INTRODUCTION TO MATROID THEORY
EXERCISE SHEET

The solutions can be handed in during the class of March 28th, if you wish me to look over them during the Easter break.

1. Let M be a matroid on the ground set E and let E° denote the set of loops of M. Recall that two elements $e, f \in E$ are called *parallel* if $\mathrm{rk}(\{e\}) = \mathrm{rk}(\{f\}) = \mathrm{rk}(\{e, f\}) = 1$.

   Prove that parallelism defines an equivalence relation on the set $E \setminus E^\circ$.

2. Let $G = (V, E, a, b)$ denote a finite graph, and let A_G denote its *incidence matrix*, i.e., the matrix whose rows are indexed by elements of V and whose columns are indexed by elements of E, and where the entry corresponding to row $v \in V$ and column $e \in E$ is defined by

$$(A_G)_{v,e} := \begin{cases} 1 & \text{if } b(e) \neq a(e) = v \\ -1 & \text{if } a(e) \neq b(e) = v \\ 0 & \text{otherwise} \end{cases}$$

   Prove that the cycle matroid of G is the matroid of linear dependencies of A_G (over every field!), i.e.,

$$M(G) \simeq M(A_G).$$

   (Thus you will have proved that every graphic matroid is representable over every field.)

   Hint: beware the hint given in the lecture!

3.
   (a) Draw the Hasse diagram of $\mathcal{L}(U_{3,5})$, the lattice of flats of the uniform matroid $U_{3,5}$.
   (b) Let M be any matroid on the ground set E. Prove:
      * For all $X \subseteq E$, $\mathcal{L}(M/X) \simeq \mathcal{L}(M)_{\geq \mathrm{cl}(X)}$;
      * For every *closed* set $X \subseteq E$, $\mathcal{L}(M[X]) = \mathcal{L}(X)_{\leq X}$.