Discrete Morse functions for complexified arrangements

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For a flat $X \in \mathcal{L}(\mathcal{A})$: $\mathcal{A}^X := \big\{X \cap H \mid H \in \mathcal{A} \setminus \mathrm{supp}(X)\big\}$.

Complexified arrangements

Let $\mathcal{A}$ be as before, and for every $i = 1, \dots n$ choose $\alpha_i \in \mathbb{R}^d$ normal to H_i.

The complexification of $\mathcal{A}$ is the set

$$\mathcal{A}_{\mathbb{C}} := \{H_1^{\mathbb{C}}, \dots, H_n^{\mathbb{C}}\} \text{ where } H_i^{\mathbb{C}} := \{z \in \mathbb{C}^d \mid \langle z \mid \alpha_i \rangle = 0\}.$$

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Theorem. [Randell '02, Dimca & Papadima '03] *$\mathcal{M}(\mathcal{A})$ is a minimal space. I.e., it has the homotopy type of a CW-complex with as many (k-)cells as there are generators in the (k-)th homology.*

Motivation

Two constructive approaches to minimality:

1) M. Salvetti, S. Settepanella; *Combinatorial Morse theory and minimality of hyperplane arrangements.* Geometry and Topology 11 (2007): 1733-1766.

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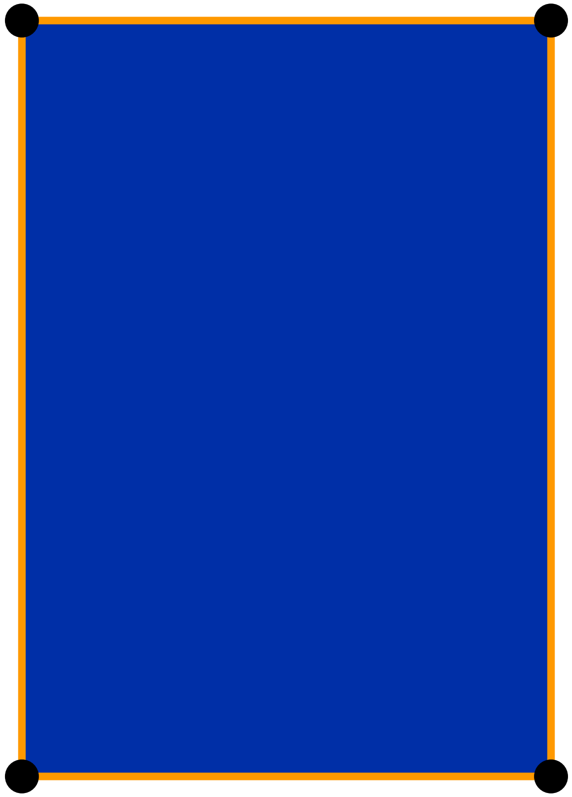
2) E. D.; *Shelling-type orderings of regular CW-complexes and acyclic matchings of the Salvetti complex.* International Mathematics Research Notices, 2008 (electronic), 39 pp.

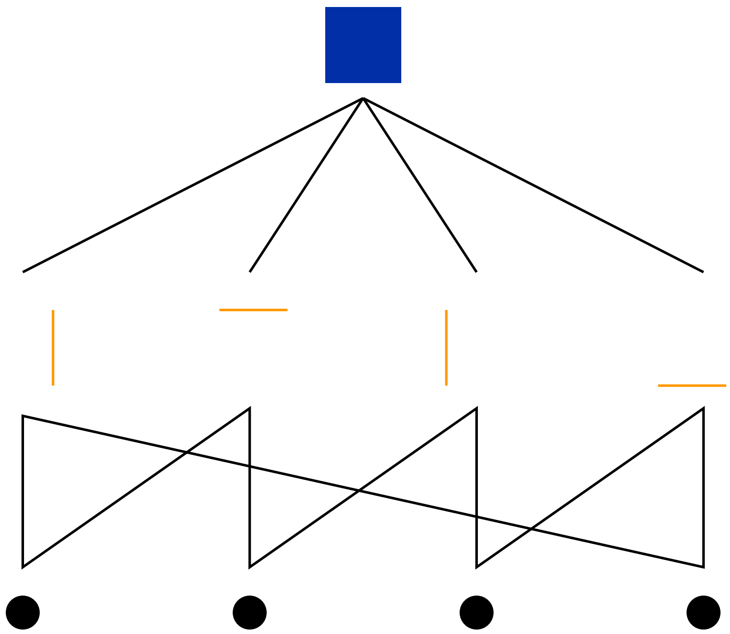
Both approaches apply Discrete Morse Theory to the Salvetti complex of $\mathcal{A}$

Discrete Morse theory

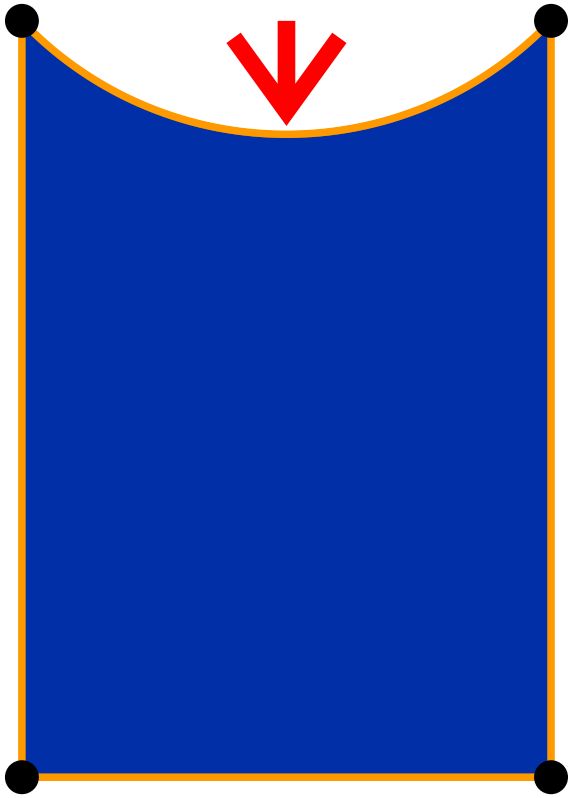
Here is a regular CW complex



with its poset of cells:



Elementary collapses...



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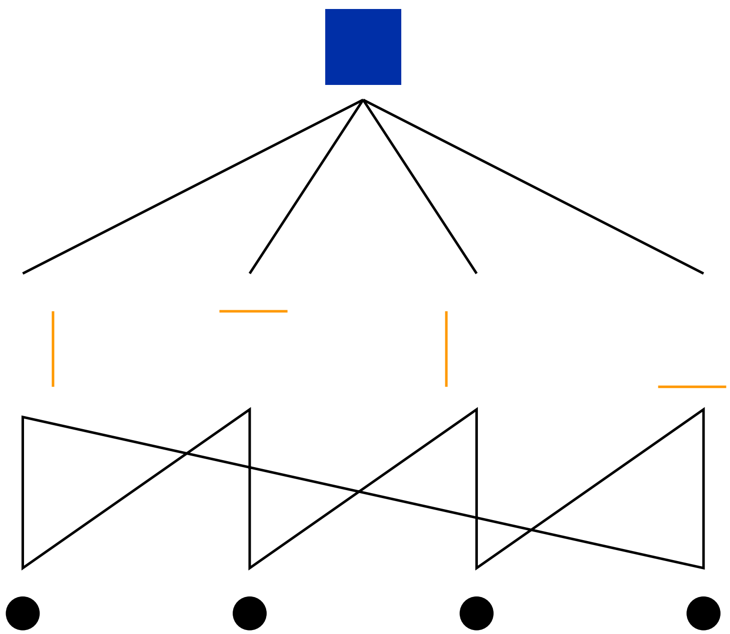

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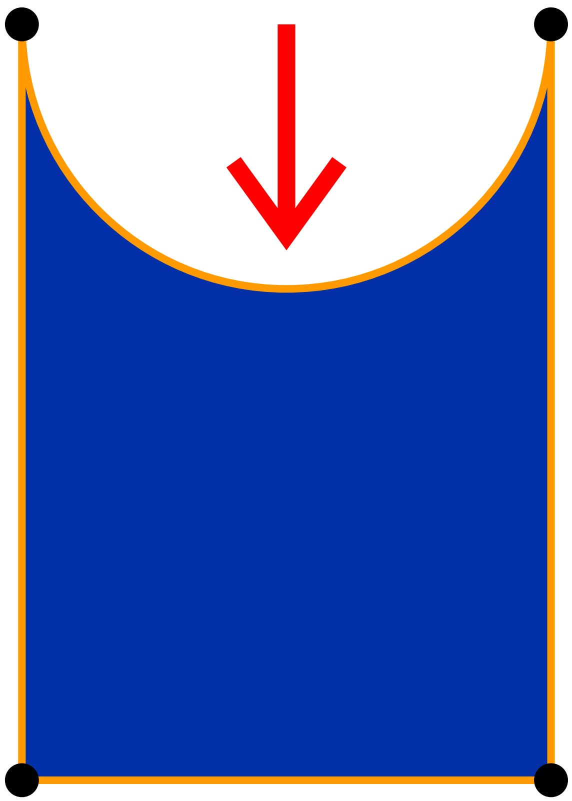

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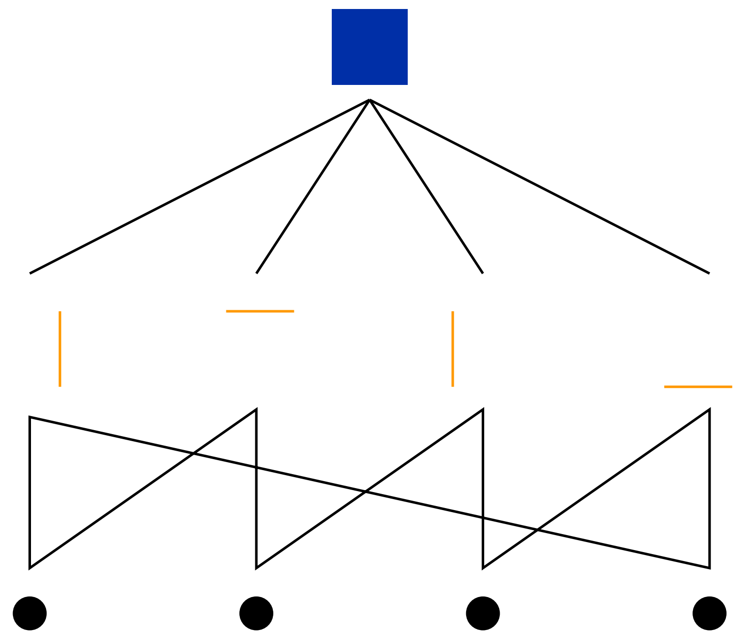

