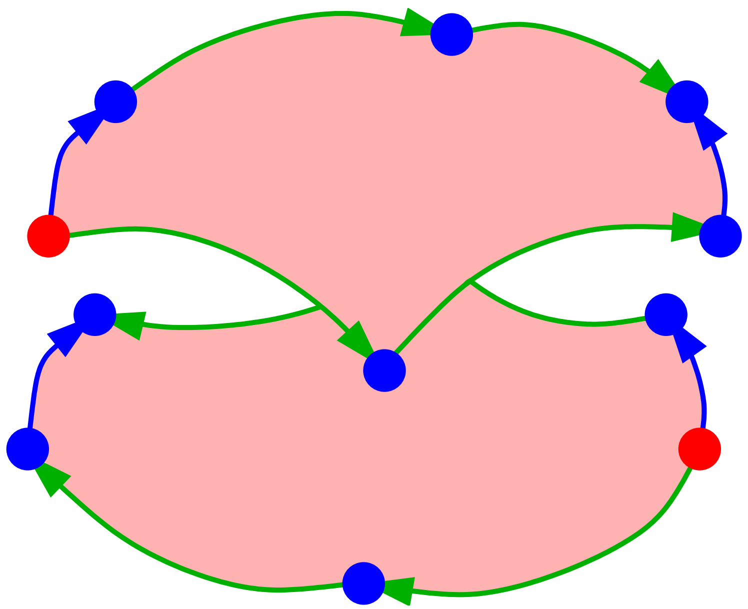


The set of shared faces is then

$$\mathcal{S}_{C_2} \cap \mathcal{S}_{C_1} =$$
$$\{[F, C] \in \mathcal{S}_{C_1} \mid \mathrm{supp}(F) \cap S(C_1, C_2) = \emptyset\}$$

# Order on chambers

**Definition.** *Let $C_1$, $C_2$ be chambers of $\mathcal{A}$, $F$ any face.*
$S(C_1, C_2) \subset \mathcal{A}$: *the set of hyperplanes separating $C_1$ from $C_2$,*
$\text{supp}(F)$: *the set of hyperplanes containing $F$.*



The set of shared faces is then

$$\mathcal{S}_{C_2} \cap \mathcal{S}_{C_1} =$$
$$\{[F, C] \in \mathcal{S}_{C_1} \mid \text{supp}(F) \cap S(C_1, C_2) = \emptyset\}$$

Fix a chamber $B$. The partial order

$$C_1 \leq_B C_2 \Leftrightarrow S(B, C_1) \subseteq S(B, C_2)$$

defines the poset of regions $\mathcal{P}_B(\mathcal{A})$ based at $B$.

# A THEOREM

Let $\dashv$ be any linear extension of $\mathcal{P}_B(\mathcal{A})$.

Let us consider

$$\mathcal{S}_C \setminus \bigcup_{K \dashv C} S_K = \{[F, C] \in \mathcal{S}_C \mid \mathsf{supp}(F) \cap S(C, K) \neq \emptyset \, \forall K \dashv C\}$$

(this is the subcomplex added when ‘glueing in’ $\mathcal{S}_C$)

# A THEOREM

Let $\dashv$ be any linear extension of $\mathcal{P}_B(\mathcal{A})$.

Let us consider

$$\mathcal{S}_C \setminus \bigcup_{K \dashv C} S_K = \{[F, C] \in \mathcal{S}_C \mid \mathrm{supp}(F) \cap S(C, K) \neq \emptyset \, \forall K \dashv C\}$$

(this is the subcomplex added when ‘glueing in’ $\mathcal{S}_C$)

**Theorem. [D.‘07]** *For all $C$ there is a flat $X_C \in \mathcal{L}$ such that*

$$\mathcal{S}_C \setminus \bigcup_{K \dashv C} \mathcal{S}_K \simeq \mathcal{F}(\mathcal{A}^{X_C})$$

(Remember that, for a flat $X \in \mathcal{L}(\mathcal{A})$, $\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathrm{supp}(X)\}$.)

