# HYPERPLANE ARRANGEMENTS "CUT TO THE BONE".

Emanuele Delucchi
MSRI / SUNY Binghamton

Algebra and combinatorics seminar
UC Davis
November 19. 2007

# What is this?

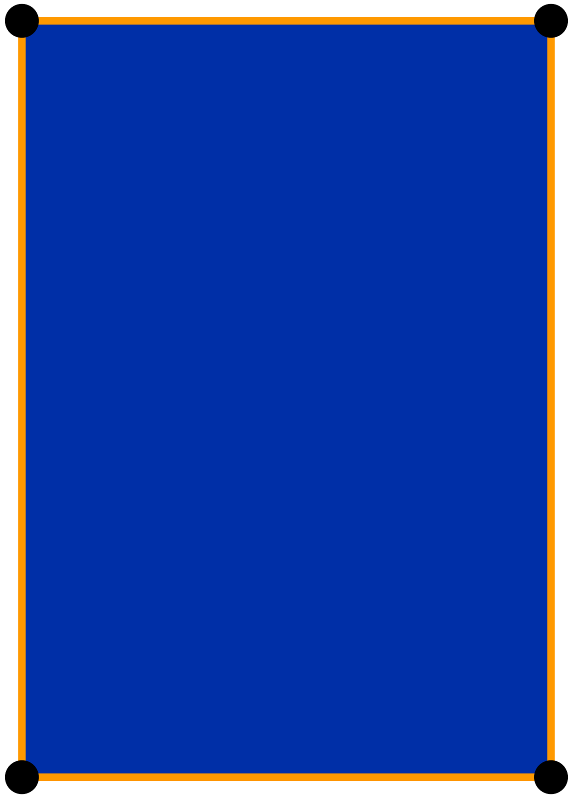
# What is this?

(1) Klein's "blue".

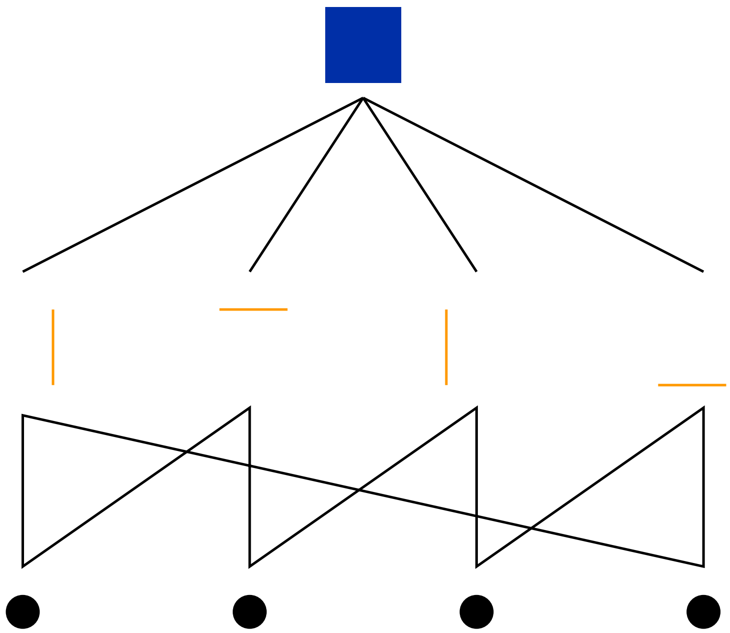
(2) A rectangle.

(3) A contractible regular CW-complex.

# CW-COMPLEXES...



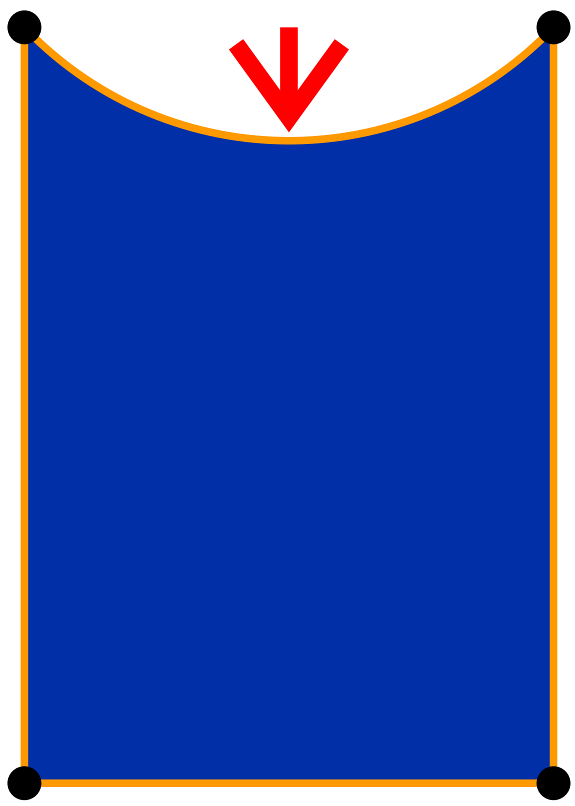
...have got cells:



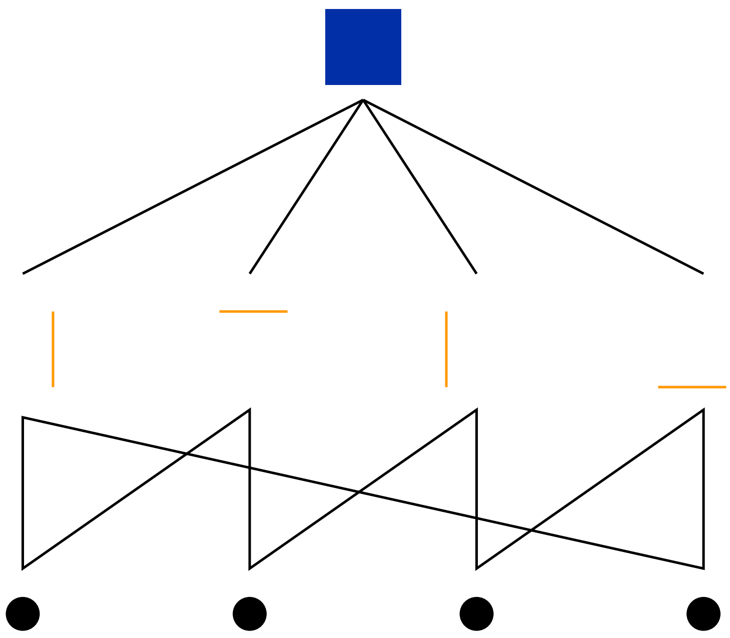
Regular: the boundary of every cell is attached by a homeomorphism.

