WARM-UP 1

1. About vectors

Think back to your Linear Algebra 1 class. If you have your textbook or lecture notes handy, look for "Steinitz's exchange theorem"[1] The statement should be a variation of the following.

(**S**) *Let* $B = \{b_1, \ldots, b_r\}$ *be a basis and* $I = \{v_1, \ldots, v_m\}$ *an independent set of the (finite-dimensional) vectorspace* V*. Then there are* $i_1, \ldots, i_m \in \{1, \ldots, r\}$ *such that* $(B \setminus \{b_{i_1}, \ldots, b_{i_m}\}) \cup I$ *is a basis of* V*.*

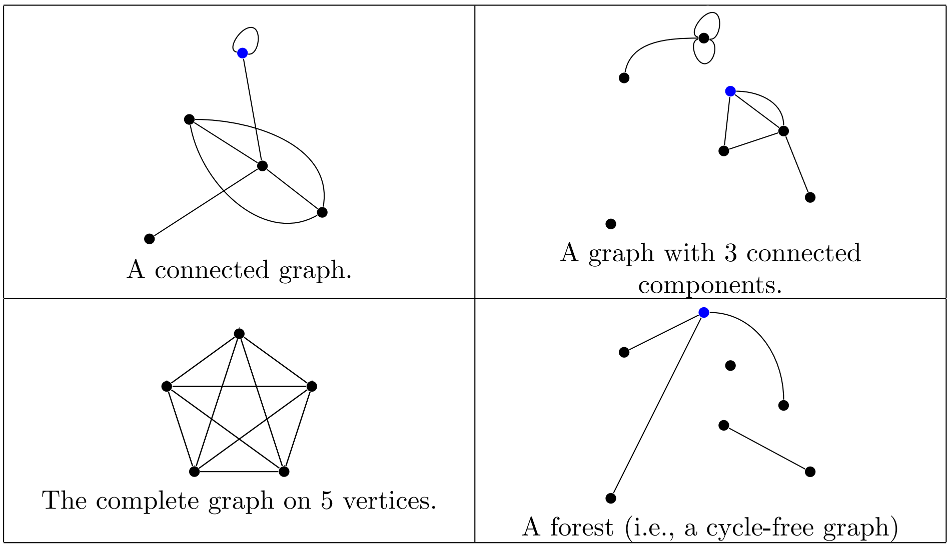
(i) Try to prove (**S**) and/or look it up in your old Linear Algebra materials. What definition or properties of "bases" are you using here?

(ii) Use (**S**) in order to prove the following statement.

Let I, J *be linearly independent, finite subsets of a vectorspace* V*. If* $|I| > |J|$*, then there is* $v \in I \setminus J$ *such that* $J \cup \{v\}$ *is linearly independent in* V*.*

2. About graphs

Graphs are often used in an intuitive fashion – they consist of "vertices joined by edges". Below are pictures of some possible examples of graphs.



A connected graph. | A graph with 3 connected components.

The complete graph on 5 vertices. | A forest (i.e., a cycle-free graph)

Notice the possible presence of *loops* (edges "with only one endpoint") and "multiple edges" (often called *parallel*, as they share their endpoints). A graph without loops and without parallel edges is called *simple*. Notice that several standard graph theory textbooks only consider simple graphs. Here, we will need the more general notion.

A *Trail* in a graph is any finite sequence of (not necessarily distinct) edges in which any two consecutive members share a vertex. A *path* is a walk with no repeated edges or vertices. A *cycle* is any nonempty walk that is "closed" (i.e., such that the last vertex equals the starting vertex). A *circuit* is a nonempty closed path.

[1] "Austauschsatz von Steinitz".

(i) Try to find a rigorous definition of “graph with finitely many edges and vertices” that can model all of the above examples and, with it, give a formal definition of trail, path, cycle, circuit. I recommend that you try this yourself first. You can then compare your notion with one that can be found in the literature, and that I have summarized below after some white pages. Do not look yet!

(ii) The *degree* of a vertex in a graph is the number of edges that “touch” the given vertex, with loops counting two (example: all blue vertices in the pictures of the previous page have degree 3). Define the degree of a vertex in your formalism. Prove that if every vertex in a finite graph has degree at least two, then the graph must contain a cycle.