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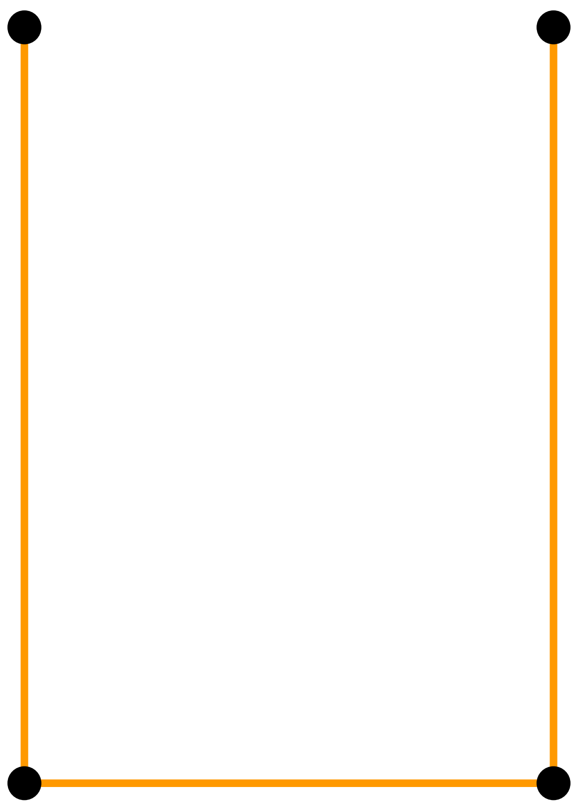

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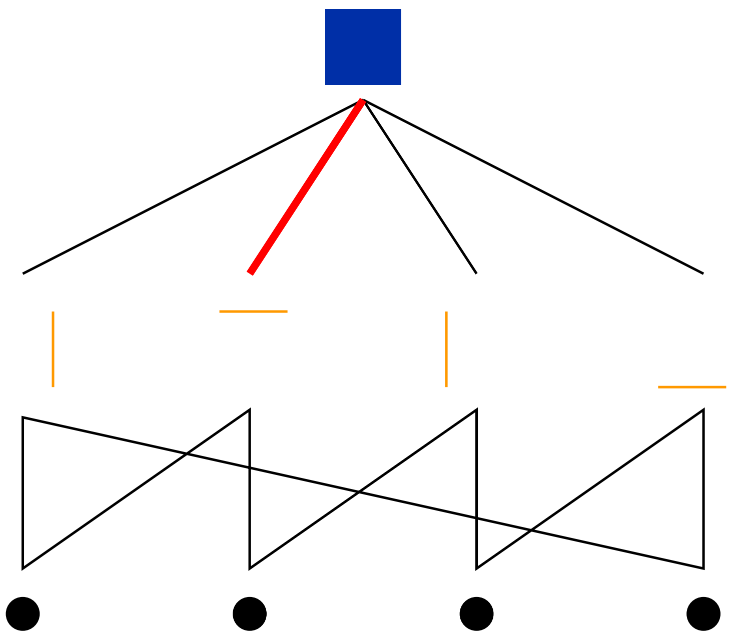

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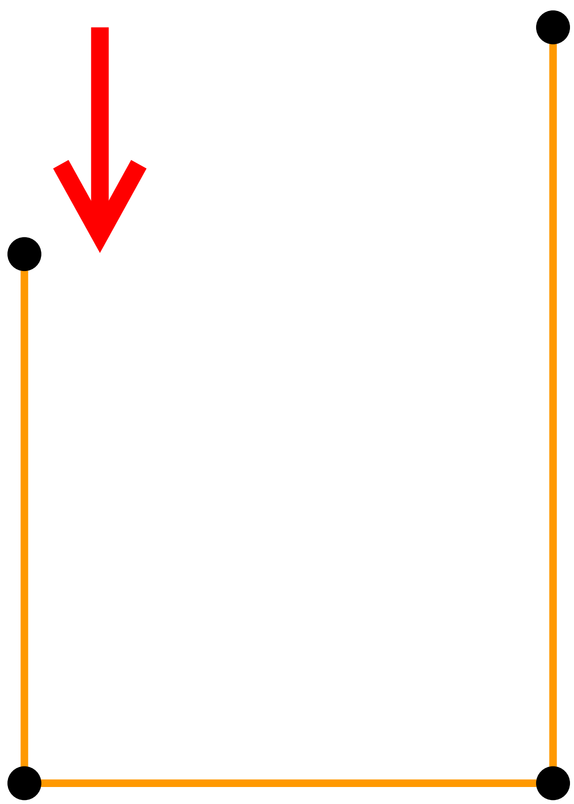

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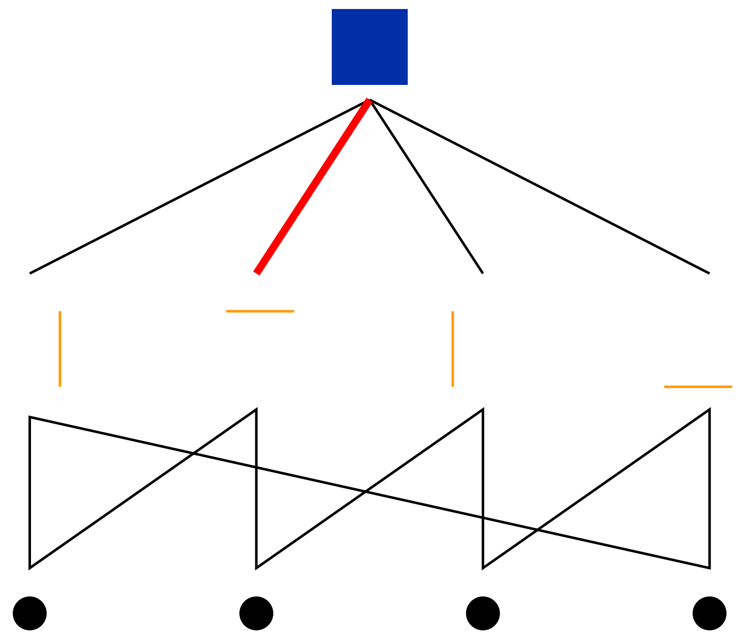

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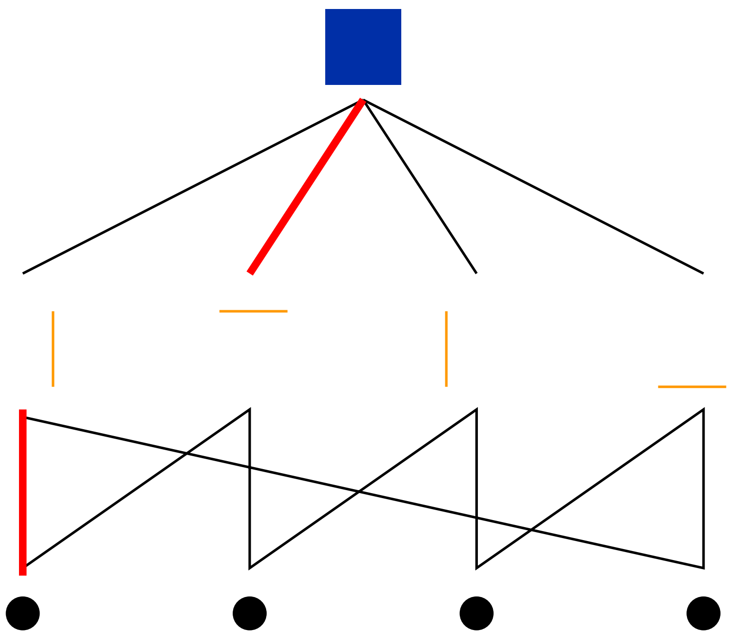

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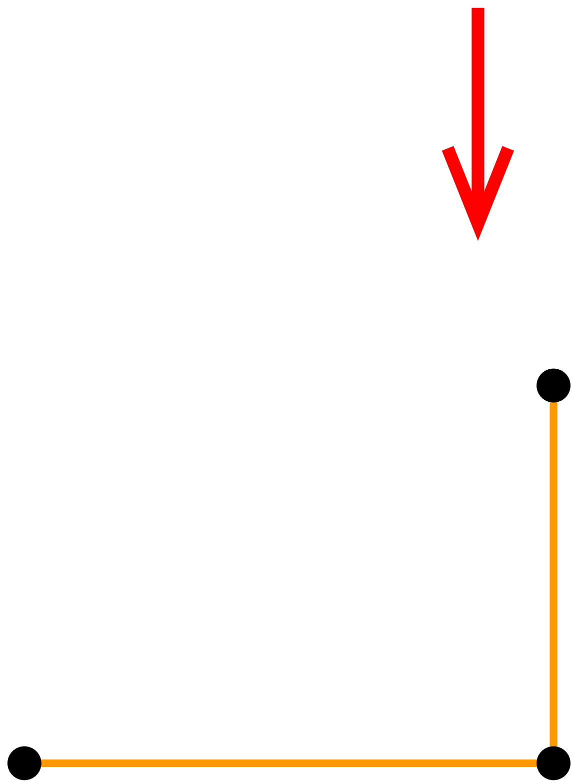

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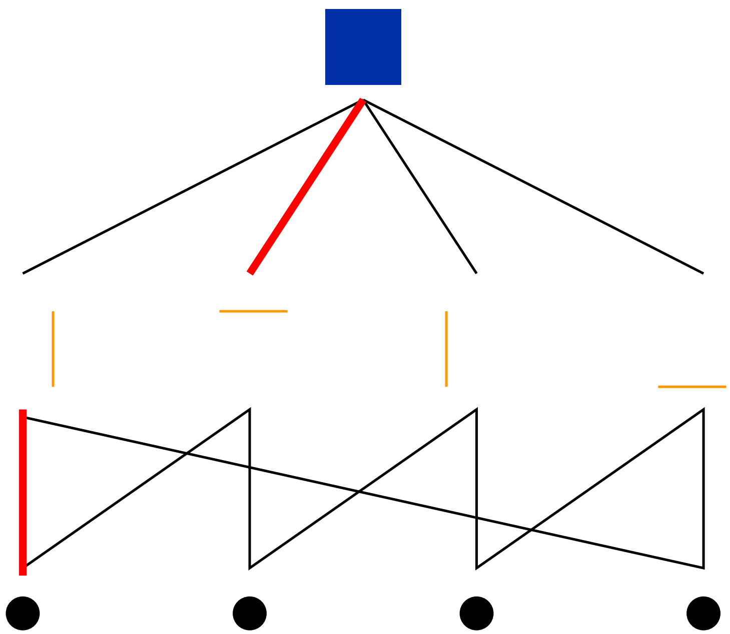

