14.* (Brylawski 1973, Greene 1973, Woodall 1974) Let B and B' be bases of a matroid M and $(B_1, B_2, \ldots, B_k)$ be a partition of B. Prove that there is a partition $(B'_1, B'_2, \ldots, B'_k)$ of B' such that $(B - B_i) \cup B'_i$ is a basis for all i in $\{1, 2, ..., k\}$.

1.5 Geometric representations of matroids of small rank

One attractive feature of graphic matroids is that one can determine many properties of such matroids from the pictures of the graphs. In this section we show that all matroids of small rank have a geometric representation that is similarly useful.

We begin our discussion by introducing another class of matroids. A multiset $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$, the members of which are in $V(m, F)$, is *affinely dependent* if $k \geq 1$ and there are elements $a_1, a_2, \ldots, a_k$ of F, not all zero, such that $\sum_{i=1}^{k} a_i \underline{v}_i = \underline{0}$ and $\sum_{i=1}^{k} a_i = 0$. Equivalently, $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ is affinely dependent if the multiset $\{(1, \underline{v}_1), (1, \underline{v}_2), \ldots, (1, \underline{v}_k)\}$ is linearly dependent in $V(m + 1, F)$, where $(1, \underline{v}_i)$ is the $(m + 1)$-tuple of elements of F whose first entry is 1 and whose remaining entries are the entries of $\underline{v}_i$. A multiset of elements from $V(m, F)$ is *affinely independent* if it is not affinely dependent. Clearly an affinely independent multiset must be a set.

1.5.1 Proposition. *Suppose that E is a set that labels a multiset of elements from $V(m, F)$. Let $\mathcal{I}$ be the collection of subsets X of E such that X labels an affinely independent subset of $V(m, F)$. Then $(E, \mathcal{I})$ is a matroid.*

Proof. Suppose that E labels the multiset $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$. Then, from the second definition of affine dependence, we deduce that $(E, \mathcal{I}) = M[A]$ where A is the $(m + 1) \times n$ matrix over F, the ith column of which is $(1, \underline{v}_i)^T$. Alternatively, one can prove this result directly by using the first definition of affine dependence, and we leave this to the reader (Exercise 2). □

The matroid $(E, \mathcal{I})$ in the last proposition is called the *affine matroid* on E, and if M is isomorphic to such a matroid, we say that M is affine *over* F.

1.5.2 Example. Let E be the subset $\{(0,0), (1,0), (2,0), (0,1), (0,2), (1,1)\}$ of $V(2, \mathbb{R})$ and consider the affine matroid M on E. The six elements of E can be represented as points in the Euclidean plane $\mathbb{R}^2$ as in Figure 1.6. It is not difficult to check that the dependent sets of M consist of all subsets of E with four or more elements together with all 3-element subsets of E such that the corresponding three points in Figure 1.6 are collinear. □

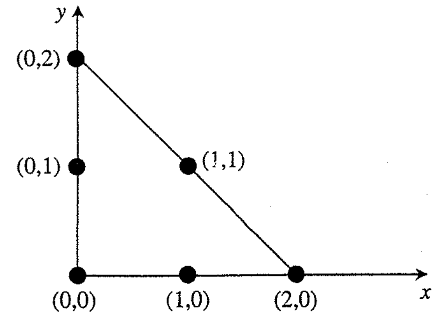

Fig. 1.6. A rank-3 affine matroid.

In general, if M is an affine matroid over $\mathbb{R}$ of rank $m+1$ where $m \leq 3$, then a subset X of $E(M)$ is dependent in M if, in the representation of X by points in $\mathbb{R}^m$, there are two identical points, or three collinear points, or four coplanar points, or five points in space. Hence the flats of M of ranks one, two, and three are represented geometrically by points, lines, and planes, respectively. A typical geometric representation of such an affine matroid is given in the next example.

1.5.3 Example. Consider the affine matroid M on the subset E of $V(3, \mathbb{R})$ where $E = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1)\}$. M has the representation shown in Figure 1.7. From that diagram, we see that the only dependent subsets of E with fewer than five elements are the three planes $\{(0,0,0), (1,0,0), (0,1,0), (1,1,0)\}$, $\{(0,0,0), (0,1,0), (0,0,1), (0,1,1)\}$, and $\{(1,0,0), (1,1,0), (0,0,1), (0,1,1)\}$. □

We now have a geometric way to represent real affine matroids of rank at most four. Next we show how to extend the use of this type of diagram to represent arbitrary matroids of rank at most four.