Contractible: is homotopy equivalent to a point $\bullet$

# Elementary collapses...

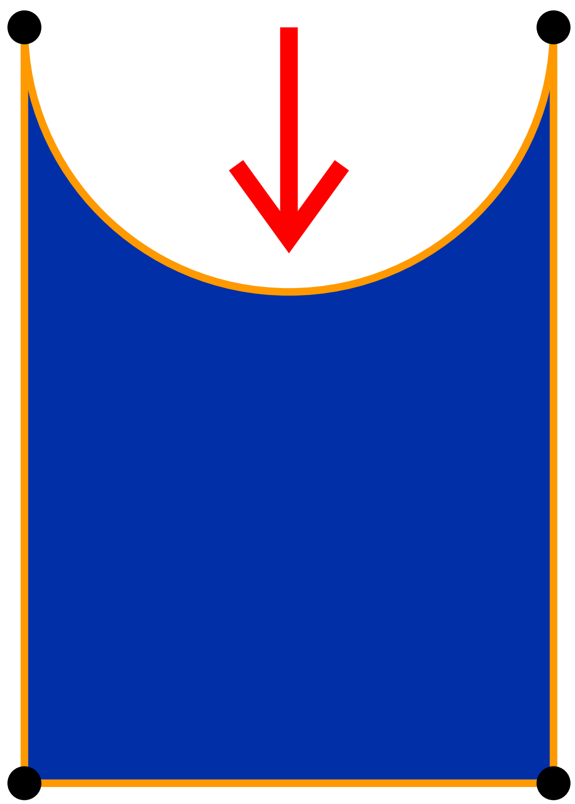


Cells:

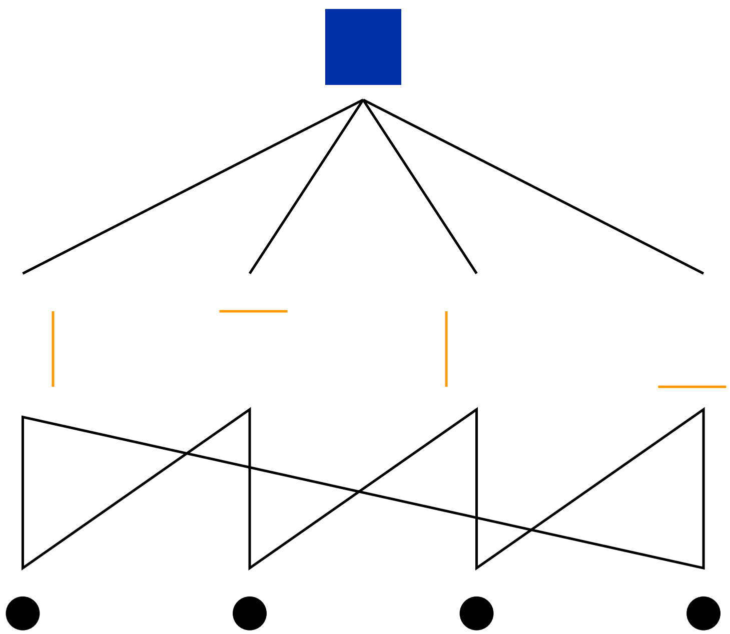


... are homotopy equivalences.

# Elementary collapses

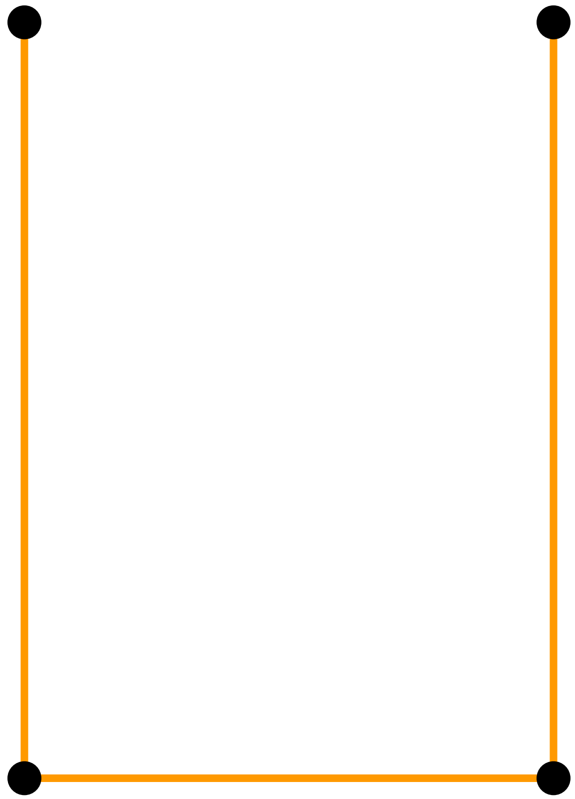


Cells:

