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BeNeFri, Fall 2020

WORKSHEET 4

Recall:

- We have set up an **on-line meeting on December 10, at 15:30**, in order to discuss your questions about the course and the exercises.
- I will also present the exact exam format on that day, after the institutional rules will be cleared. The exam format will also be communicated by email.

Please let me anyway know in advance of the zoom-meeting if there is anything special you would like me to discuss in that occasion. This helps me prepare for it and gauge what, if anything, I might adjust in preparing the remainder of the lecture.

1. Reading

Read the parts of the Lecture Notes that we did not discuss explicitly together, including proofs. Note: the final paragraph about abstract simplicial complexes and topology is thought of as an "appendix" in view of future things to come.

2. Solving

(i) A standard definition for geometric lattices is the following:

"A finite lattice L is geometric if

(a) it is ranked (meaning: it possesses a rank function),

(b) it is atomic (meaning: every element $x \in L$ can be expressed as a join of atoms)[1]*,*

(c) for all $x, y \in L$, if $x, y \gtrdot x \wedge y$, then $x \vee y \gtrdot x, y$. "

Prove that this definition is equivalent to the one given in class.

(ii) Let M be the cycle matroid of the complete graph on 4 vertices (thus, this matroid has 6 elements). Draw the Bergman complex $\mathscr{B}(M)$.

(iii) Consider the 3-dimensional linear subspace V of $\mathbb{R}^5$ defined by the equations:

$$2x_1 + 2x_2 - x_3 - x_4 + x_5 = 0 \qquad x_1 + x_2 - x_4 + x_5 = 0$$

Let M the matroid represented by V (according to the correspondence we studied in Lecture 4). Write the set of circuits, bases and the rank function of M. Moreover, represent M via an "affine diagram". Is M graphic? Can you draw the Bergman complex $\mathscr{B}(M)$? What is the intersection lattice of the hyperplane arrangement induced on V by the coordinate hyperplanes of $\mathbb{R}^5$?

(iv) Let $\mathscr{A}$ be an arrangement of hyperplanes in $\mathbb{R}^n$. Find a characterization in terms of $\mathscr{A}$ of when the geometric lattice $\mathcal{L}(\mathscr{A})$ decomposes in a direct sum of two geometric lattices.

[1] More precisely: for every $x \in L$ there is $\ell \in \mathbb{N}$ and atoms $a_1, \dots, a_\ell \in A(L)$ such that $x = a_1 \vee a_2 \vee \dots \vee a_\ell$