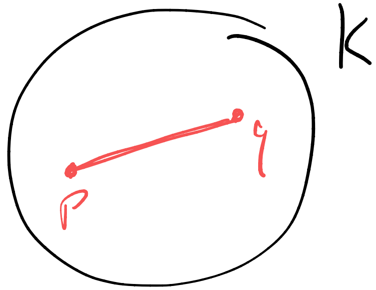
$P \subseteq \mathbb{R}^n$ is a convex polytope if there are $v_1 \dots v_k \in \mathbb{R}^n$ $(k \in \mathbb{N})$ with

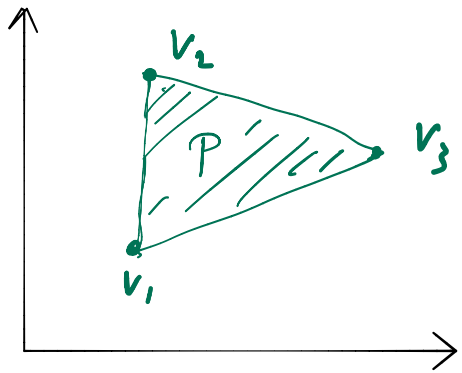
$$P = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \sum_i \lambda_i = 1,\ \lambda_i \geq 0 \ \forall i \right\} = \mathrm{conv}\{v_1, \dots, v_k\}$$



In general, $K \subseteq \mathbb{R}^n$ convex if it contains the segment $tp + (1-t)q$, $0 \leq t \leq 1$ between any $p, q \in K$

The convex hull of $X \subseteq \mathbb{R}^n$ is the smallest convex set containing X

In fact: $P \subseteq \mathbb{R}^n$ is a convex polytope if and only if it is the convex hull of a finite set of points

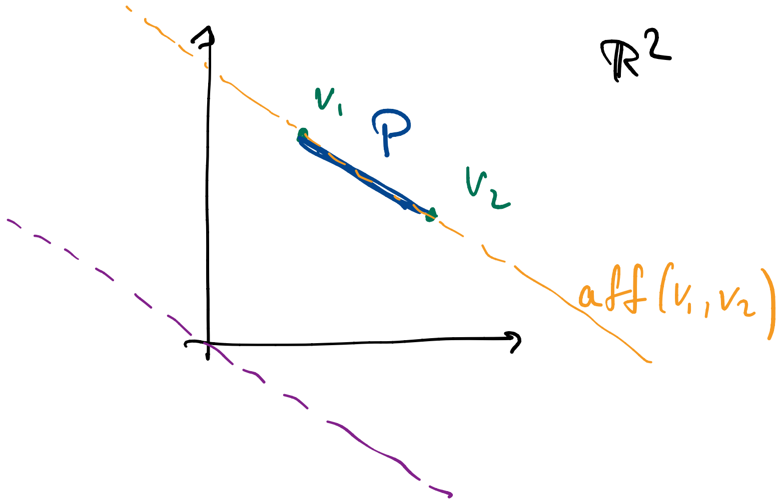


$P \subseteq \mathbb{R}^n$ is a convex polytope if there are $v_1 \cdots v_k \in \mathbb{R}^n$ $(k \in \mathbb{N})$ with

$$P = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \sum_i \lambda_i = 1,\ \lambda_i \geq 0 \ \forall i \right\} = \operatorname{conv}\{v_1, \dots, v_k\}$$

The affine hull of $v_1 \cdots v_k$ is

$$\operatorname{aff}\{v_1 \cdots v_k\} = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \sum_{i=1}^{k} \lambda_i = 1 \right\}$$



The dimension of P is

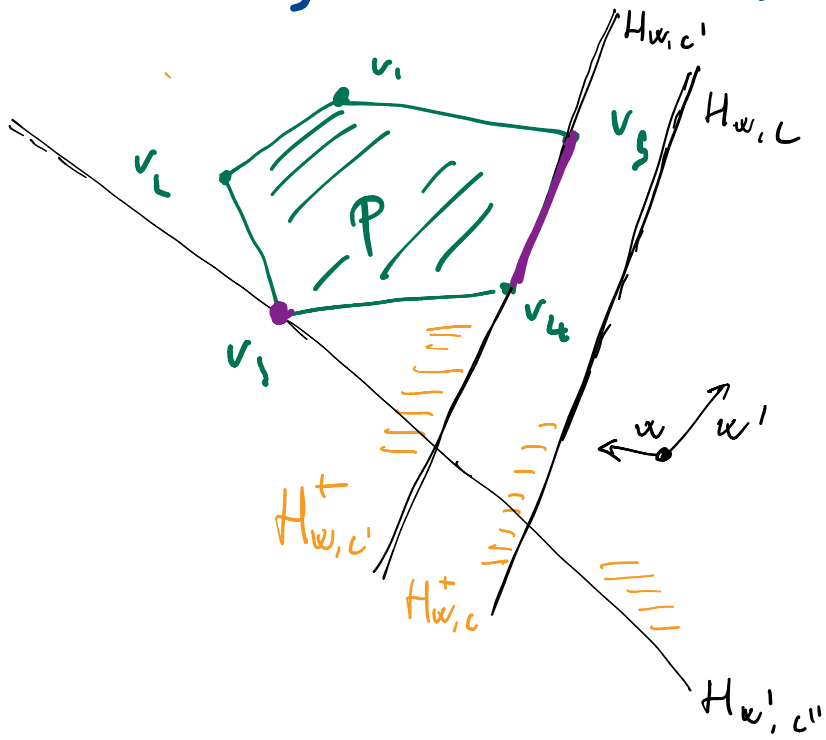
$$\dim(P) = \dim(\operatorname{aff}(v_1 \cdots v_k)),$$