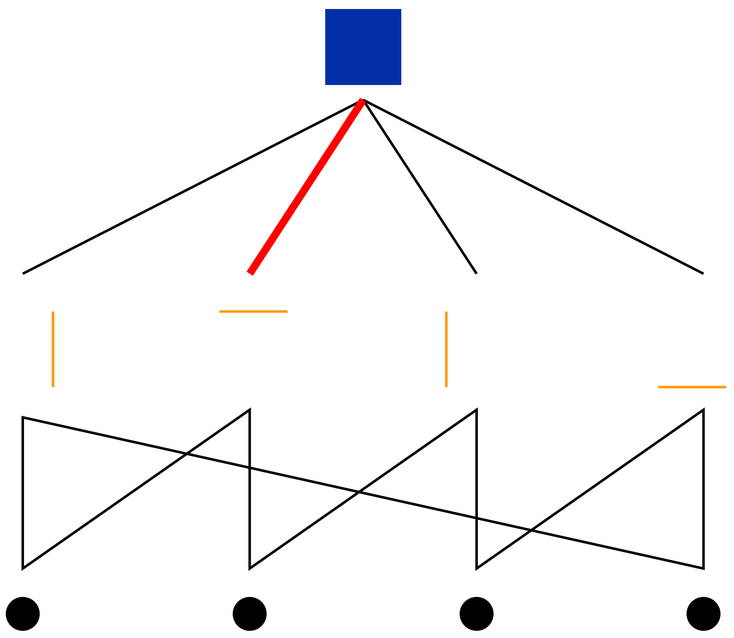


… are homotopy equivalences

# Elementary collapses

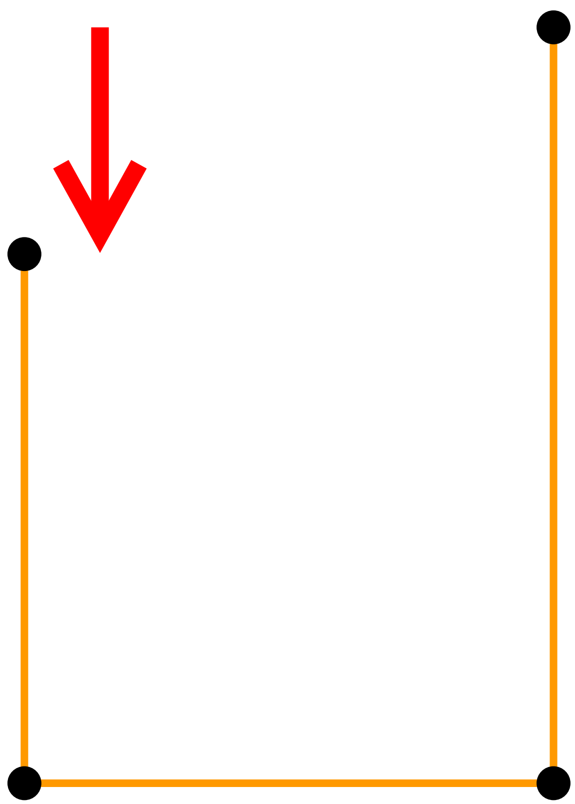


Cells:

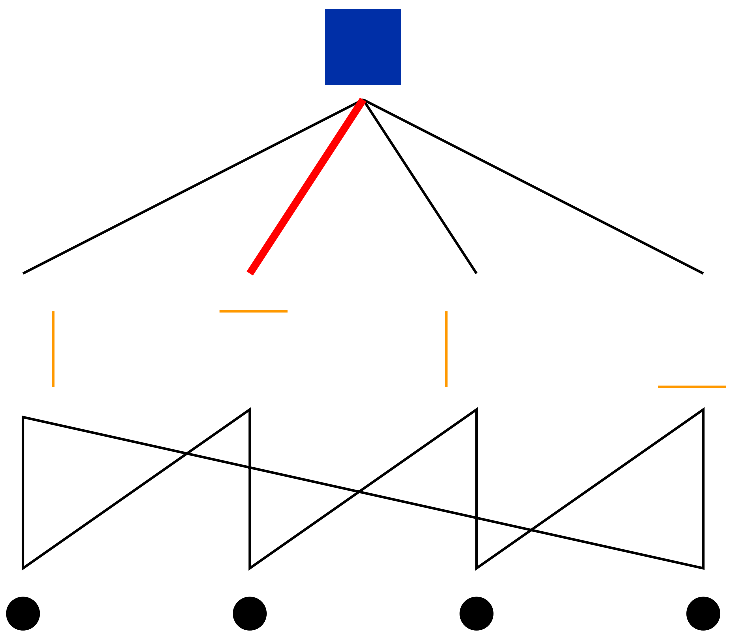


… are homotopy equivalences

# Elementary collapses



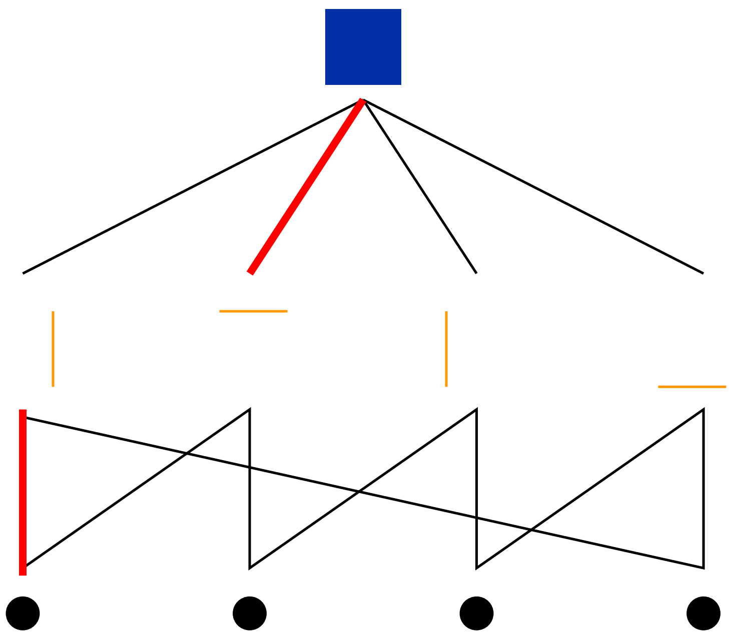
Cells:



… are homotopy equivalences

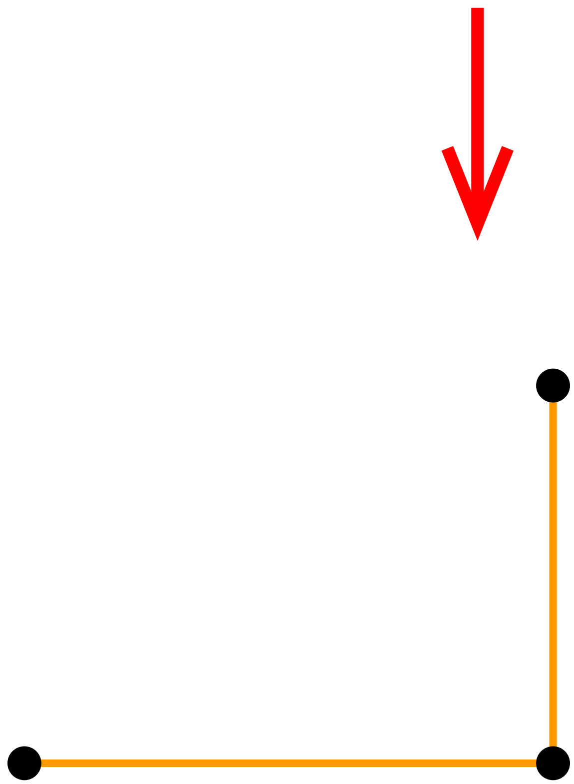
# Elementary collapses

Cells:

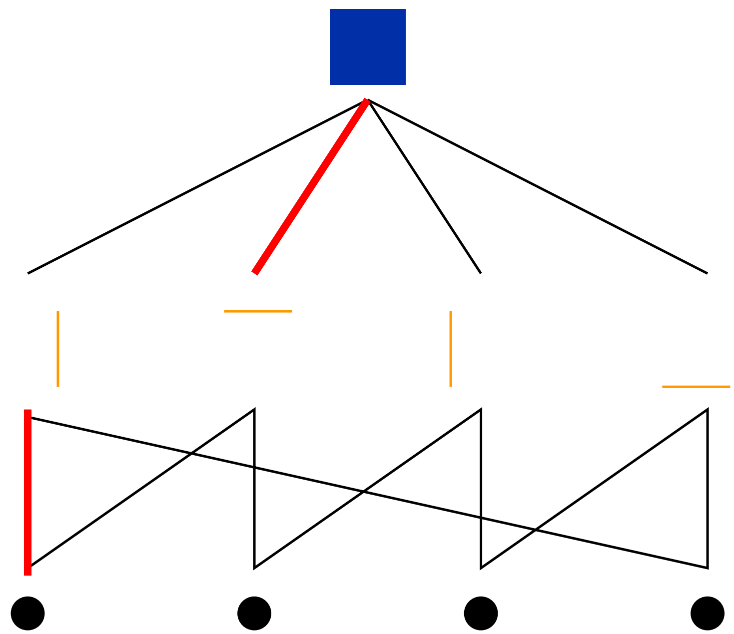


… are homotopy equivalences

# Elementary collapses



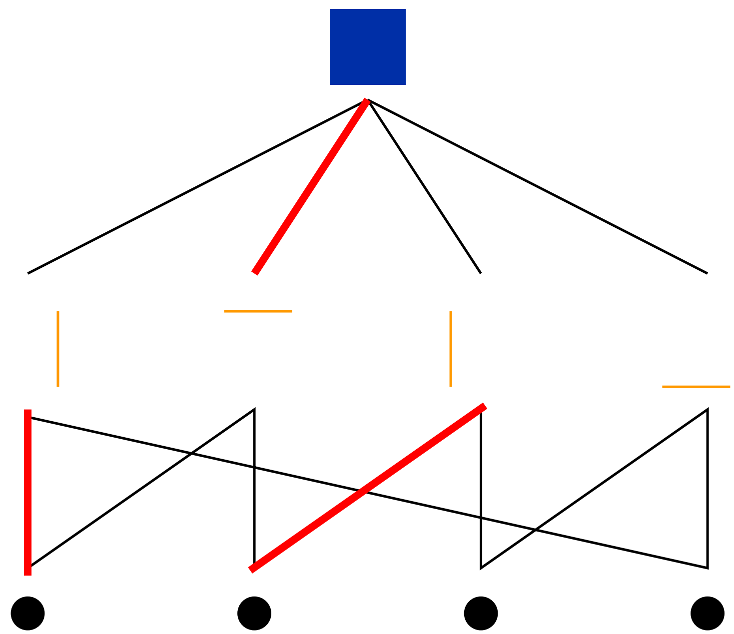
Cells:



... are homotopy equivalences

# Elementary collapses

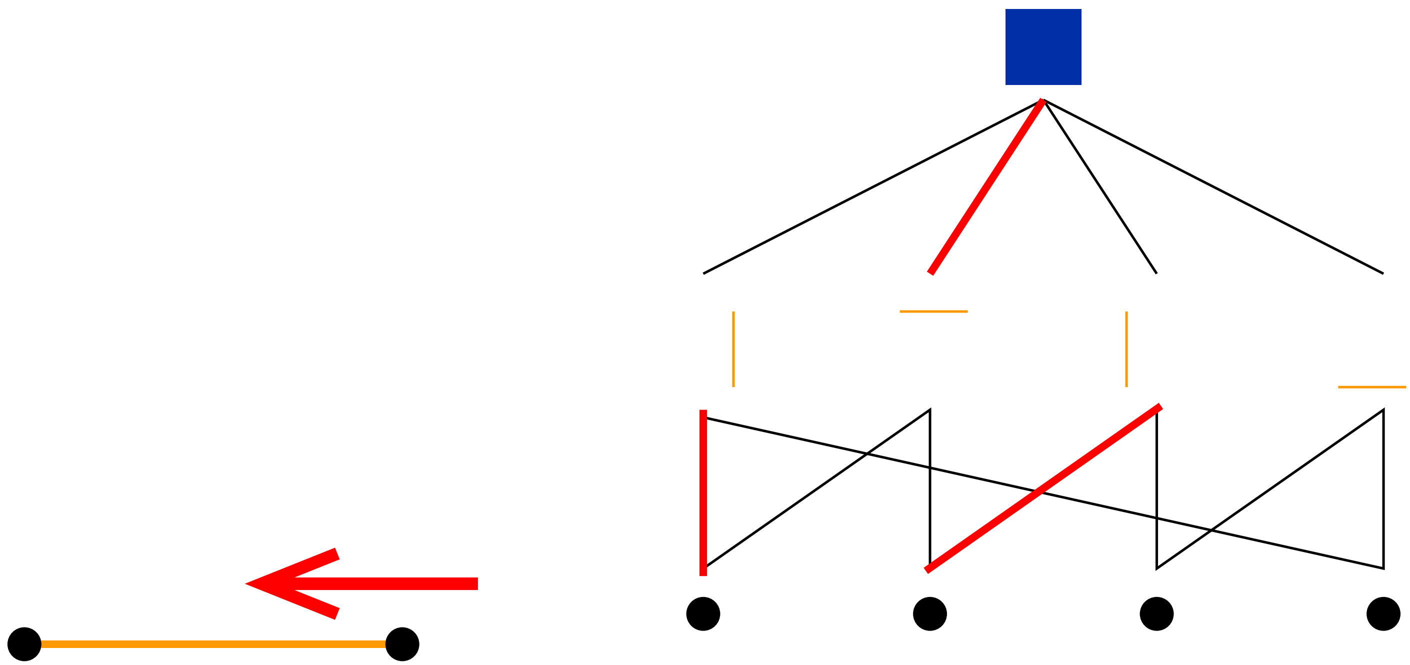
Cells:



… are homotopy equivalences

# Elementary collapses

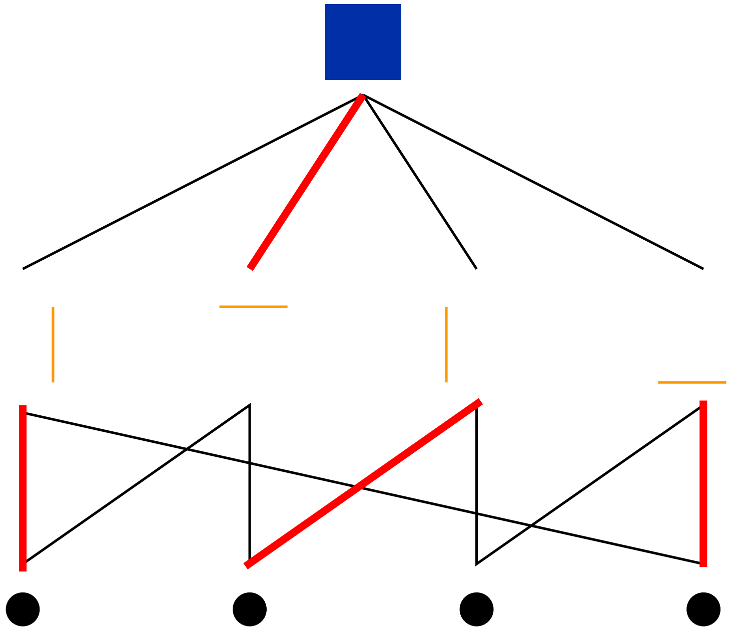
Cells:



... are homotopy equivalences

# Elementary collapses

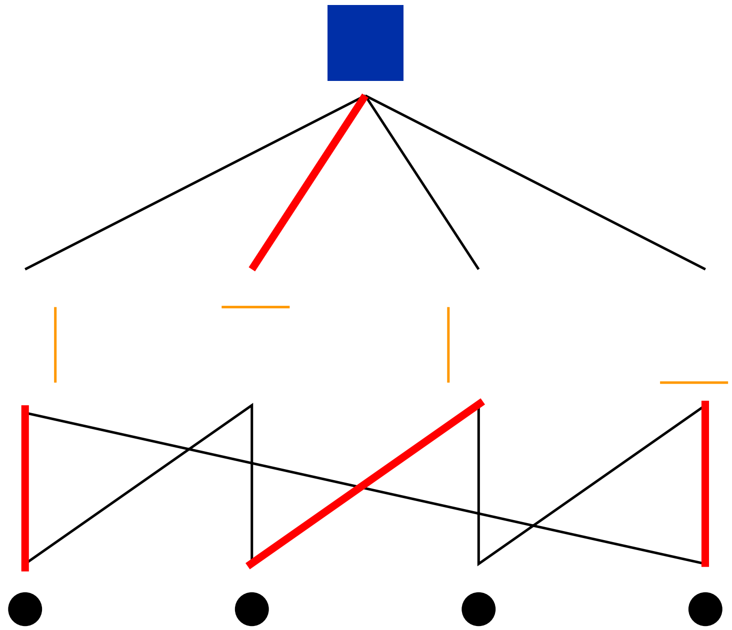
Cells:



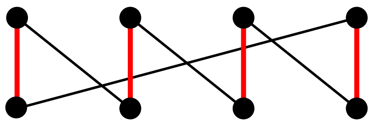
... are homotopy equivalences

# Acyclic matchings

The sequence of collapses is encoded in a matching of the Hasse diagram of the poset of cells.



**Question:** Does any matching encode such a sequence?

**Answer:** No. For example, "cycles" like  cannot appear.

# Discrete Morse theory

$K$: a regular CW-complex,

$\mathcal{K}$: the poset of cells of $K$,

$\mathfrak{M}$: an acyclic matching of the Hasse diagram of $\mathcal{K}$,

$B_i$: set of unmatched cells of dimension $i$.

**Theorem. [Foreman '98]** *$K$ is homotopy equivalent to a CW-complex $K'$ with $|B_i|$ cells of dimension $i$.*
*Moreover, the incidence coefficient $[\sigma : \tau]_{K'}$ of $\sigma \in B_{i+1}$ and $\tau \in B_i$ can be computed as*

$$[\sigma : \tau]_{K'} = \sum_{\sigma=\sigma_1>\tau_1<\sigma_2>\cdots<\sigma_n>\tau_n=\tau} \prod_{i=1}^{n-1} [\sigma_i : \tau_i]_K [\sigma_{i+1} : \tau_i]_K$$

*where the sum is over all alternating paths*

$$\sigma_1 - \tau_1 - \sigma_2 - \tau_2 - \ldots \ \tau_{n-1} - \sigma_n - \tau_n = \tau.$$

# Some useful facts

Recall that a linear estension of a partial order $<$ is a total order $\sqsubset$ such that $x < y$ imples $x \sqsubset y$.