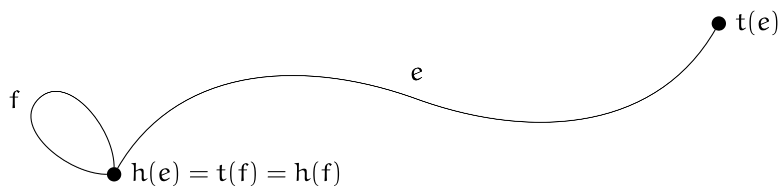
(iii) Let E be the set of edges of a finite graph G. Prove that for every $C \subseteq E$ the following are equivalent: (i) C is the set of edges of a circuit, (ii) C is the set of edges of an inclusion-minimal cycle (i.e., every nontrivial subset of C is not the set of edges of any cycle).

(iy) Prove that, if C_1 and C_2 are the set of edges of two distinct circuits of a finite graph G, and if $e \in C_1 \cap C_2$ is a common edge, then there is a circuit C_3 of G that is contained in $C_1 \cup C_2$ but does not contain e (i.e., $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$).

Graphs

Definition 0.1. A graph $G = (V, E, h, t)$ is a quadruple consisting of a (finite) set of vertices V, a (finite) set of edges E and two functions $h, t : E \to V$ that assign to every edge its "ends". Given any set $A \subseteq E$ of edges we let $V(A) := h(A) \cup t(A)$ be the set of all ends of edges in A.

We will often omit braces when designing one-element sets, if no need for specification arises. For instance, given $e \in E$ we will write $V(e)$ for $V(\{e\})$.



Two edges with their ends.

A *loop* in G is any $e \in E$ with $|V(e)| = 1$. Two edges $e, e' \in E$ are called *parallel* if $V(e) = V(e')$. The graph G is called *simple* if it has no loops nor parallel edges. A *trail* in G is any sequence $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that $\{v_{i-1}, v_i\} = V(e_i)$ for all $i = 1, \ldots, k$. It is called *closed* (or a "cycle") if $k > 0$ and $v_0 = v_k$. A *path* is a trail where all edges and all vertices are pairwise distinct (in this case we will talk about a "path from v_0 to v_k". A *circuit* in G is a minimal closed trail, i.e., a closed trail which, after removal of any edge, is a path (in particular, every loop is a circuit).

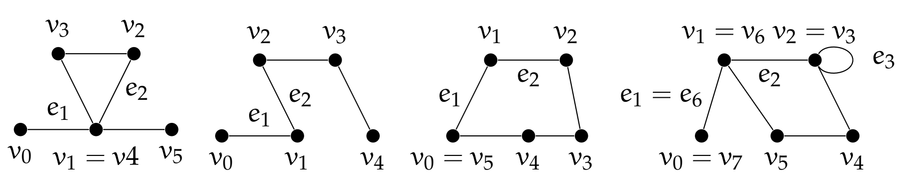


FIGURE 1. A trail, a path, a circuit and a closed trail that is not a circuit.

Let $T \subseteq V$ be a set of vertices of G the *vertex-induced subgraph* defined by T is the graph $G(T) := (T, E', h, t)$ where $E' = \{e \in E \mid \{h(e), t(e)\} \subseteq T\}$ is the set of edges with both endpoints in T.

Definition 0.2. Let G be a graph. We call G *connected* if for any two vertices $v, w \in V$ there is a path from v to w in G. A *connected component* of G is any maximal connected vertex-induced subgraph, i.e., any vertex-induced subgraph $G(T)$ that is connected and such that, for every $v \in V \setminus T$, $G(T \cup \{v\})$ is not connected . We define

$$c(G) := \text{the number of connected components of } G.$$

Definition 0.3. The *degree* of a vertex v is the number

$$\deg(v) := |t^{-1}(v)| + |h^{-1}(v)|.$$

Definition 0.4 (Edge deletion)**.** Let $G = (V, E, h, t)$ be a graph and let $A \subseteq E$. The *deletion* of A from G is the graph $G \setminus A := (V, E \setminus A, h_{|E \setminus A}, t_{|E \setminus A})$ on the same vertex set as G but without the edges in A, and with the functions h, t restricted accordingly. If $A = \{e\}$ consists of a single element, we sometimes write $G \setminus e$ for $G \setminus \{e\}$. The “restriction” of G to A is $G[A] := G \setminus (E \setminus A)$.

Remark 0.5 (On the word “subgraph”)*.* Every graph of the form $G[A]$ we will call a “subgraph” of G. Notice the difference with the notion of "vertex induced subgraph" discussed earlier on. The latter will not appear in the following, so we feel safe in our terminological choice.