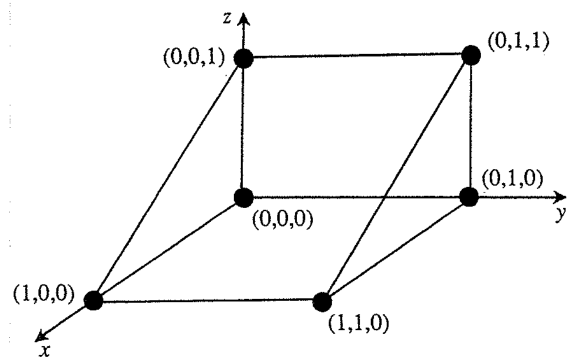

Fig. 1.7. A rank-4 affine matroid.

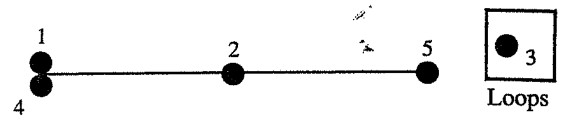

Fig. 1.8. A geometric representation of a rank-2 matroid.

1.5.4 **Example.** The matroid M in Example 1.1.2 can be represented by the diagram in Figure 1.8. In such a diagram, we represent a 2-element circuit by two touching points, and a 3-element circuit by a line through the corresponding points. Loops, which cannot occur in an affine matroid, are represented in an inset as shown. □

In general, such diagrams are governed by the following rules. All loops are marked in a single inset. Parallel elements are represented by touching points, or sometimes by a single point labelled by all the elements in the parallel class. Corresponding to each element that is not a loop and is not in a non-trivial parallel class, there is a distinct point in the diagram which touches no other points. If three elements form a circuit, the corresponding points are collinear. Likewise, if four elements form a circuit, the corresponding points are coplanar. In such a diagram, the lines need not be straight and the planes may be twisted. Moreover, sometimes, to simplify the diagram, certain lines and planes will be listed rather than drawn. At other times, certain lines with fewer than three points on them will be marked as part of the indication of a plane, or as construction lines. We call such a diagram a *geometric representation* for the matroid. The reader is warned that such a representation is not to be confused with the diagram of a graph. Where ambiguity could arise in what follows, we shall always indicate how a particular diagram is to be interpreted.

One needs to be careful not to assume that an arbitrary diagram involving points, lines, and planes is actually a geometric representation for some matroid.

1.5.5 **Example.** 'The Escher matroid' (Brylawski and Kelly 1980). Consider the diagram shown in Figure 1.9, the dependent lines being $\{1,2,3\}$

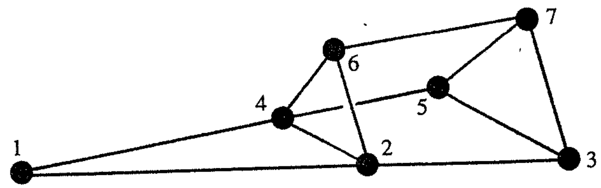

Fig. 1.9. The Escher matroid.

and $\{1,4,5\}$, and the dependent planes $\{1,2,3,4,5\}, \{1,2,3,6,7\}$, and $\{1,4,5,6,7\}$. With the rules that govern diagrams being as specified above, this diagram does not represent a matroid on $E = \{1,2,\dots,7\}$. To see this, assume the contrary and let $X = \{1,2,3,6,7\}$ and $Y = \{1,4,5,6,7\}$. Then $r(X) = 3 = r(Y)$ and $r(X \cup Y) = 4$. Thus, by (R3), $r(\{1,6,7\}) = r(X \cap Y) \leq 2$. But 1, 6, and 7 are distinct non-collinear points, so $r(\{1,6,7\}) = 3$; a contradiction. If we make 1, 6, and 7 collinear as in Figure 1.10, the resulting diagram does represent a rank-4 matroid. We leave it to the reader to check this. □