**Theorem. [....]** *A matching $\mathfrak{M}$ of a poset $\mathcal{K}$ is acyclic if and only if there is a linear extension $\sqsubset$ of $\mathcal{K}$ where every matched pair consists of consecutive elements.*

For instance, if the CW-comples $K$ happens to be shellable :

**Theorem. [Chari, Babson & Hersh, D.'07]** *Any Recursive Coatom Ordering of the poset $\mathcal{K}$ defines a linear extension and a corresponding acyclic matching of $\mathcal{K}$ such that the only nonmatched cells are the homology cells of the given shelling.*

In particular, recall the classical shellings of oriented matroids.

**Corollary. [D. '07]** *For every chlice of a tope $T$ of an oriented matroid there is a linear extension and an acyclic matching of the poset of faces such that $T$ is the only unmatched element.*

# Arrangements of hyperplanes

An arrangement of real hyperplanes is a set

$$\mathcal{A} := \{H_1, \ldots, H_n\}$$

of linear hyperplanes in $\mathbb{R}^d$.

$\mathcal{A}$ defines a stratification of $\mathbb{R}^d$ into open polyhedral strata, called faces .

$\mathcal{F}(\mathcal{A})$ is the set of all strata, ordered by reverse inclusion of the topological closures: $F_1 > F_2 \Leftrightarrow \overline{F_1} \subseteq \overline{F_2}$ .

$\mathrm{supp}(F) :=\{H \in \mathcal{A} \mid H \supset F\}$ be the set of all hyperplanes containing $F$.

$\mathcal{L}(\mathcal{A})$ is the lattice of all intersections $X = \bigcap_{H \in \mathcal{K}} H$ for $\mathcal{K} \subset \mathcal{A}$, ordered by reverse inclusion: $X > Y \Leftrightarrow X \subset Y$ .

For a flat $X \in \mathcal{L}(\mathcal{A})$:

$$\mathrm{supp}(X) := \{H \in \mathcal{A} \mid X \subset H\},\ \mathcal{A}^X := \left\{X \cap H \mid H \in \mathcal{A} \setminus \mathrm{supp}(X)\right\}.$$

# Order on chambers

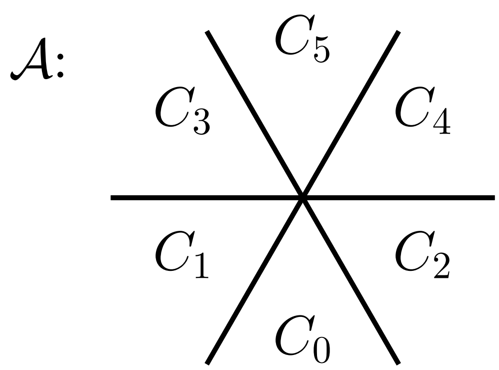
$\mathcal{C}(\mathcal{A})$ is the set of connected components of $\mathbb{R}^d \setminus \mathcal{A}$, called chambers of $\mathcal{A}$.

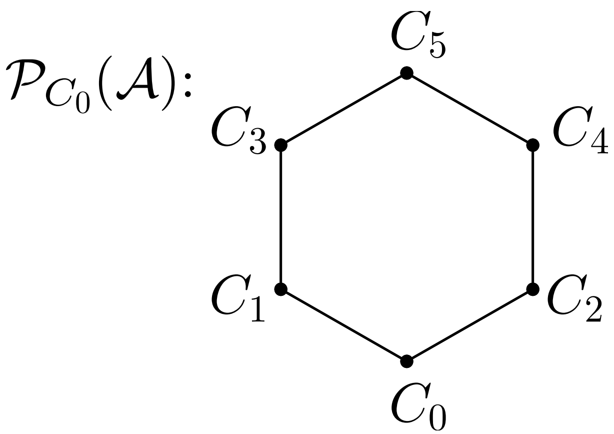
**Definition.** *Let $C_1$, $C_2$ be chambers of $\mathcal{A}$.*
$S(C_1, C_2) \subset \mathcal{A}$*: the set of hyperplanes separating $C_1$ from $C_2$.*
*Fix a chamber $C_0$. The partial order*

$$C_1 \leq_{C_0} C_2 \Leftrightarrow S(C_0, C_1) \subseteq S(C_0, C_2)$$

*defines the poset of regions $\mathcal{P}_{C_0}(\mathcal{A})$ based at $C_0$.*

The Hasse diagram of this poset 'is' a plane projection of the 1-skeleton of the polar polytope.



# Complexified arrangements

Let $\mathcal{A}$ be as before, and for every $i = 1, \dots n$ choose $\alpha_i \in \mathbb{R}^d$ normal to $H_i$.

The complexification of $\mathcal{A}$ is the set

$$\mathcal{A}_{\mathbb{C}} := \{H_1^{\mathbb{C}}, \dots, H_n^{\mathbb{C}}\} \text{ where } H_i^{\mathbb{C}} := \{z \in \mathbb{C}^d \mid \langle z \mid \alpha_i \rangle = 0\} .$$

The topological space we are interested in is

$$\mathcal{M}(\mathcal{A}) := \mathbb{C}^d \setminus \bigcup_{i=1}^{n} H_i^{\mathbb{C}}.$$

**Theorem. [Randell '02, Dimca & Papadima '03]** *$\mathcal{M}(\mathcal{A})$ is a minimal space. I.e., it has the homotopy type of a CW-complex with as many ($k$-)cells as there are generators in the ($k$-)th homology.*


BREAKING NEWS
MINIMAL MODELS
FOR COMPLEXIFIED ARRANGEMENTS
LIVE
12:51 ET
MSNBC
NAS 22.40
OM: STAY CURRENT ON POLITICAL NE

# Minimal models

Two constructive approaches to minimality, both via discrete Morse theory:

1) **arXiv:0705.2874** M. Salvetti, S. Settepanella; *Combinatorial Morse theory and minimality of hyperplane arrangements.*