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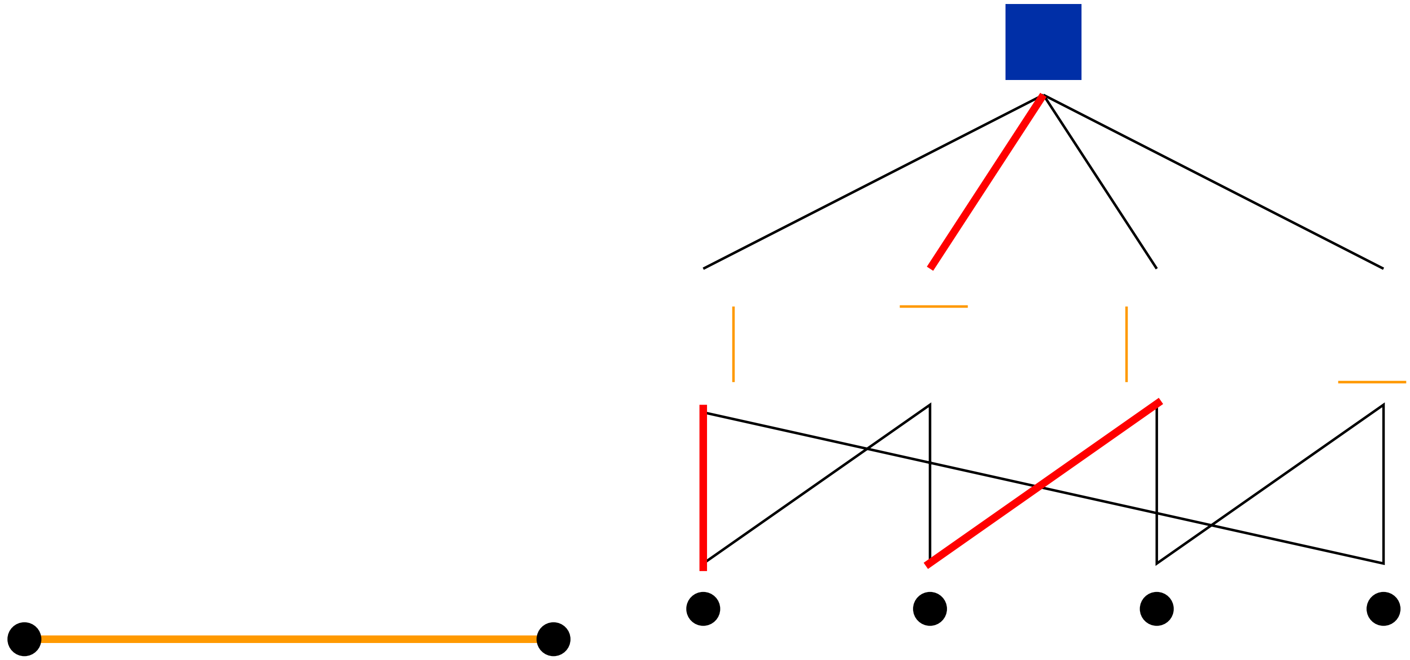

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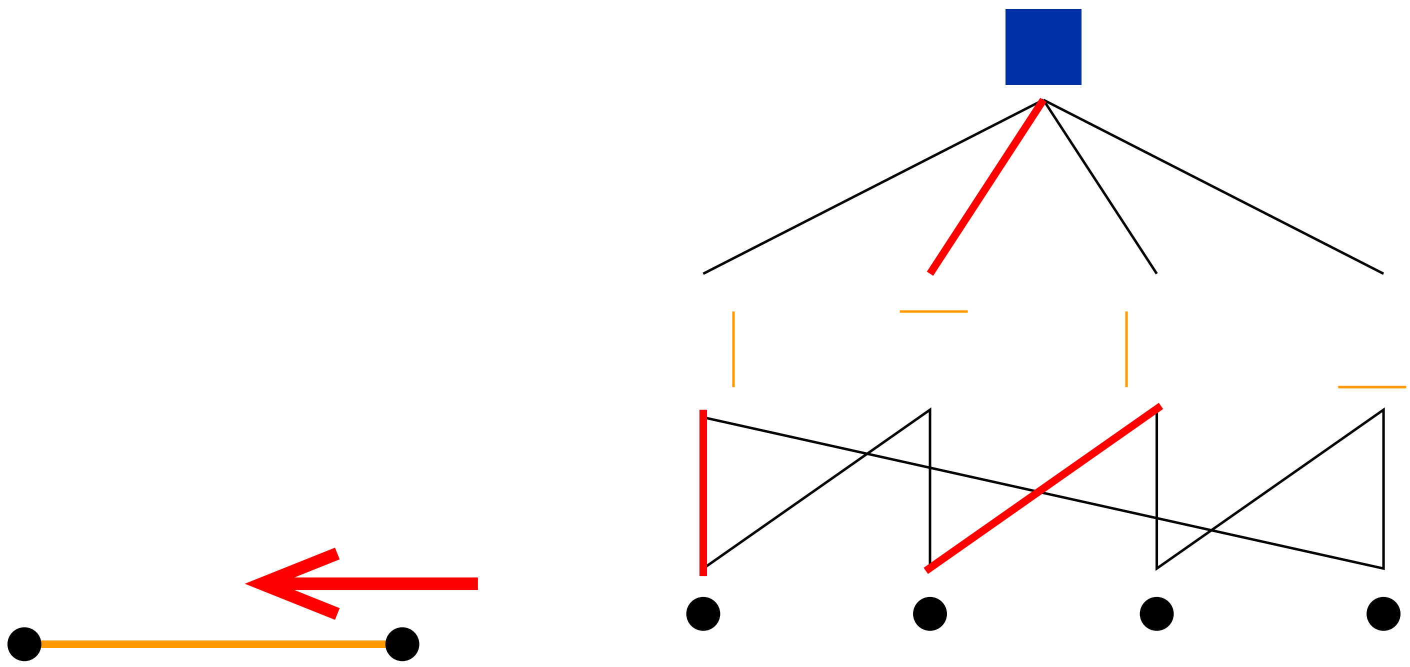

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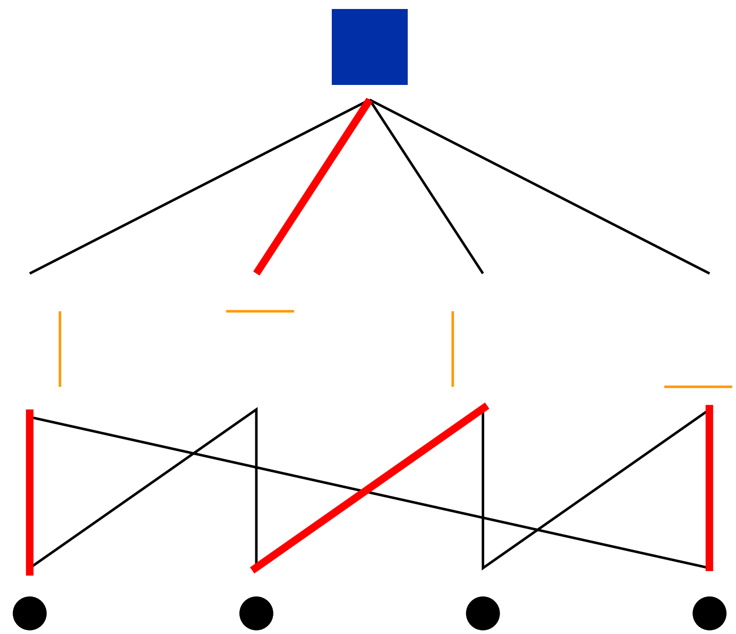

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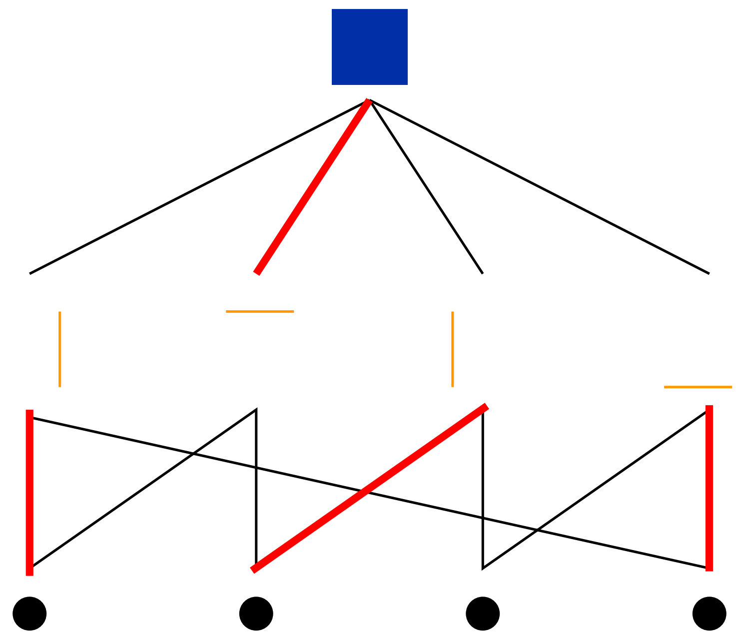
Acyclic matchings

The sequence of collapses is encoded in a matching of the Hasse diagram of the poset of cells.