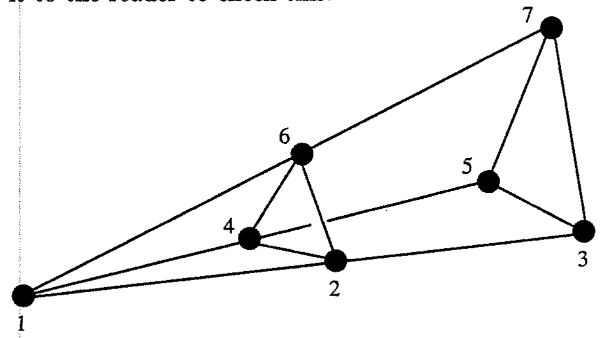

Fig. 1.10. The Escher matroid should have an extra line.

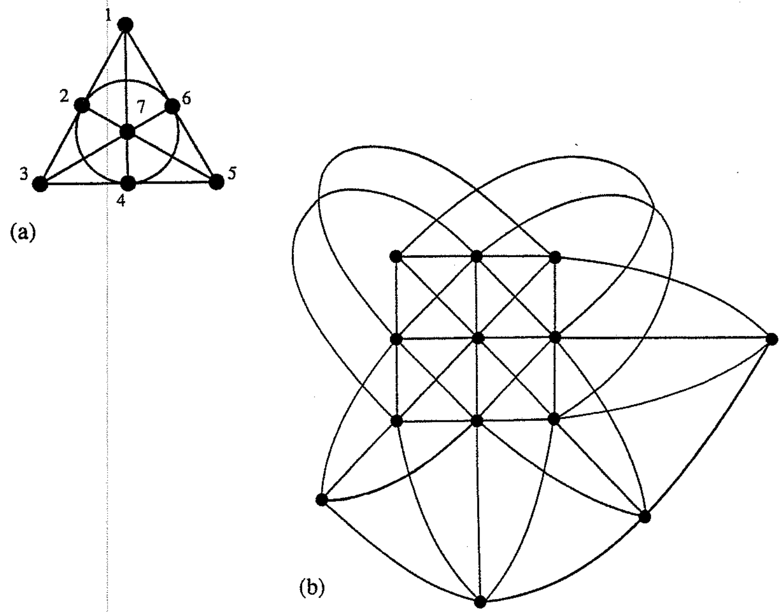

Fig. 1.11. (a) $PG(2,2)$ and (b) $PG(2,3)$.

1.5.6 **Example.** The pictures shown in Figure 1.11 are representations of the 7-point and 13-point projective planes, $PG(2,2)$ and $PG(2,3)$. Interpreting these pictures as diagrams subject to the above rules, it is not difficult to check that each represents a rank-3 matroid. The 7-point projective plane is called the *Fano plane*. The corresponding matroid, the *Fano matroid*, will be denoted by F_7 or $PG(2,2)$. This matroid is of fundamental importance and will occur frequently throughout this book. Indeed, all projective geometries play an important role in matroid theory; we shall discuss this in detail in Chapter 6.

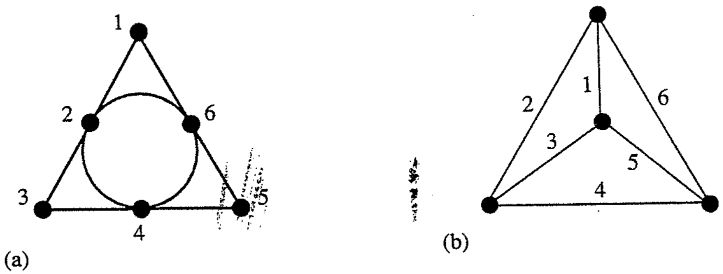

Fig. 1.12. (a) $F_7\backslash 7$. (b) A graph whose cycle matroid is $F_7\backslash 7$.

The diagram in Figure 1.12(a) represents the matroid that is obtained from F_7 by deleting the element 7. Notice that no line has been drawn through 3 and 6, or through 2 and 5, or through 1 and 4, even though $\{3,6\}$, $\{2,5\}$, and $\{1,4\}$ are rank-2 flats of the matroid. Such 2-point lines are usually omitted from these diagrams, as are 3-point planes.