# So what?

$\mathcal{F}(\mathcal{A})$ is the (augmented) face poset of the polytope polar to the arrangement. Thus, it is the face poset of a shellable complex.

# So what?

$\mathcal{F}(\mathcal{A})$ is the (augmented) face poset of the polytope polar to the arrangement. Thus, it is the face poset of a shellable complex.

**Theorem. [D.'07]** *Every shelling of a regular CW-complex induces an acyclic matching on the poset of faces where the critical cells are precisely the homology facets of the shelling.*
*(generalizes [Babson and Hersh,'05], [Chari,'00])*

# So what?

$\mathcal{F}(\mathcal{A})$ is the (augmented) face poset of the polytope polar to the arrangement. Thus, it is the face poset of a shellable complex.

**Theorem. [D.'07]** *Every shelling of a regular CW-complex induces an acyclic matching on the poset of faces where the critical cells are precisely the homology facets of the shelling.*
*(generalizes [Babson and Hersh,'05], [Chari,'00])*

In particular, for every $C$ we have an

acyclic matching $\mathfrak{M}_C$ of $\mathcal{F}(\mathcal{A}^{X_C})$ with only one critical cell.

## So what?

$\mathcal{F}(\mathcal{A})$ is the (augmented) face poset of the polytope polar to the arrangement. Thus, it is the face poset of a shellable complex.

**Theorem. [D.'07]** *Every shelling of a regular CW-complex induces an acyclic matching on the poset of faces where the critical cells are precisely the homology facets of the shelling.*

*(generalizes [Babson and Hersh,'05], [Chari,'00])*

In particular, for every $C$ we have an

acyclic matching $\mathfrak{M}_C$ of $\mathcal{F}(\mathcal{A}^{X_C})$ with only one critical cell.

Let

$$\mathfrak{M} := \bigcup_{C \in \mathcal{C}(\mathcal{A})} \mathfrak{M}_C$$

**Corollary.** By the "patchwork lemma", $\mathfrak{M}$ is an acyclic matching of the poset of cells of $Sal(\mathcal{A})$.

# No-broken-circuit sets

Recall $\mathcal{A} := \{H_1, \ldots, H_n\}$ and the normal vectors $\alpha_i$. A set $U \subset \mathcal{A}$ is called *dependent / independent* if the corresponding set of vectors is.

# No-broken-circuit sets

Recall $\mathcal{A} := \{H_1, \ldots, H_n\}$ and the normal vectors $\alpha_i$. A set $U \subset \mathcal{A}$ is called *dependent / independent* if the corresponding set of vectors is.

• A circuit of $\mathcal{A}$ is any minimally dependent subset $U \in \mathcal{A}$. Any set of the form $U \setminus \min U$ for a circuit $U$ is called broken circuit.

# No-broken-circuit sets

Recall $\mathcal{A} := \{H_1, \ldots, H_n\}$ and the normal vectors $\alpha_i$. A set $U \subset \mathcal{A}$ is called *dependent / independent* if the corresponding set of vectors is.

• A circuit of $\mathcal{A}$ is any minimally dependent subset $U \in \mathcal{A}$. Any set of the form $U \setminus \min U$ for a circuit $U$ is called broken circuit.

• A no-broken-circuit (nbc-) set is any independent set that does not contain any broken circuit.

# No-broken-circuit sets

Recall $\mathcal{A} := \{H_1, \ldots, H_n\}$ and the normal vectors $\alpha_i$. A set $U \subset \mathcal{A}$ is called *dependent / independent* if the corresponding set of vectors is.

• A circuit of $\mathcal{A}$ is any minimally dependent subset $U \in \mathcal{A}$. Any set of the form $U \setminus \min U$ for a circuit $U$ is called broken circuit.

• A no-broken-circuit (nbc-) set is any independent set that does not contain any broken circuit.
$\mathrm{nbc}_i(\mathcal{A})$ is the family of all the nbc-sets of $\mathcal{A}$ with $i$ elements.

# No-broken-circuit sets

Recall $\mathcal{A} := \{H_1, \ldots, H_n\}$ and the normal vectors $\alpha_i$. A set $U \subset \mathcal{A}$ is called *dependent / independent* if the corresponding set of vectors is.