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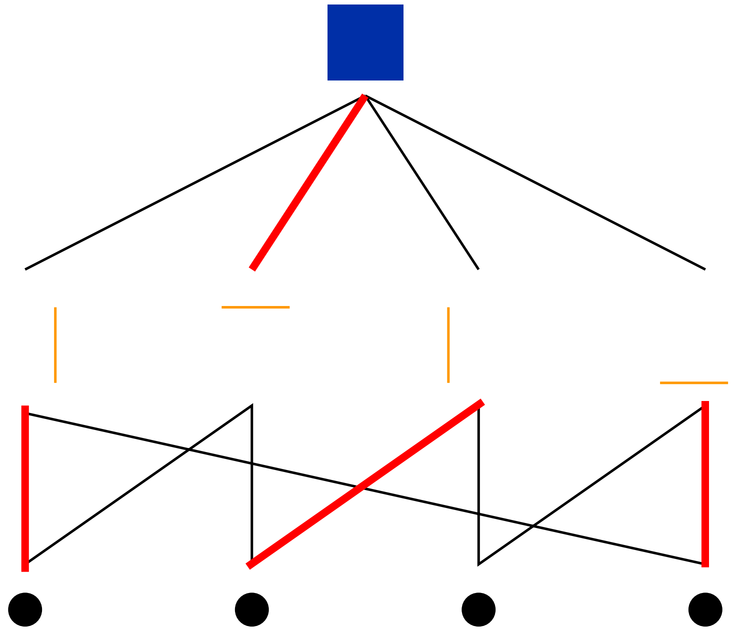

Question: Does any matching encode such a sequence?

ACYCLIC MATCHINGS

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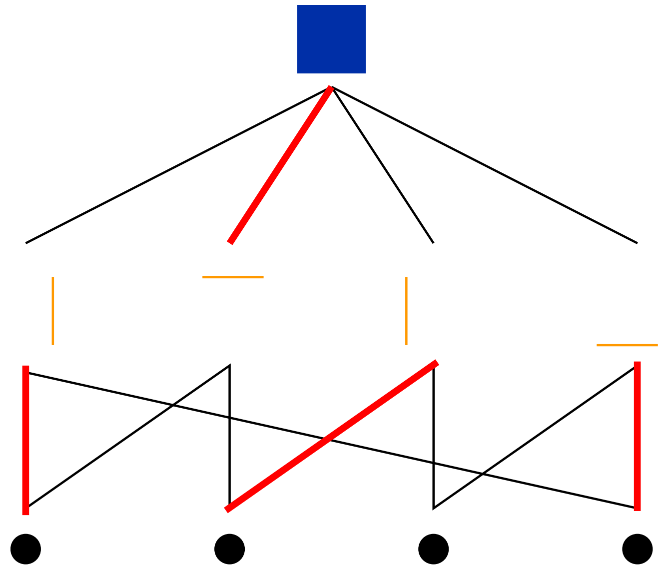
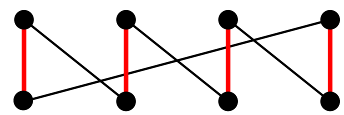


Question: Which matchings encode such a sequence?

Answer: Those whitout "cycles" like .

Acyclic matchings $\leftrightarrow$ discrete Morse functions.

The Salvetti complex $Sal(\mathcal{A})$

[Salvetti '87]

Given a complexified arrangement $\mathcal{A}$, consider the oriented graph $\Gamma(\mathcal{A})$ with:

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$E(\Gamma(\mathcal{A}))$ contains (C_1, C_2)
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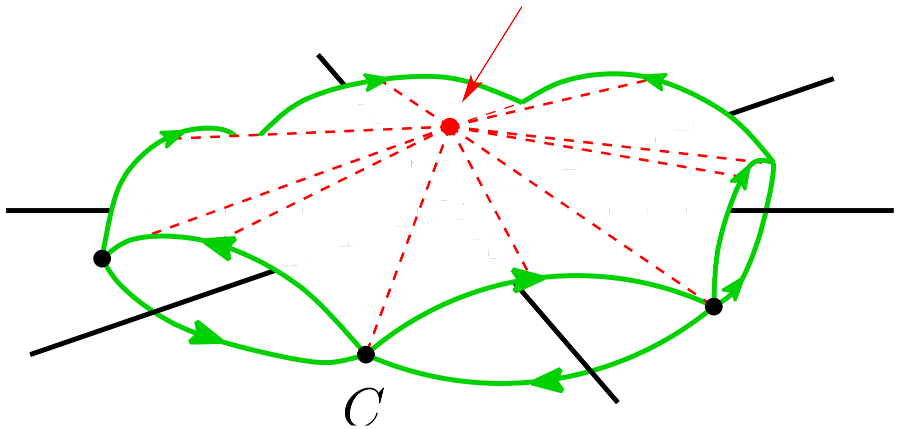

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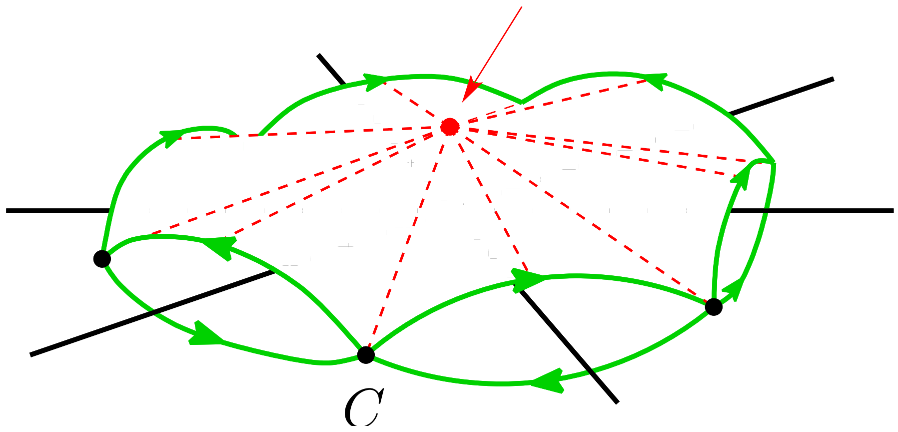

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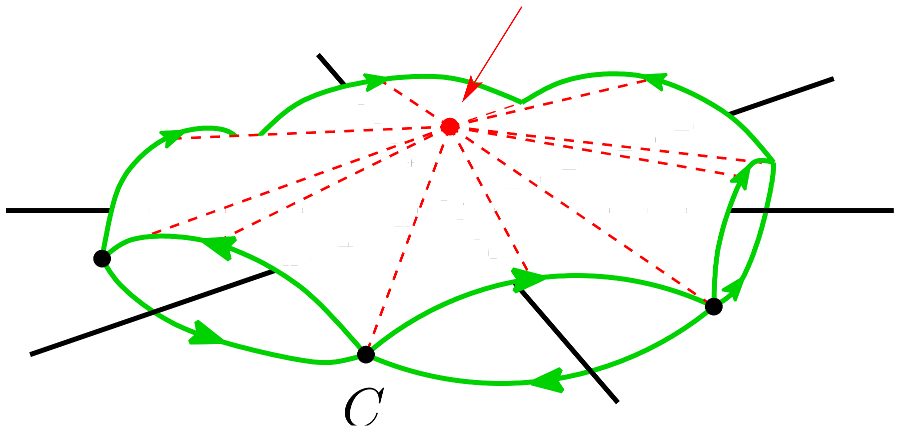

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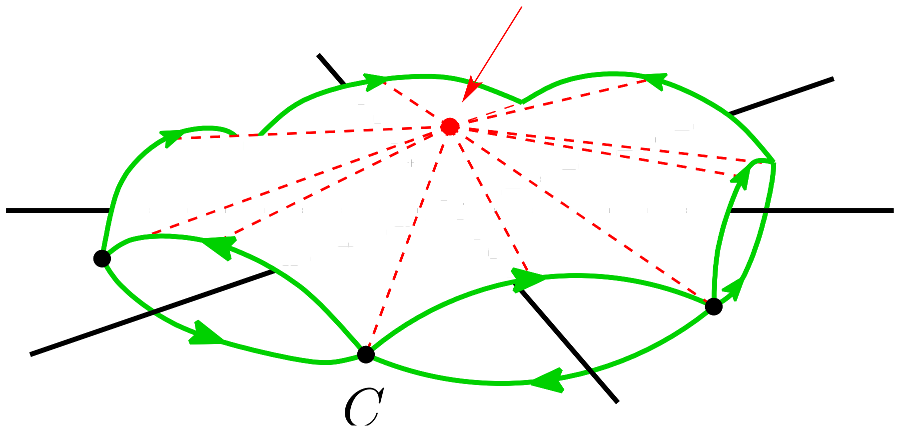

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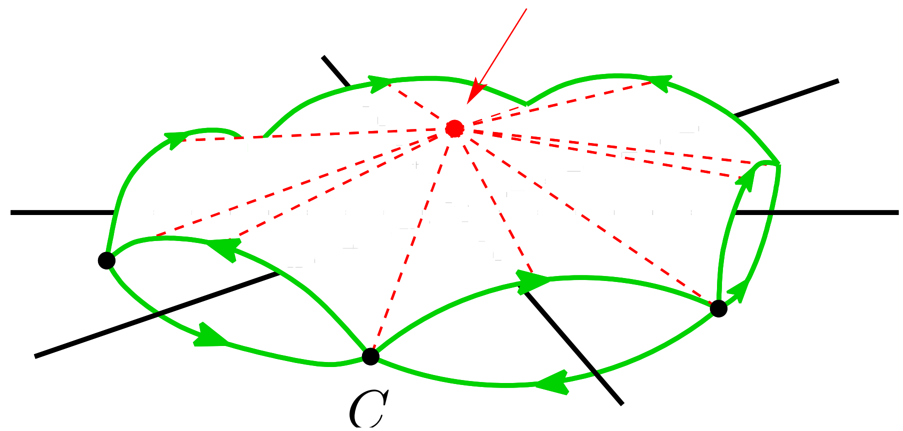

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Theorem. [Salvetti '87] $Sal(\mathcal{A}) \simeq \mathcal{M}(\mathcal{A})$.

