INTRODUCTION TO MATROID THEORY
SUGGESTED EXERCISES
(CLASS OF MARCH 15TH, 2018)

1. Consider the matroid $U_{2,4}$.
   (a) Compute the characteristic polynomial $\chi_{U_{2,4}}(t)$ using the definition.
   (b) Let e be an element of the ground set of $U_{2,4}$.
   Prove that $U_{2,4} \setminus e \simeq U_{2,3}$ and $U_{2,4}/e \simeq U_{1,3}$.
   (c) Compute $\chi_{U_{2,3}}(t)$ and $\chi_{U_{1,3}}(t)$ using the definition.
   (d) Check your computations in (a), (b), (c) by verifying that $\chi_{U_{2,4}}(t) = \chi_{U_{2,3}}(t) - \chi_{U_{1,3}}(t)$.
   (e) Use the formula from (a) in order to show that $U_{2,4}$ is not graphic.
   (Hint: it's easier than you think).

For the following exercise, recall from Exercise 7 on page 14 of the book (you solved this in the 3d week of classes!): *Let M_1, M_2 be matroids on disjoint sets E_1, E_2 with independent sets $\mathcal{I}_1$, resp. $\mathcal{I}_2$. Let $E := E_1 \cup E_2$. Then the* direct sum $M_1 \oplus M_2$ *is the matroid M with independent sets given by*

$$\mathcal{I}(M) := \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$$

2. Prove that

$$(*) \qquad \chi_{M_1 \oplus M_2}(t) = \chi_{M_1}(t)\chi_{M_2}(t).$$

   In order to do so, one possibility is to follow the following path (where, for all $A \subseteq E$, we let $A_1 := A \cap E_1$ and $A_2 := A \cap E_2$).
   (i) Prove that, for all $A \subseteq E$,
   $$M[A] = M[A_1] \oplus M[A_2].$$
   (ii) Prove that, for all $A \subseteq E$,
   $$\mathrm{rk}(A) = \mathrm{rk}_1(A_1) + \mathrm{rk}_2(A_2),$$
   where rk, resp. rk_1, rk_2 denote the rank functions of M, resp. M_1, M_2.
   (iii) Prove Equality (*).