• A circuit of $\mathcal{A}$ is any minimally dependent subset $U \in \mathcal{A}$. Any set of the form $U \setminus \min U$ for a circuit $U$ is called broken circuit.

• A no-broken-circuit (nbc-) set is any independent set that does not contain any broken circuit.
$\mathsf{nbc}_i(\mathcal{A})$ is the family of all the nbc-sets of $\mathcal{A}$ with $i$ elements.

It is then natural to set formally $\mathsf{nbc}_0(\mathcal{A}) := \{\emptyset\}$, and let

$$\mathsf{nbc}(\mathcal{A}) := \bigcup_{i=0,\ldots n} \mathsf{nbc}_i(\mathcal{A})$$

# Minimality

**Theorem. [Orlik, Solomon '80]** *For all* $i = 0, \ldots, n$, *the set* $\mathsf{nbc}_i(\mathcal{A})$ *indexes a basis of* $H_i(\mathcal{M}(\mathcal{A}), \mathbb{Z})$.

# MINIMALITY

**Theorem. [Orlik, Solomon '80]** *For all* $i = 0, \dots, n$, *the set* $\mathsf{nbc}_i(\mathcal{A})$ *indexes a basis of* $H_i(\mathcal{M}(\mathcal{A}), \mathbb{Z})$.

**Theorem. [Zaslavsky '75, Brylawski '77]** *For any real arrangement* $\mathcal{A}$*:*

$$|\mathcal{C}(\mathcal{A})| = |\mathsf{nbc}(\mathcal{A})|$$

# Minimality

**Theorem. [Orlik, Solomon '80]** *For all $i = 0, \ldots, n$, the set $\mathsf{nbc}_i(\mathcal{A})$ indexes a basis of $H_i(\mathcal{M}(\mathcal{A}), \mathbb{Z})$.*

**Theorem. [Zaslavsky '75, Brylawski '77]** *For any real arrangement $\mathcal{A}$:*

$$|\mathcal{C}(\mathcal{A})| = |\mathsf{nbc}(\mathcal{A})|$$

Recall that our matching $\mathfrak{M}$ has one critical cell for every chamber. Thus, by the *pigeon hole principle*:

**Corollary.** The Morse complex obtained from our $\mathfrak{M}$ is minimal.

# MINIMALITY

**Theorem. [Orlik, Solomon '80]** *For all $i = 0, \ldots, n$, the set* $\mathsf{nbc}_i(\mathcal{A})$ *indexes a basis of $H_i(\mathcal{M}(\mathcal{A}), \mathbb{Z})$.*

**Theorem. [Zaslavsky '75, Brylawski '77]** *For any real arrangement $\mathcal{A}$:*

$$|\mathcal{C}(\mathcal{A})| = |\mathsf{nbc}(\mathcal{A})|$$

Recall that our matching $\mathfrak{M}$ has one critical cell for every chamber. Thus, by the *pigeon hole principle*:

**Corollary.** The Morse complex obtained from our $\mathfrak{M}$ is minimal.

...but we would like to understand better...

# A BIJECTION

**Lemma. [Jewell & Orlik, '02]** *For every choice of a base chamber* $B$ *and an ordering of* $\mathcal{A}$ *there is an explicit bijection*

$$\eta : \mathcal{C}(\mathcal{A}) \to \mathsf{nbc}(\mathcal{A}).$$

# A BIJECTION

**Lemma. [Jewell & Orlik, '02]** *For every choice of a base chamber $B$ and an ordering of $\mathcal{A}$ there is an explicit bijection*

$$\eta : \mathcal{C}(\mathcal{A}) \to \mathsf{nbc}(\mathcal{A}).$$

Recall that the construction of nbc relies upon an (arbitrary) ordering of $\mathcal{A}$.

