Emanuele Delucchi
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## WORKSHEET 1

This sheet leads you through the "distance-learning" part of the first module. The assignments marked with a (*) are also a preparation for the second module, and should therefore be completed within the week (we will discuss any questions about those during our second meeting). We will set up an online meeting towards the end of October in order to fully discuss the remaining assignments, and any other questions related to modules 1 and 2.

### 1. Reading

Read the parts of the proofs of cryptomorphisms that were not fully treated in the lecture. The notes in this part follow closely Oxley's standard textbook, with the goal of acquainting you with its style, so that you should be able to more readily access informations there, should you need more about matroids in the future. Write down your questions so we will be able to address them. Also, if some reasonings or passages seem unusual or difficult, try to think about why this is. The passages in question are the following.

(i) Proof of Theorem 2.3 from the lecture notes.

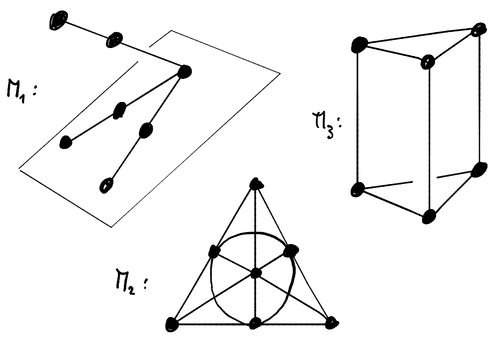
(ii) (*) Proof of the cryptomorphism for bases from the lecture notes (Proposition 3.4 and Theorem 3.5)

(iii) Proof of the cryptomorphism between bases and rank function, directly from Oxley's book [Pages 20-23, see the literature folder].

### 2. Solving

Watch the video on diagrams (*), and work on the following exercises

(i) (*) Write the set of bases, the set of circuits and (some) values of the rank function for the matroids represented in the following pictures. The matroid $M_3$ is graphic: can you find a graph $G$ such that $M_3 \simeq M(G)$?




(ii) (*) Prove that the Uniform matroid $U_{2,4}$ is not graphic.

(iii) Write the diagram representation of the graphic matroid of the complete graph on 4 vertices.

(iv) What is the diagram representation of the uniform matroid $U_{3,k}$ of rank 3 on $k$ elements, for $k \geq 3$?

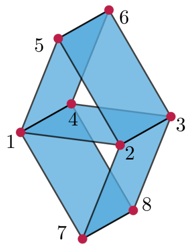



FIGURE 1. The Vámos matroid. Blue squares denote the 4-elements circuits. In particular, $\{5,6,7,8\}$ is an independent set. Credit: Wikipedia/Creative Commons.

(vi) In this exercise we prove, stepwise, that the Vàmos matroid is not representable over any field. Try to solve the steps without help, and only look at the hints in the footnotes after trying yourself.

First, notice that $V_8$ has rank 4 (why is that?). Now suppose, by way of contradiction, that the Vámos matroid is representable over some field $\mathbb{F}$. Then, there is a $4 \times 8$ matrix $A$ with entries in $\mathbb{F}$ whose columns $a_1, \dots, a_8 \in \mathbb{F}^4$ realize the linear dependencies represented in the diagram of Figure 1.

**Shorthand notation.** We use angled brackets in order to denote linear span, so $\langle A \rangle = \operatorname{span}_{\mathbb{F}} A$ for every $A \subseteq \mathbb{F}^4$. Moreover, for $I \subseteq \{1, \dots, 8\}$ we will write $\langle I \rangle := \langle \{a_i \mid i \in I\} \rangle$. For instance, $\langle 1, 5 \rangle = \mathbb{F}a_1 + \mathbb{F}a_5$. Notice that then $\dim\langle I \rangle = \operatorname{rk}(I)$, where rk is the rank function of $V_8$.

(i) Prove[1] that $\langle 5,6 \rangle \cap \langle 1,2,3,4 \rangle \neq \emptyset$. Choose and fix $v \in \langle 5,6 \rangle \cap \langle 1,2,3,4 \rangle$, $v \neq 0$.

(ii) Prove[2] that $\langle v \rangle \subseteq \langle 1,4 \rangle$ and $\langle v \rangle \subseteq \langle 2,3 \rangle$.

(iii) Prove that $\langle v \rangle = \langle 1,4 \rangle \cap \langle 2,3 \rangle$.

Let then $V_8'$ be the matroid of linear dependencies of the vectors $a_1, \dots, a_8, v$.

(iv) Prove that, in a diagram of $V_8'$, the points $v, 5, 6$ are on the same line, as are the points $v, 1, 4$ and $v, 2, 3$.

(v) Prove that also the triple $v, 7, 8$ is colinear (that is the dashed line in Figure 2) below.

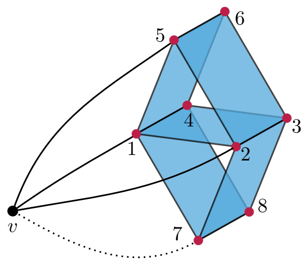



FIGURE 2. The matroid $V_8'$, i.e., the Vámos matroid "enhanced" by the new element $v$. Credit: Wikipedia/Creative Commons.

(vi) Prove that, then, $5, 6, 7, 8$ is a dependent set in $V_8'$.

Item (vi) implies in particular that the elements $5, 6, 7, 8$ form a dependent set – indeed, a circuit – in $V_8$ as well, a contradiction!

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[1]Hint: use the rank axioms in order to commpute the dimension of the intersection.

[2]Hint: for the first containment, notice that by definition $v \in \langle 1,4,5,6 \rangle$.