2) **arXiv:0705.3107** E. D.; *Shelling-type orderings of regular CW-complexes and acyclic matchings of the Salvetti complex.*

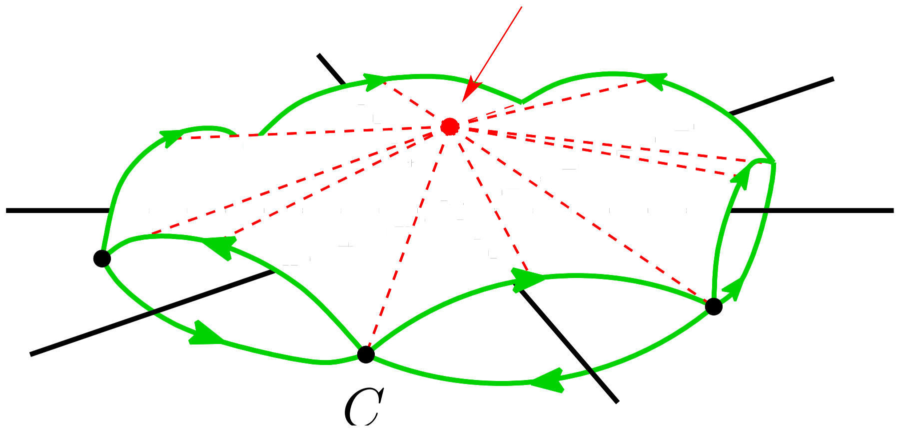
# THE SALVETTI COMPLEX $Sal(\mathcal{A})$

[Salvetti '87]

Given a complexified arrangement $\mathcal{A}$, consider the oriented graph $\Gamma(\mathcal{A})$ with:

$V(\Gamma(\mathcal{A}))$ : chambers of $\mathcal{A}$

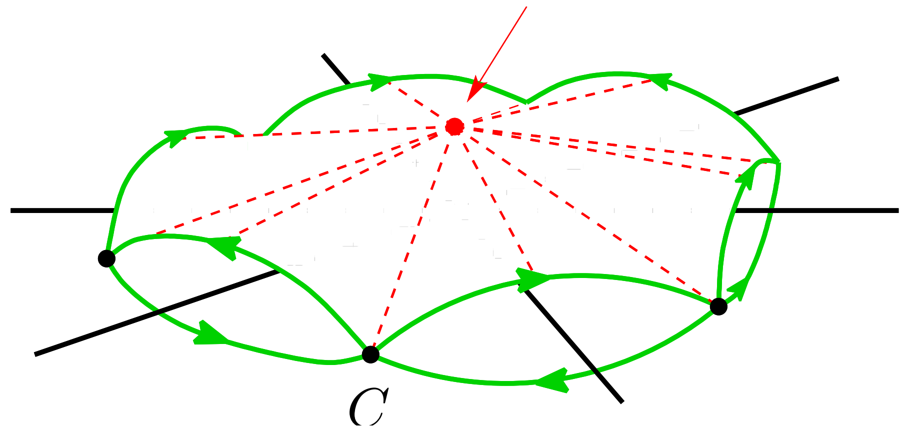
$E(\Gamma(\mathcal{A}))$ contains $(C_1, C_2)$ iff $C_1$ is adjacent to $C_2$.



# The Salvetti complex $Sal(\mathcal{A})$

[Salvetti '87]

Given a complexified arrangement $\mathcal{A}$, consider the oriented graph $\Gamma(\mathcal{A})$ with:




$V(\Gamma(\mathcal{A}))$ : chambers of $\mathcal{A}$

$E(\Gamma(\mathcal{A}))$ contains $(C_1, C_2)$ iff $C_1$ is adjacent to $C_2$.

- $Sal(\mathcal{A})^1 = \Gamma(\mathcal{A})$

- A $k$-cell $[F, C]$ for every codimension-$k$ face $F$ and every chamber $C < F$.

- $[F, C]^1$: all oriented *(positive)* paths starting at $C$ and passing exactly once *(minimal)* through all $H \supset F$.

**Theorem. [Salvetti '87]** $Sal(\mathcal{A}) \simeq \mathcal{M}(\mathcal{A})$.

# POSET OF CELLS OF $Sal(\mathcal{A})$

$Sal(\mathcal{A})$ is a regular CW-complex. The cells correspond to pairs $\langle F, C \rangle$ where $F \in \mathcal{F}, C \in \mathcal{C}, F > C$.

The boundary relations are such that

$$\langle F_2, C_2 \rangle \subset \partial \langle F_1, C_1 \rangle \text{ if and only if}$$

$$F_1 < F_2 \text{ and } \text{supp}(F_1) \cap S(C_1, C_2) = \emptyset \ .$$

Let us call $\mathcal{S}(\mathcal{A})$ the poset of cells of $Sal(\mathcal{A})$, ordered by inclusion. Then

**Theorem. [Salvetti '87]** $\Delta(\mathcal{S}(\mathcal{A})) \simeq \mathcal{M}(\mathcal{A})$.

$\mathcal{S}(\mathcal{A})$ contains many copies of $\mathcal{F}(\mathcal{A})$. There is an isomorphism of posets

$$\mathcal{F}(\mathcal{A}) \simeq \mathcal{S}_C := \{\langle F, K \rangle \mid \langle F, K \rangle < \langle P, C \rangle\}$$

for every $C \in \mathcal{C}$ (here and in the following $P := \bigcap \mathcal{A} = \min \mathcal{F}$).