**Definition.** *An ordering of $\mathcal{A}$ satisfies the cut property with respect to a chamber $B$ if it is obtained from the sequence in which a maximal chain in $\mathcal{P}_B(\mathcal{A})$ traverses the hyperplanes.*

# A bijection

**Lemma. [Jewell & Orlik, '02]** *For every choice of a base chamber $B$ and an ordering of $\mathcal{A}$ there is an explicit bijection*

$$\eta : \mathcal{C}(\mathcal{A}) \to \mathsf{nbc}(\mathcal{A}).$$

Recall that the construction of nbc relies upon an (arbitrary) ordering of $\mathcal{A}$.

**Definition.** *An ordering of $\mathcal{A}$ satisfies the cut property with respect to a chamber $B$ if it is obtained from the sequence in which a maximal chain in $\mathcal{P}_B(\mathcal{A})$ traverses the hyperplanes.*

**Theorem. [D. '07]** *Every ordering of $\mathcal{A}$ satisfying the cut property with respect to $B$ defines a ("lexicographic") linear extension $\dashv^{lex}$ of $\mathcal{P}_B(\mathcal{A})$.*

# A BIJECTION

**Lemma. [Jewell & Orlik, '02]** *For every choice of a base chamber $B$ and an ordering of $\mathcal{A}$ there is an explicit bijection*

$$\eta : \mathcal{C}(\mathcal{A}) \to \mathsf{nbc}(\mathcal{A}).$$

Recall that the construction of nbc relies upon an (arbitrary) ordering of $\mathcal{A}$.

**Definition.** *An ordering of $\mathcal{A}$ satisfies the cut property with respect to a chamber $B$ if it is obtained from the sequence in which a maximal chain in $\mathcal{P}_B(\mathcal{A})$ traverses the hyperplanes.*

**Theorem. [D. '07]** *Every ordering of $\mathcal{A}$ satisfying the cut property with respect to $B$ defines a ("lexicographic") linear extension $\dashv^{lex}$ of $\mathcal{P}_B(\mathcal{A})$. With respect to this linear extension, we have*

$$X_C = \bigcap \eta(C).$$

# SUMMARY

Let us write $[F_c, C]$ for the critical cell of $\mathfrak{M}_C$, and recall that $|F_C| = X_C$.

**Lemma. [D. '07]** $\dim(\overline{C} \cap X_C) = |\eta(C)|$.

# Summary

Let us write $[F_c, C]$ for the critical cell of $\mathfrak{M}_C$, and recall that $|F_C| = X_C$.

**Lemma. [D. '07]** $\dim(\overline{C} \cap X_C) = |\eta(C)|$.

We can then summarize.

**Proposition. [D. '07]** *Let $\mathcal{A}$ be a finite real arrangement of linear hyperplanes. Every choice of a base chamber $B$ and of an ordering $\mathcal{A} = \{H_1, \ldots, H_n\}$ satisfying the cut property with respect to $B$ defines a discrete Morse function on $Sal(\mathcal{A})$ with the minimal number of critical cells. The faces of the associated Morse $CW$-complex are*

$$\Big[\ \underbrace{\overline{C} \cap \bigcap \eta(C)}_{\text{``}F_C\text{''}},\quad C\ \Big].$$

# Bibliography

M. Salvetti; *Topology of the complement of real hyperplanes in $C^N$.* Invent. Math. **88** (1987), no. 3, 603-618.

D. N. Kozlov; *Combinatorial algebraic topology.* Algorithms and Computation in Mathematics, 21. Springer, Berlin, 2008. xx+389 pp.

M. Salvetti, S. Settepanella; *Combinatorial Morse theory and minimality of hyperplane arrangements.* Geom. Topol. 11 (2007), 1733-1766.

E. Delucchi, S. Settepanella; *Combinatorial polar orderings and follow-up arrangements.* Preprint, ArXiv:0711.1517 (but ask us for the updated version).

A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. M. Ziegler; *Oriented Matroids.* Encyclopedia of Mathematics and its Applications, 46. Cambridge University Press, Cambridge, 1999. xii+548 pp.

E. Delucchi; *Shelling-type orderings of regular CW-complexes and acyclic matchings of the Salvetti complex.* IMRN 2008, 39 pp.