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[Salvetti and Settepanella '07]

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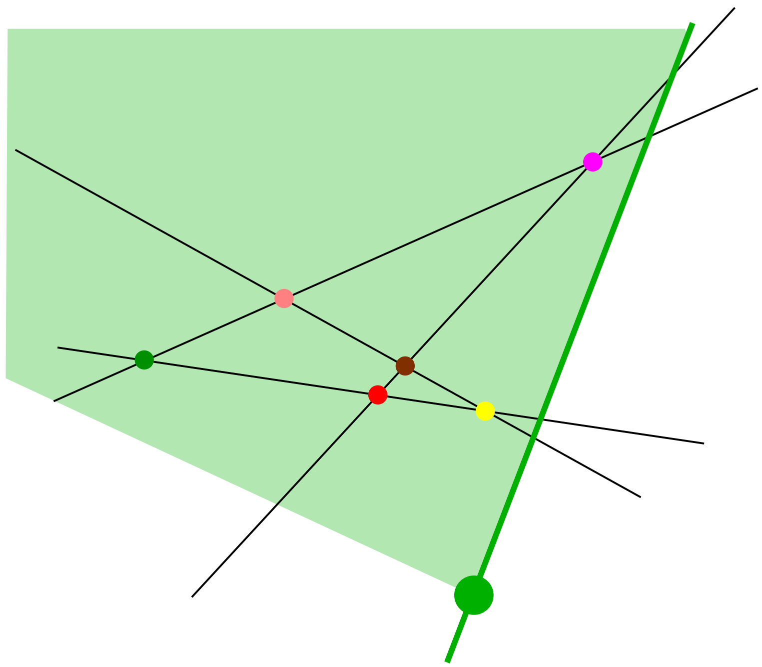

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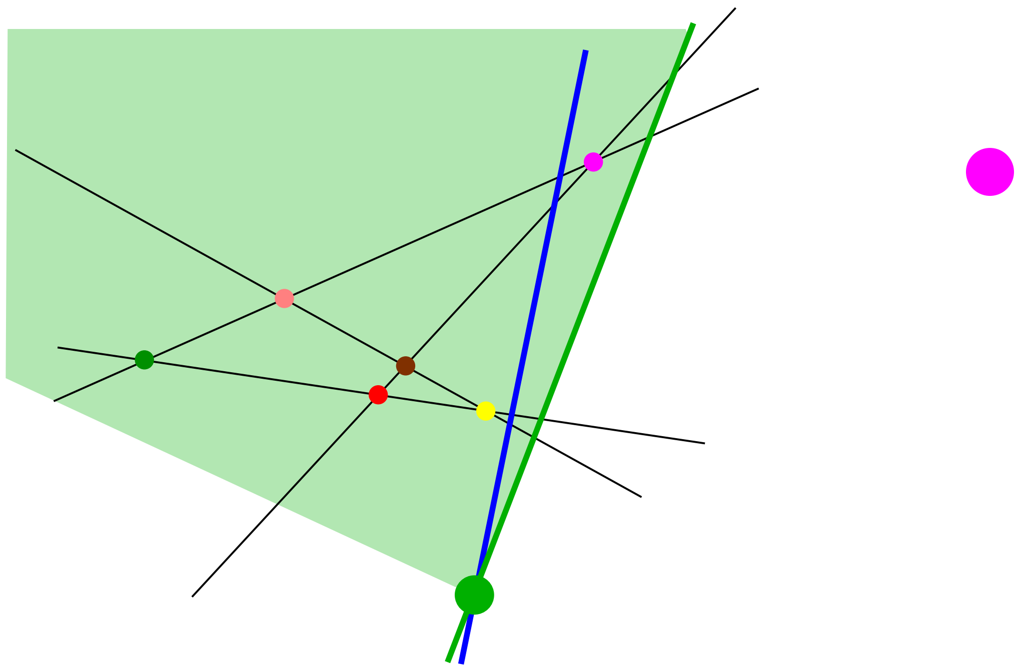

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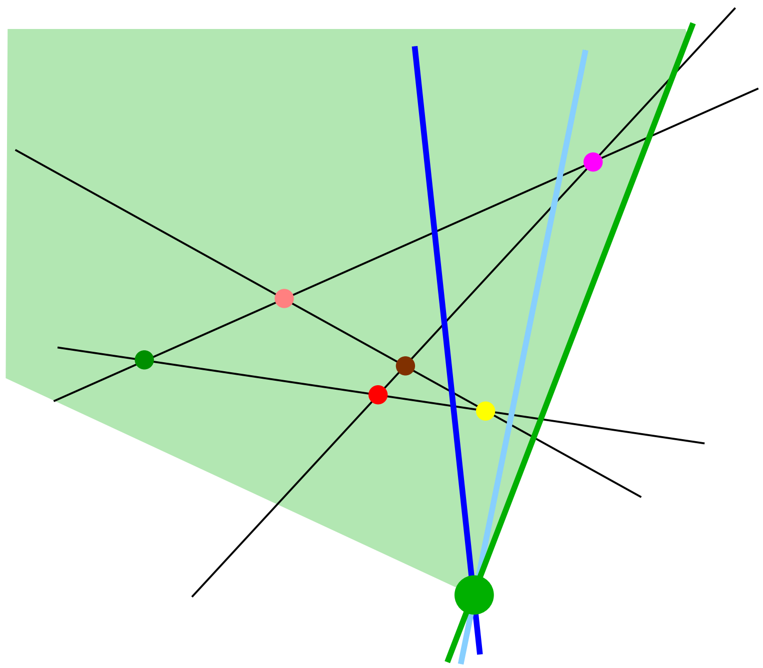

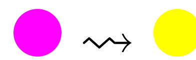

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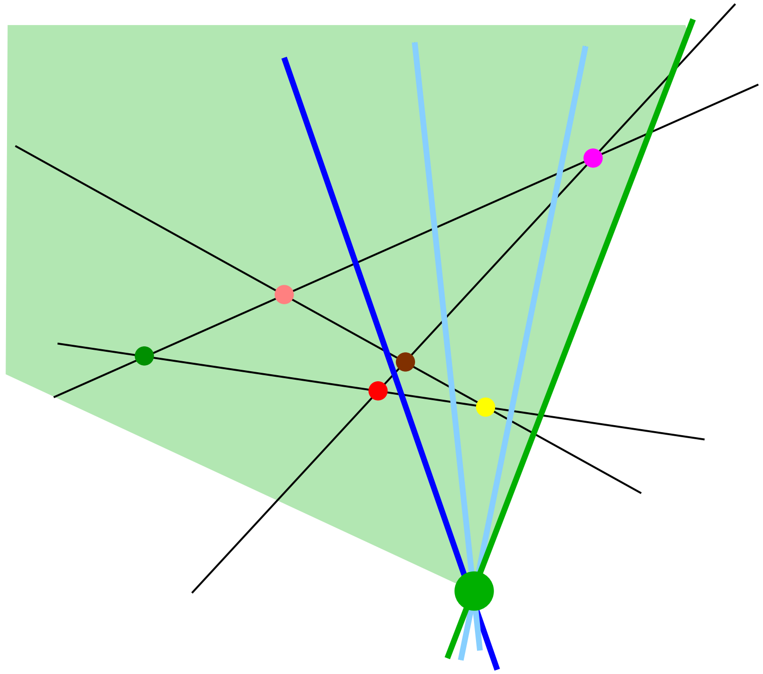

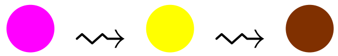

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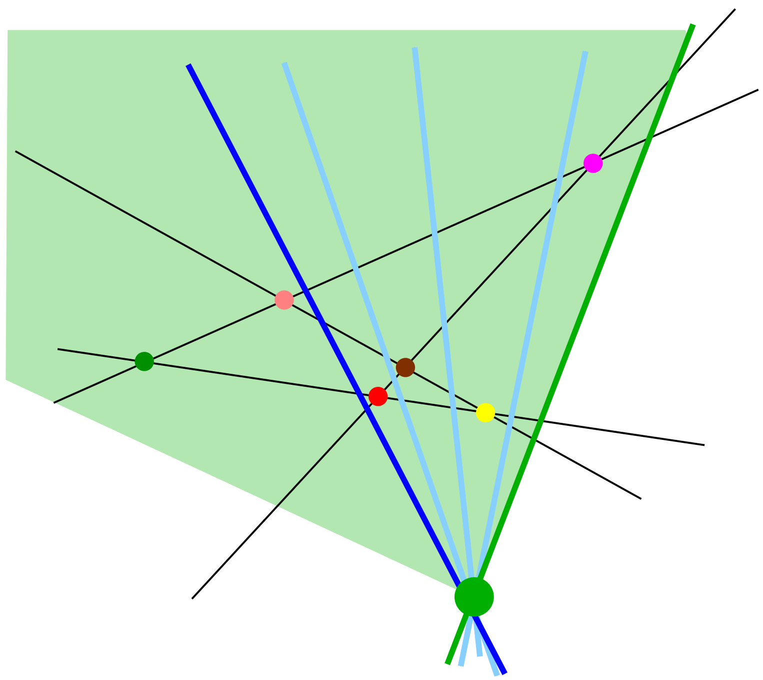

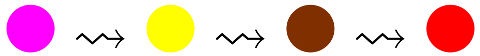

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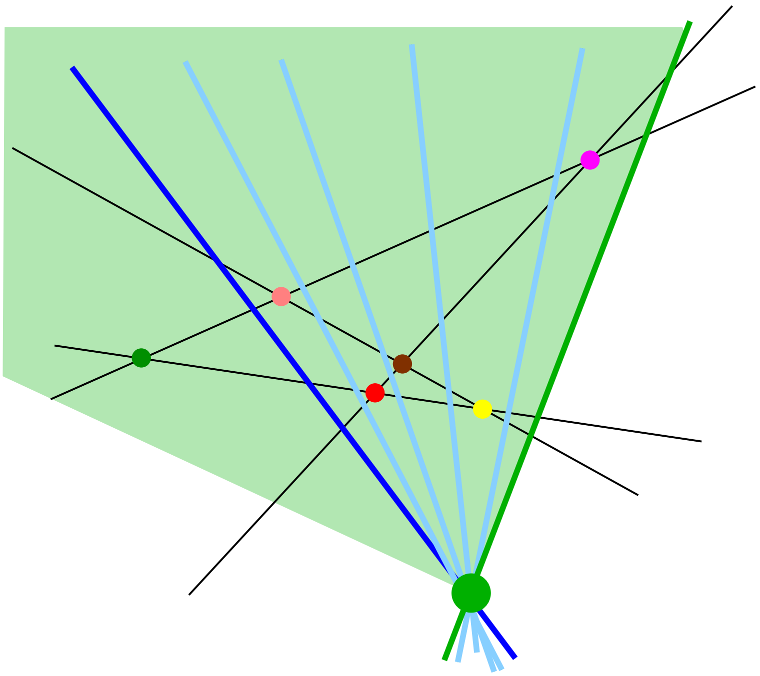