It is not difficult to check that $F_7\backslash 7 \cong M(K_4)$, where the edges of the graph K_4 are labelled as in Figure 1.12(b). Similarly, $F_7\backslash 2$, for which a geometric representation is shown in Figure 1.13, is also isomorphic to $M(K_4)$. Indeed, as the reader can easily check, $F_7\backslash x \cong M(K_4)$ for all x in $E(F_7)$. This is one of the many attractive features of F_7. □

1.5.7 **Example.** Consider the *affine* matroid on the full vector space $V(3,2)$. We denote this matroid by $AG(3,2)$. It has eight elements, corresponding to the eight points in Figure 1.14(a). In addition, it has fourteen 4-point planes, not all of which are marked in Figure 1.14(a). These planes consist of the six faces of the cube, the six diagonal planes such as $\{(0,0,0),(1,0,0),(1,1,1),(0,1,1)\}$, and the two twisted planes, $\{(0,0,0),(1,0,1),(1,1,0),(0,1,1)\}$ and its complement. Note that if Figure 1.14(a) is viewed as an affine matroid over $\mathbb{R}$ instead of over $GF(2)$, then we get twelve rather than fourteen planes, the two twisted planes disappearing.

An alternative representation for $AG(3,2)$ can be obtained from the 11-point matroid shown in Figure 1.14(b). This 11-point matroid is obtained by 'sticking together' two copies of F_7 along a line. The restriction of this matroid to the set $\{1,2,\ldots,8\}$ is isomorphic to the binary affine cube in Figure 1.14(a). We leave it to the reader to check the details of this. To see the fourteen 4-point planes in the second representation for $AG(3,2)$, we first note that $\{1,2,3,4\}$ and $\{5,6,7,8\}$ are two of the fourteen. The other twelve break into three groups of four according to whether the corresponding planes in the original 11-point matroid contain a, b, or c. For example, the four planes containing a arise by taking the union of two lines containing a, one from each copy of F_7, and neither equal to $\{a,b,c\}$. □

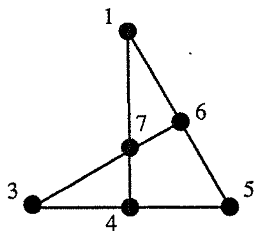

Fig. 1.13. $F_7\backslash 2$.

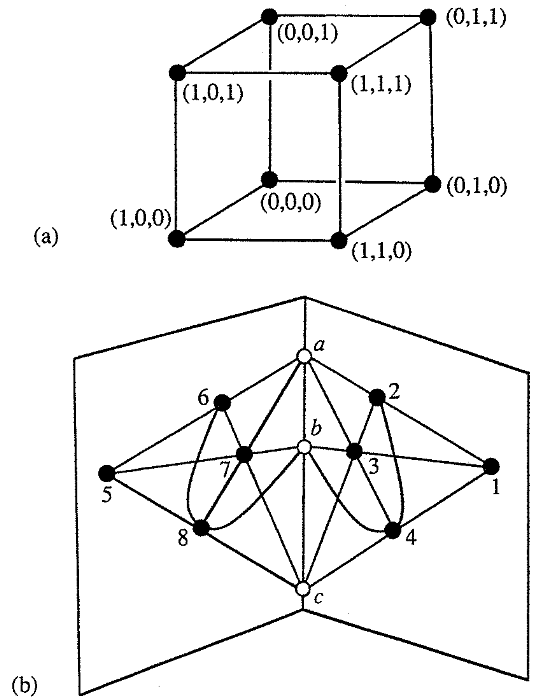

Fig. 1.14. Two geometric representations of $AG(3,2)$.