## How do the $\mathcal{S}_C$ interact with each other?

By definition, given $C_1, C_2 \in \mathcal{C}(\mathcal{A})$:

$$\mathcal{S}_{C_1} \cap \mathcal{S}_{C_2} = \{\langle F, K\rangle \in \mathcal{S}_{C_2} \mid S(C_1, C_2) \cap \mathsf{supp}(F) = \emptyset\}$$

Choose $B \in \mathcal{C}(\mathcal{A})$ and consider a linear extension $\dashv$ of $\mathcal{P}_B(\mathcal{A})$ . Then:

$$\mathcal{S}_C \setminus \bigcup_{K \dashv C} S_K = \left\{\langle F, C'\rangle \in \mathcal{S}_C \,\middle|\, \mathsf{supp}(F) \cap S(C,K) \neq \emptyset \text{ for all } K \dashv C\right\}.$$

Let $J(C) := \{X \in \mathcal{L} \mid \mathsf{supp}(X) \cap S(C,K) \neq \emptyset \text{ for all } K \dashv C\}$

**Lemma. [D. '07]** *For all $C \in \mathcal{P}_B(\mathcal{A})$, $J(C)$ is an upper interval in $\mathcal{L}(\mathcal{A})$.*

With $X_C := \min J(C)$, we obtain:

**Theorem. [D. '07]**

$$\mathcal{S}_C \setminus \bigcup_{K \dashv C} S_K \simeq \mathcal{F}(\mathcal{A}^{X_C}).$$

# THE DISCRETE MORSE VECTOR FIELD

We have thus a decomposition

$$\mathcal{S}(\mathcal{A}) = \coprod_{C \in \mathcal{P}_B(\mathcal{A})} \mathcal{F}(\mathcal{A}^{X_C})$$

and we know how to construct

a total ordering $\sqsubset^C$ and an acyclic matching $\mathfrak{M}_C$ of every $\mathcal{F}(\mathcal{A}^{X_C})$.

Let now

$$\mathfrak{M} := \biguplus_{C \in \mathcal{P}_B(\mathcal{A})} \mathfrak{M}_C.$$

**Corollary. [D. '07]** $\mathfrak{M}$ *defines a discrete Morse vector field on* $Sal(\mathcal{A})$.

**Proof:** Acyclicity is checked by concatenating the $\sqsubset^C$'s.

# No broken circuits

Recall $\mathcal{A} := \{H_1, \ldots, H_n\}$ and the normal vectors $\alpha_i$. A set $U \in \mathcal{A}$ is called *dependent / independent* if the corresponding set of vectors is.

- A circuit of $\mathcal{A}$ is any minimally dependent subset $U \in \mathcal{A}$.
- Any set of the form $U \setminus \min U$ for a circuit $U$ is called broken circuit .
- A no-broken-circuit (nbc-) set is any independent set that does not contain any broken circuit.
- $\mathrm{nbc}_i(\mathcal{A})$ is the family of all the nbc-sets of $\mathcal{A}$ with $i$ elements.

**Theorem. [Orlik, Solomon '80]** *For all $i = 1, \ldots, n$, the set $\mathrm{nbc}_i$ indexes a basis of $H_i(\mathcal{M}(\mathcal{A}), \mathbb{Z})$.*

It is then natural to set formally $\mathrm{nbc}_0(\mathcal{A}) := \{\emptyset\}$, and let

$$\mathrm{nbc}(\mathcal{A}) := \bigcup_{i=0,\ldots n} \mathrm{nbc}_i(\mathcal{A})$$

# POINCARÉ POLYNOMIAL

...so that the Poincaré polynomial of $\mathcal{M}(\mathcal{A})$ is

$$\pi_{\mathcal{M}(\mathcal{A})}(t) = \sum_{i=0}^{n} |\mathsf{nbc}_i| t^i$$

**Theorem. [Zaslavsky '75]** *For any real arrangement $\mathcal{A}$:*

$$|\mathcal{C}(\mathcal{A})| = |\pi_{\mathcal{M}(\mathcal{A})}(-1)|.$$

Equivalently, $|\mathcal{C}(\mathcal{A})| = |\mathsf{nbc}(\mathcal{A})|$.

By the *pigeon hole principle*:

**Corollary. [D. '07]** *The Morse complex obtained from our $\mathfrak{M}$ is minimal.*

# A BIJECTION

In 2002, Jewell and Orlik gave a bijective proof of Zaslavsky's result.

**Lemma. [Jewell & Orlik, '02]** *For every choice of a base chamber $B$ and an ordering of $\mathcal{A}$ there is a bijection $\eta : \mathcal{C}(\mathcal{A}) \to \mathsf{nbc}(\mathcal{A})$.*

Recall that the construction of nbc relies upon an (arbitrary) ordering of $\mathcal{A}$.

**Definition.** *An ordering of $\mathcal{A}$ satisfies the cut property with respect to a chamber $B$ if it is obtained from the sequence in which a maximal chain in $\mathcal{P}_B(\mathcal{A})$ traverses the hyperplanes.*

**Lemma. [D. '07]** *Every ordering of $\mathcal{A}$ satisfying the cut property with respect to $B$ defines a ("lexicographic") linear extension $\dashv^{lex}$ of $\mathcal{P}_B(\mathcal{A})$.*

**Theorem. [D. '07]** *For the linear extension $\dashv^{lex}$ we have*

$$X_C = \bigcap \eta(C).$$

# THE FINAL MIX

- $\mathcal{S}(\mathcal{A}) \simeq \biguplus_{C\in\mathcal{C}} \mathcal{F}(\mathcal{A}^{X_C})$,
- For every $C$ an acyclic matching $\mathfrak{M}_C$ of $\mathcal{F}(\mathcal{A}^{X_C})$,
- $X_C = \bigcap \eta(C)$,
- **Lemma. [D. '07]** $\dim(\overline{C} \cap X_C) = |\eta(C)|$.
- $|\mathcal{C}(\mathcal{A})| = \sum_{i=1}^{n} \mathrm{rank}(H^i(\mathcal{M}(\mathcal{A})))$

Then:

**Proposition. [D. '07]** *Let $\mathcal{A}$ be a finite real arrangement of linear hyperplanes. Every choice of a base chamber $B$ and of an ordering $\mathcal{A} = \{H_1, \dots, H_n\}$ satisfying the cut property with respect to $B$ defines a discrete Morse function on $Sal(\mathcal{A})$. The associated Morse $CW$-complex has faces indexed by*

$$\langle \overline{C} \cap \bigcap \eta(C), C \rangle$$

*and is therefore minimal.*