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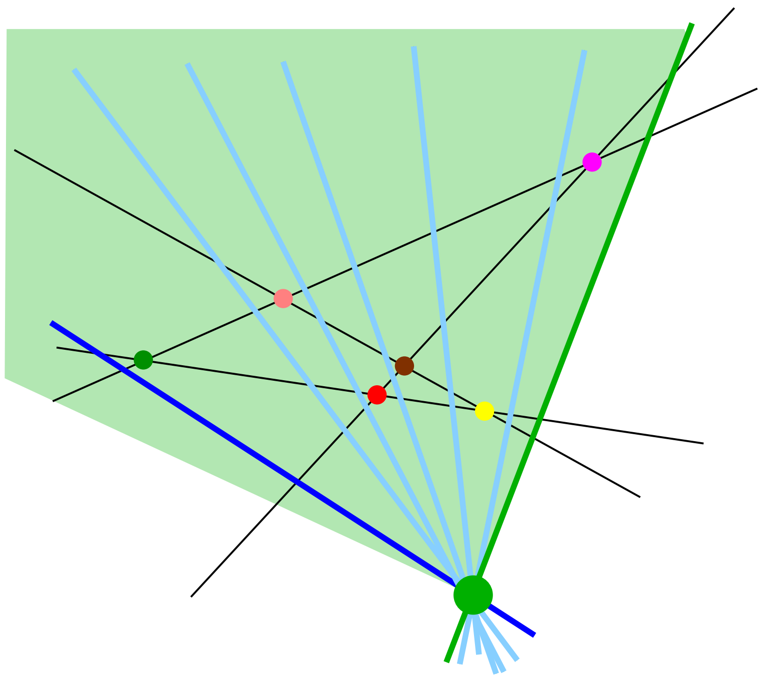

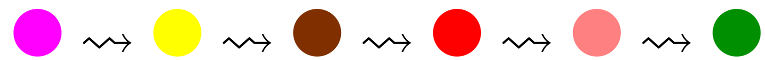

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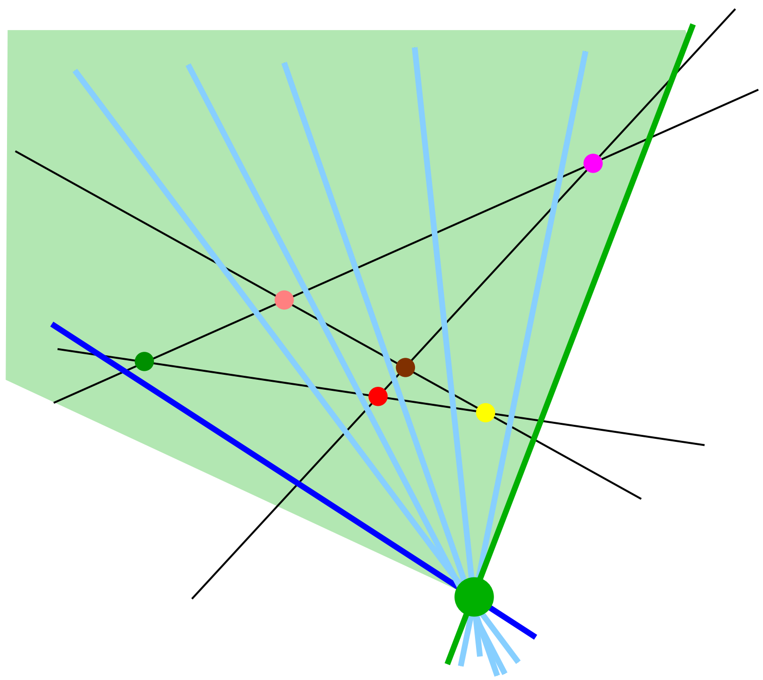

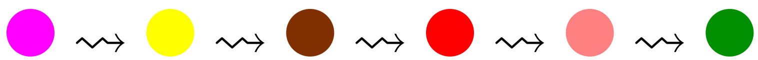

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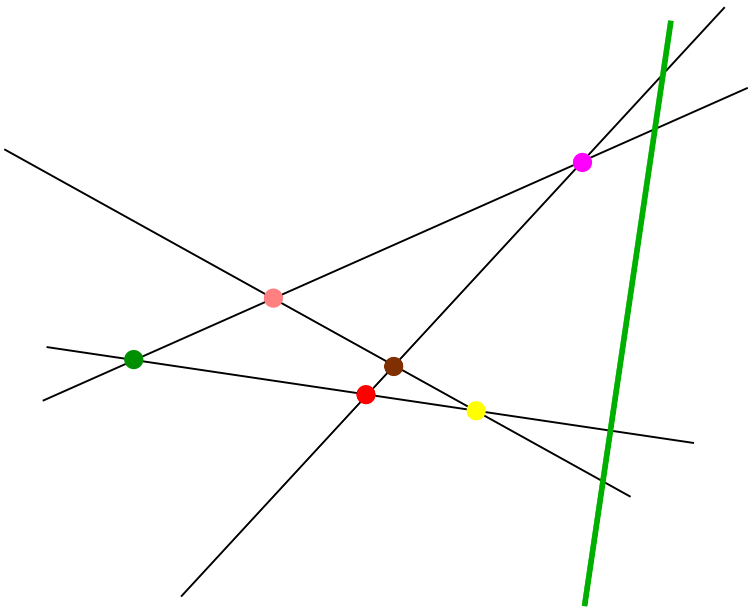
There is an algorithm that starts from these data to construct a maximum acyclic matching for $Sal(\mathcal{A})$ and the boundary maps of the corresponding minimal complex.

Combinatorial polar ordering

[D. and Settepanella '07]

Let $V_0 \subset V_1 \subset \ldots\ldots V_d = \mathbb{R}^d$ be a flag of affine subspaces of $\mathbb{R}^d$ in general position with respect to $\mathcal{A}$.



For all $k = 1, \ldots, d$ let the pseudohyperplane V_{k-1} sweep through the arrangement $V_k \cap \mathcal{A}$.

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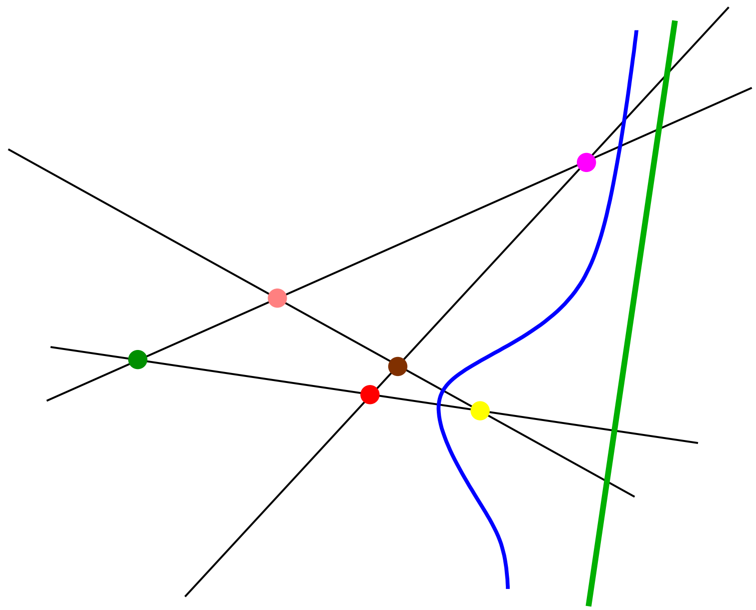

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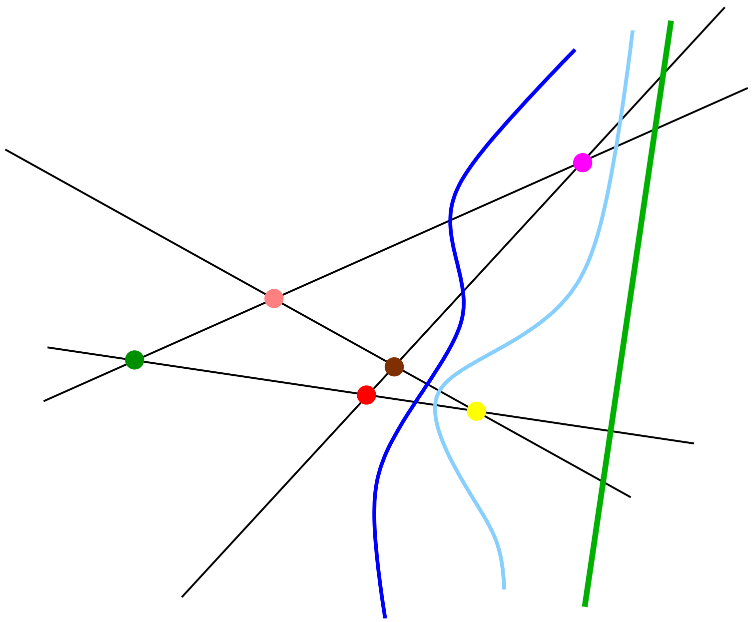

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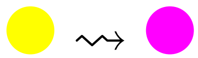

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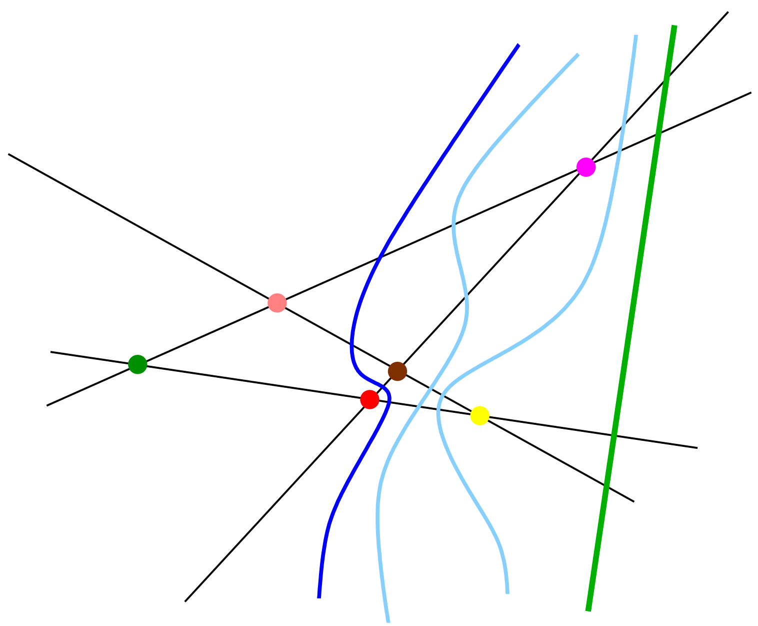

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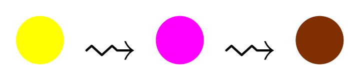