We have already seen in Example 1.5.5 that a diagram involving points, lines, and planes need not correspond to a matroid. Next we state necessary and sufficient conditions under which such a diagram is actually a geometric representation for a simple matroid of rank at most four in which the rank-1, rank-2, and rank-3 flats correspond to the points, lines, and planes in the diagram. These rules are stated just for simple matroids because we already know how to recognize loops and parallel elements in such a diagram. The rules, all of which are very natural from our current geometric perspective, include the following straightforward non-degeneracy conditions: the sets of points, lines, and planes are disjoint; there are no sets of touching points; every line contains at least two points; any two distinct points lie on a line; every plane contains at least three non-collinear points; and any three distinct non-collinear points lie on a plane. For a diagram having at most one plane, there is only one other condition:

1.5.8 *Any two distinct lines meet in at most one point.*

For a diagram having two or more planes, there are three rules in addition to the non-degeneracy conditions (Mason 1971):

1.5.9 *Any two distinct planes meeting in more than two points do so in a line.*

1.5.10 *Any two distinct lines meeting in a point do so in at most one point and lie on a common plane.*

1.5.11 *Any line not lying on a plane intersects it in at most one point.*

We leave the proofs of these results to the reader (Exercises 3 and 4). Using geometric representations, the reader should be able to check that there are 17 non-isomorphic matroids on a 4-set, and 35 non-isomorphic matroids on a 5-set.

1.5.12 **Example.** The diagram in Figure 1.15 obeys 1.5.8 and is therefore a geometric representation for a matroid N. Comparing Figure 1.15 with the geometric representation for F_7 in Figure 1.11(a), we see that $\{2,4,6\}$ is both a circuit and a hyperplane in F_7, whereas, in N, this set is a

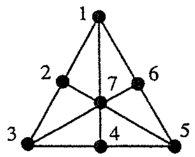

Fig. 1.15. F_7^-.

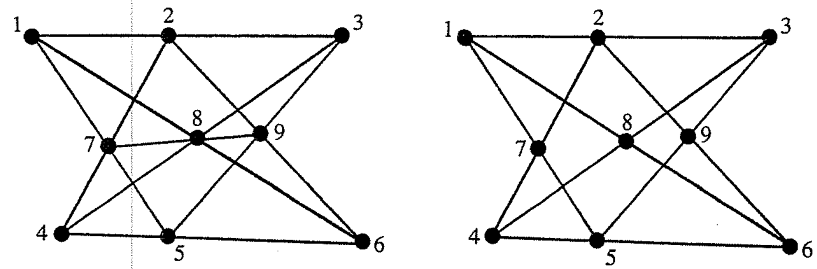

Fig. 1.16. The Pappus and non-Pappus matroids.

basis. We say that N has been obtained from F_7 by *relaxing* the circuit–hyperplane $\{2,4,6\}$. This operation can be performed on matroids in general. □

1.5.13 **Proposition.** *Let M be a matroid having a subset X that is both a circuit and a hyperplane. Let $\mathcal{B}' = \mathcal{B}(M) \cup \{X\}$. Then $\mathcal{B}'$ is the set of bases of a matroid M' on $E(M)$. Moreover,*

$$\mathcal{C}(M') = (\mathcal{C}(M) - \{X\}) \cup \{X \cup e : e \in E(M) - X\}.$$

Proof. We leave this as an exercise. □

The matroid M' in the last proposition is called a *relaxation* of M. Thus the matroid N in Figure 1.15 is a relaxation of F_7. We call N the *non-Fano matroid* and denote it by F_7^-.

1.5.14 **Example.** The diagrams in Figure 1.16 obey 1.5.8 and are therefore geometric representations for rank-3 matroids. We call these matroids the *Pappus* and *non-Pappus matroids*, respectively, because of their relationship to the well-known Pappus configuration in projective geometry. As we shall see in Chapter 6, the non-Pappus matroid is not representable over any field. A smallest matroid with this property can be obtained from $AG(3,2)$ by relaxing a circuit–hyperplane. The proof that this matroid is non-representable will also be delayed until Chapter 6. □

1.5.15 **Example.** Figure 1.17 contains another familiar object from projective geometry, the 3-dimensional Desargues configuration. One can check that the points, lines, and planes of this diagram obey 1.5.9–1.5.11 so that this diagram is indeed a geometric representation for a 10-element rank-4 matroid. Alternatively, one can show that this diagram is the geometric representation for $M(K_5)$ with the edges of K_5 being labelled as in Figure 1.5. □