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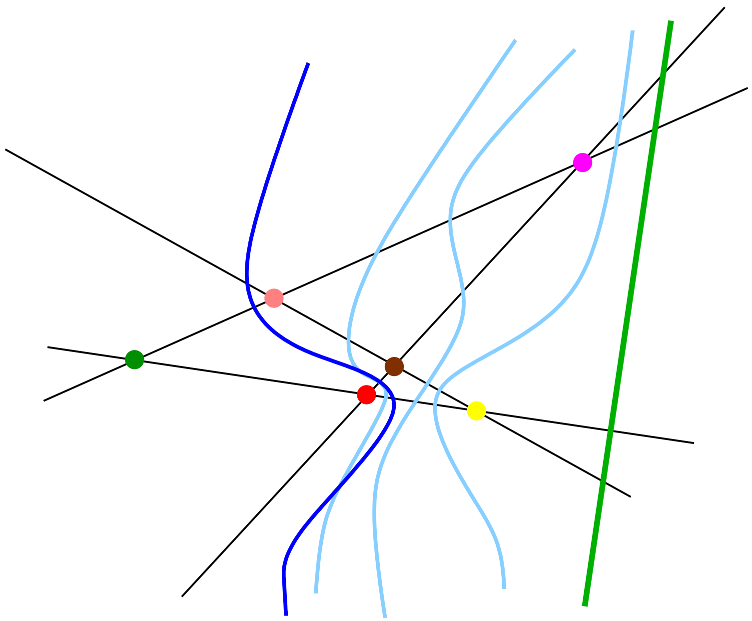

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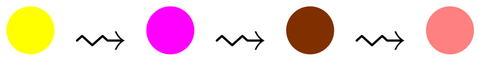

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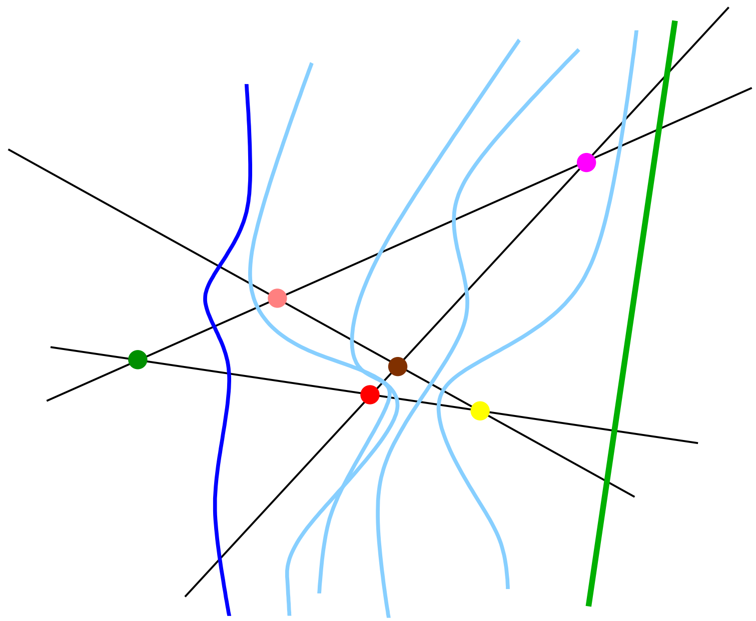

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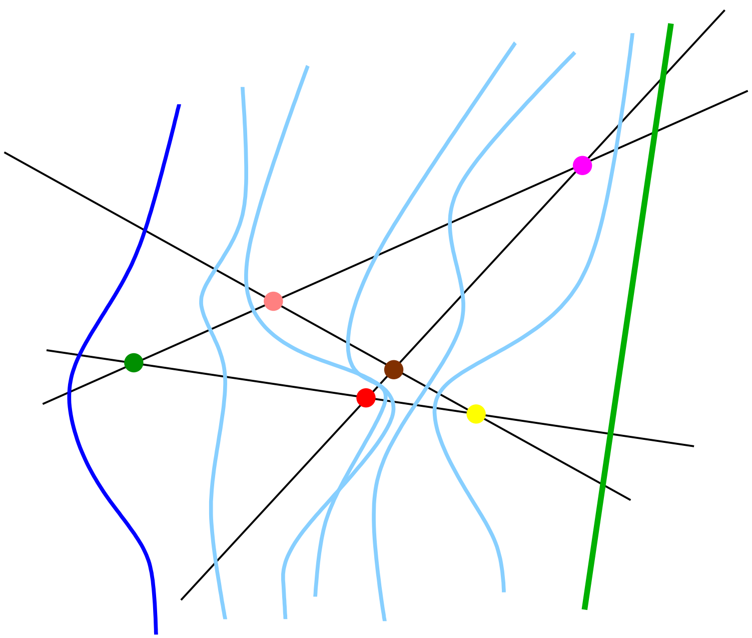

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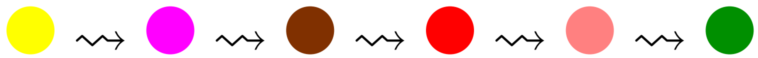

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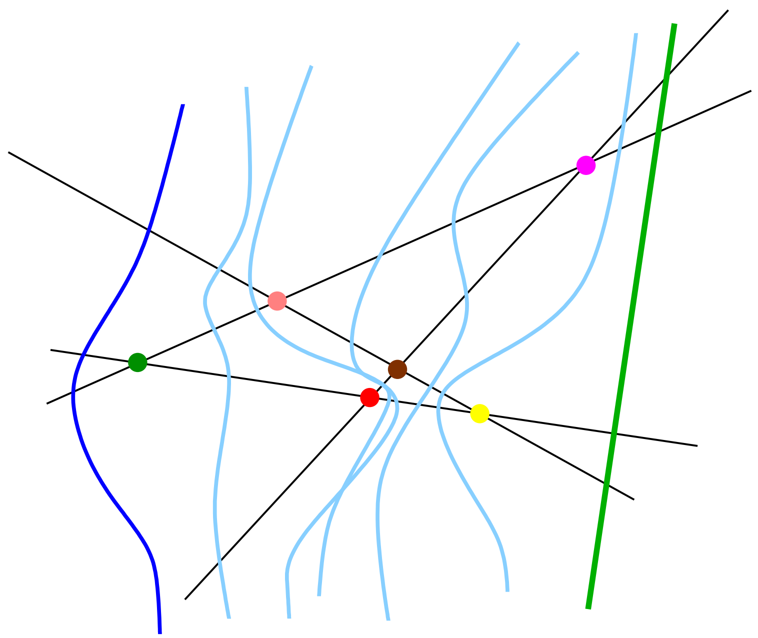

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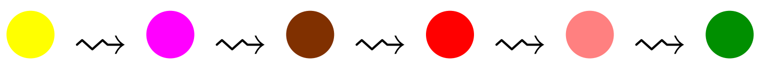

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Theorem. [D., Settepanella '07] *Combinatorial polar orderings that start with the (combinatorially) same flag induce the same discrete Morse function.*

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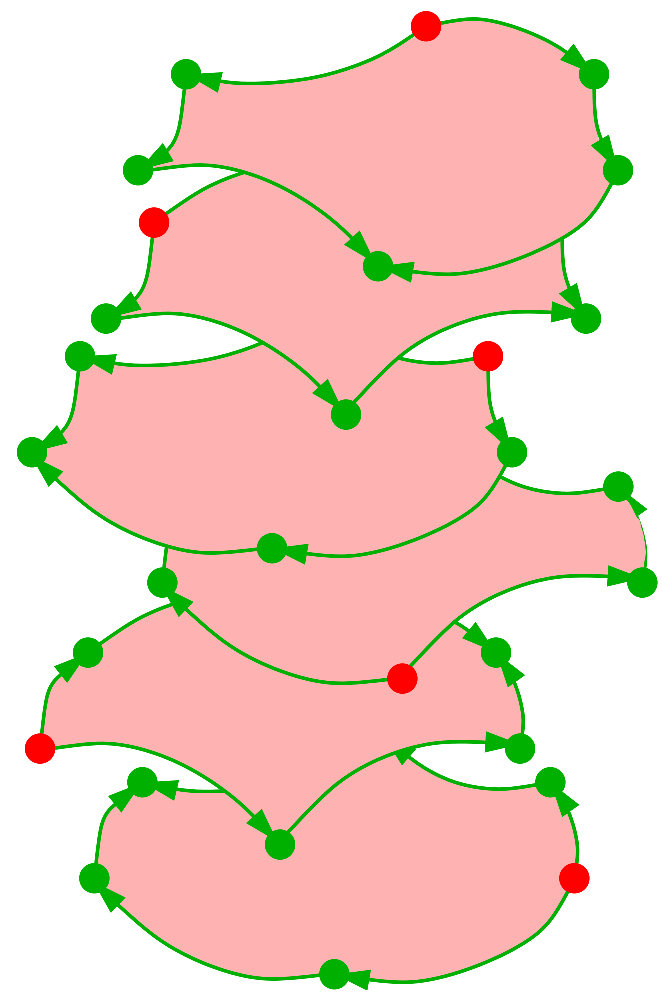
Let $\mathcal{A}$ from now be a central arrangement. Then $Sal(\mathcal{A})$ is the union of combinatorially isomorphic maximal cells, one for every chamber.



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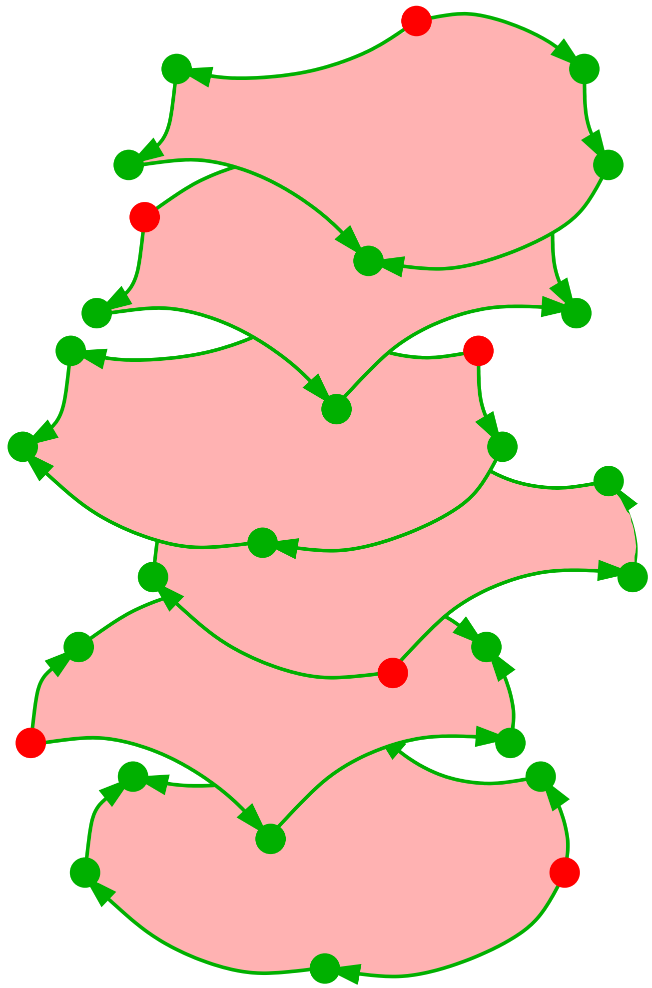

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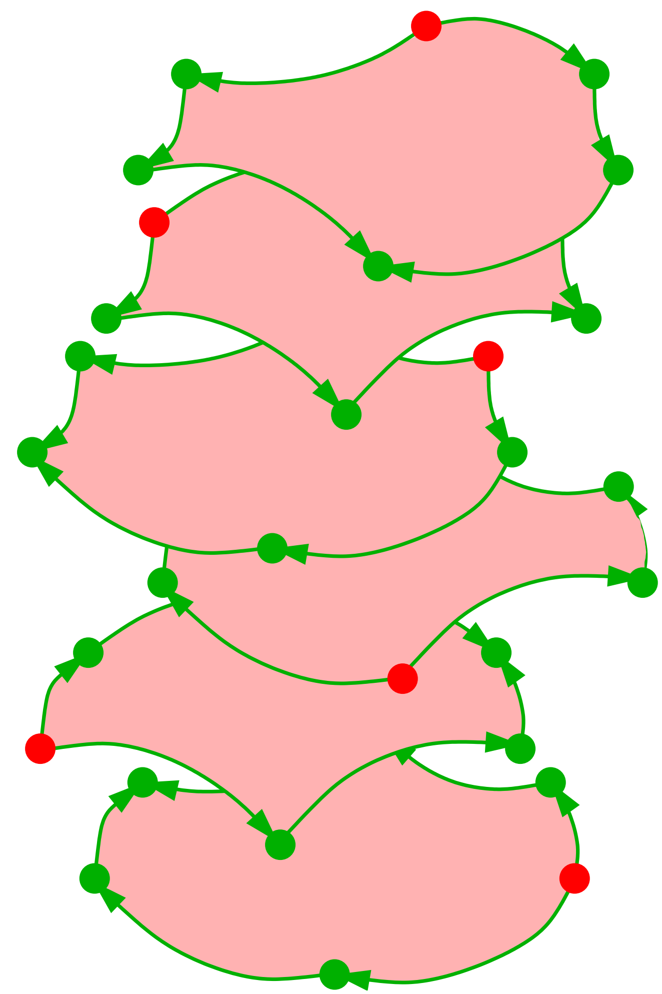

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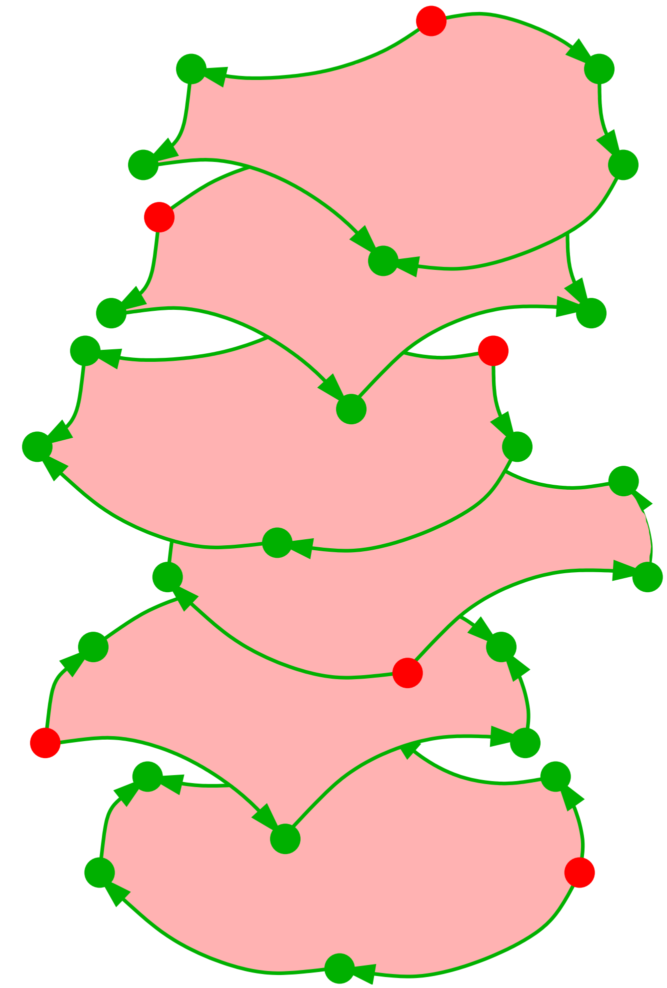
Question:

What is $\mathcal{S}_{C_1} \cap \mathcal{S}_{C_2}$ for $C_1 \neq C_2$?



Order on chambers

Definition. *Let C_1, C_2 be chambers of $\mathcal{A}$.*
$S(C_1, C_2) \subset \mathcal{A}$: *the set of hyperplanes separating C_1 from C_2.*
Fix a chamber B. The partial order

$$C_1 \leq_B C_2 \Leftrightarrow S(B, C_1) \subseteq S(B, C_2)$$

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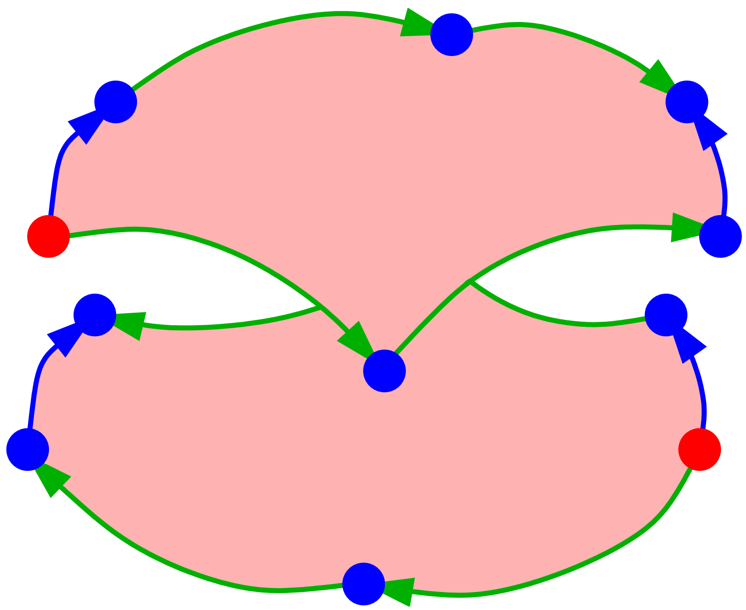

The set of shared faces is then

$$\mathcal{S}_{C_2} \cap \mathcal{S}_{C_1} =$$
$$\{[F, C] \in \mathcal{S}_{C_1} \mid \mathrm{supp}(F) \cap S(C_1, C_2) = \emptyset\}$$

The key theorem

Let $\dashv$ be any linear extension of $\mathcal{P}_B(\mathcal{A})$.

Let us consider

$$\mathcal{S}_C \setminus \bigcup_{K \dashv C} S_K = \{[F, C] \in \mathcal{S}_C \mid \mathsf{supp}(F) \cap S(C, K) \neq \emptyset \, \forall K \dashv C\}$$

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Theorem. [D.'07] *For all C there is a flat $X_C \in \mathcal{L}$ such that*

$$\mathcal{S}_C \setminus \bigcup_{K \dashv C} \mathcal{S}_K \simeq \mathcal{F}(\mathcal{A}^{X_C})$$

Discrete Morse functions

$\mathcal{F}(\mathcal{A})$ is the (augmented) face poset of the polytope polar to the arrangement. Thus, it is the face poset of a shellable complex.

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Let

$$\mathfrak{M} := \bigcup_{C \in \mathcal{C}(\mathcal{A})} \mathfrak{M}_C$$

Corollary. $\mathfrak{M}$ is an acyclic matching of the poset of cells of $Sal(\mathcal{A})$.

No broken circuits

Recall $\mathcal{A} := \{H_1, \dots, H_n\}$ and the normal vectors α_i. A set $U \subset \mathcal{A}$ is called *dependent / independent* if the corresponding set of vectors is.

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It is then natural to set formally $\mathrm{nbc}_0(\mathcal{A}) := \{\emptyset\}$, and let

$$\mathrm{nbc}(\mathcal{A}) := \bigcup_{i=0,\ldots n} \mathrm{nbc}_i(\mathcal{A})$$

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Theorem. [Orlik, Solomon '80] *For all* $i = 0, \ldots, n$, *the set* $\mathsf{nbc}_i(\mathcal{A})$ *indexes a basis of* $H_i(\mathcal{M}(\mathcal{A}), \mathbb{Z})$.

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A bijection

In 2002, Jewell and Orlik gave a bijective proof of Zaslavsky's result.

Lemma. [Jewell & Orlik, '02] *For every choice of a base chamber* B *and an ordering of* $\mathcal{A}$ *there is a bijection* $\eta : \mathcal{C}(\mathcal{A}) \to \mathsf{nbc}(\mathcal{A})$.

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Definition. *An ordering of $\mathcal{A}$ satisfies the cut property with respect to a chamber B if it is obtained from the sequence in which a maximal chain in $\mathcal{P}_B(\mathcal{A})$ traverses the hyperplanes.*

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$$X_C = \bigcap \eta(C).$$

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(and, so far, the end of the story)

Let us write $[F_C, C]$ for the critical cell of $\mathfrak{M}_C$, and recall that $|F_C| = X_C$.

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We can then summarize.

Proposition. [D. '07] *Let $\mathcal{A}$ be a finite real arrangement of linear hyperplanes. Every choice of a base chamber B and of an ordering $\mathcal{A} = \{H_1, \ldots, H_n\}$ satisfying the cut property with respect to B defines a discrete Morse function on $Sal(\mathcal{A})$. The faces of the associated Morse CW-complex are*

$$\Big[\quad \underbrace{\overline{C} \cap \bigcap \eta(C),}_{\text{``}F_C\text{''}} \quad C \quad \Big]$$

In particular, the obtained complex is minimal.