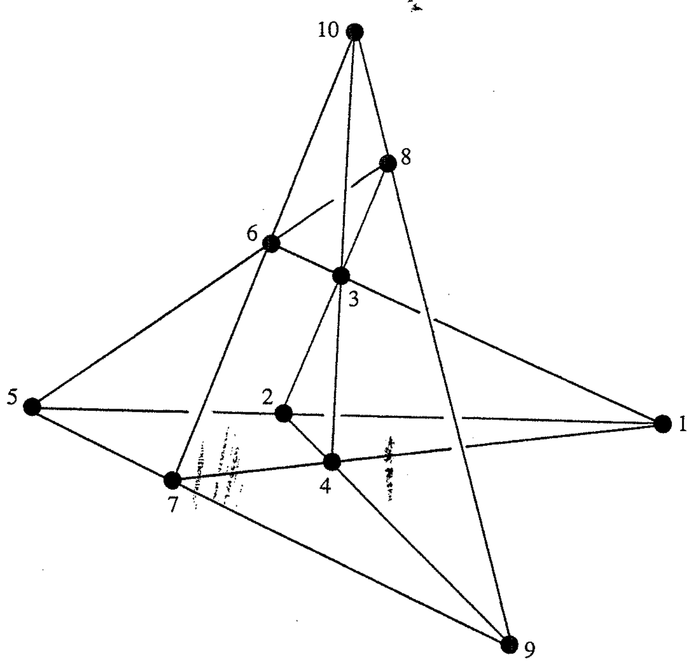

Fig. 1.17. $M(K_5)$.

In light of the geometric representations discussed in this section, it should not be surprising that the terms *point*, *line*, and *plane* are often used in an arbitrary matroid to refer to flats of ranks one, two, and three, respectively.

Exercises

1. Determine which of conditions 1.5.9–1.5.11 is violated by the diagram in Figure 1.9.

2. Give a direct proof of Proposition 1.5.1 by using the first definition of affine dependence.

3. Let D be a diagram involving points and lines in the plane and satisfying 1.5.8 as well as the non-degeneracy conditions stated before it. Prove that there is a simple matroid of rank at most three on the set of points of D whose rank-1 and rank-2 flats are the points and lines, respectively, of D.

4. Let D be a diagram involving points, lines, and planes and satisfying 1.5.9–1.5.11 as well as the non-degeneracy conditions. Prove that there is a simple matroid of rank at most four on the set of points of D that has as its rank-1, rank-2, and rank-3 flats the points, lines, and planes, respectively, of D.

5. Show that neither the Fano nor the non-Fano matroid is graphic.

6. Prove that an affine matroid over $GF(2)$ has no circuits with an odd number of elements.

7. Prove that every relaxation of F_7 is isomorphic to F_7^-.

8. Let M be a matroid and X be a circuit-hyperplane of M. Let M' be the matroid obtained from M by relaxing X. Find, in terms of M, the independent sets, the rank function, the hyperplanes, and the flats of M'.

9. Prove that the following statements are equivalent for a rank-r matroid M.

   (a) M is a relaxation of some matroid.

   (b) M has a basis B such that $C(e, B) = B \cup e$ for every e in $E(M) - B$ and neither B nor $E(M) - B$ is empty.

   (c) M has a non-empty basis B such that $B \neq E(M)$ and every $(r-1)$-element subset of B is a flat.

10. The matroids M_1 and M_2 for which geometric representations are shown in Figure 1.18 are graphic. For $i = 1, 2$, find a graph G_i for which $M(G_i) \cong M_i$.

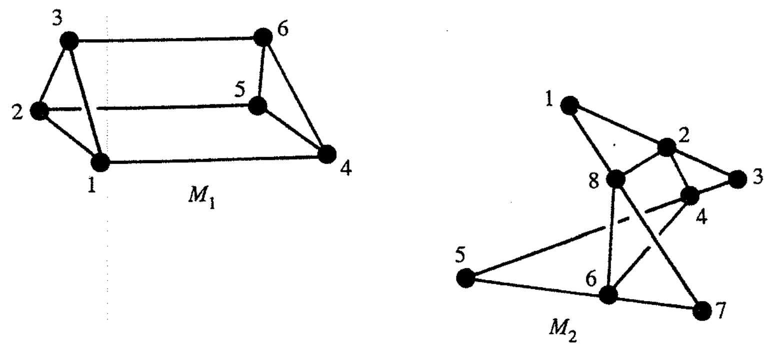

Fig. 1.18. Geometric representations of two graphic matroids.