which equals the dimension of the "translate at the origin"

$$\langle v_i - v_1 \mid i \geq 2 \rangle$$

$P \subseteq \mathbb{R}^n$ is a **convex polytope** if there are $v_1 \cdots v_k \in \mathbb{R}^n$ $(k \in \mathbb{N})$ with

$$P = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \sum_i \lambda_i = 1,\ \lambda_i \geq 0 \ \forall i \right\} = \operatorname{conv}\{v_1, \dots, v_k\}$$



Given $w \in \mathbb{R}^n$, $c \in \mathbb{R}$, let

$$H^{+}_{w,c} := \{ x \in \mathbb{R}^n \mid \langle w, x \rangle \overset{\geq}{=} c \}$$ (Affine) halfspace

Affine hyp.

$H_{w,c}$ is **valid** for P if $P \subseteq H^+_{w,c}$

A **face** of P is any $F \subseteq P$ of the form

$$F = P \cap H_{w,c} \quad \text{for valid } H_{w,c}$$

Note:

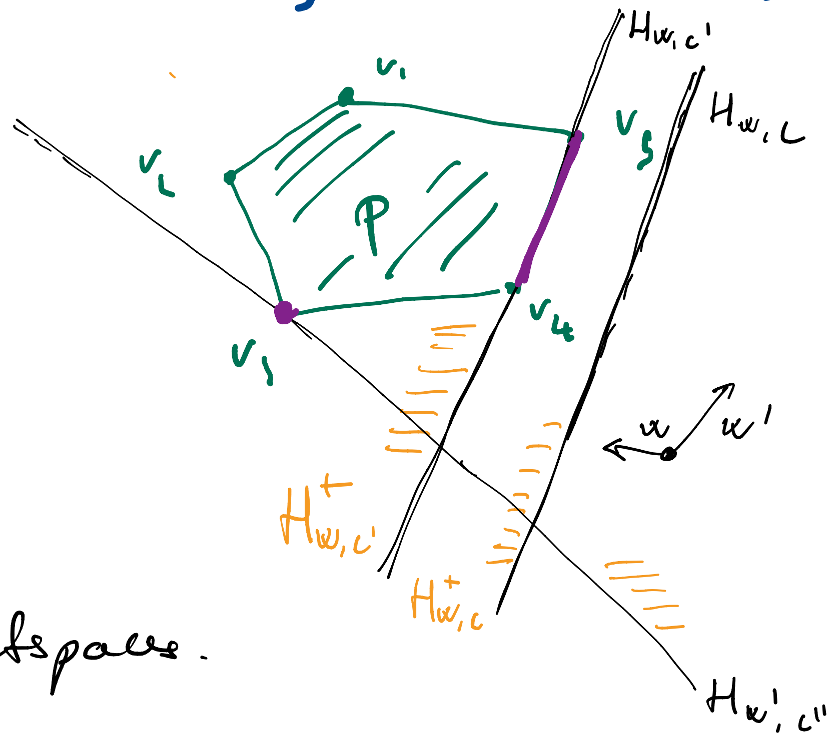
- $\emptyset \subseteq P$ and $P \subseteq P$ are faces of P

  $w = 0,\ c = -1$ $\qquad$ $w = 0,\ c = 0$

- Every face of a polytope is a polytope: $F = \operatorname{conv}(\{v_1 \cdots v_k\} \cap H_{w,c})$
- Faces of dimension 0: vertices ; dim. 1: edges ; dim $\dim(P) - 1$: facet

$P \subseteq \mathbb{R}^n$ is a convex polytope if there are $v_1 \dots v_k \in \mathbb{R}^n$ $(k \in \mathbb{N})$ with

$$P = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \sum_i \lambda_i = 1,\ \lambda_i \geq 0\ \forall i \right\} = \mathrm{conv}\{v_1, \dots, v_k\}$$



Given $w \in \mathbb{R}^n$, $c \in \mathbb{R}$, let

$$H^+_{w,c} := \{ x \in \mathbb{R}^n \mid \langle w, x \rangle \overset{\geq}{=} c \}$$

(Affine) halfspace

Affine hyp.

$H^+_{w,c}$ is valid for P if $P \subseteq H^+_{w,c}$

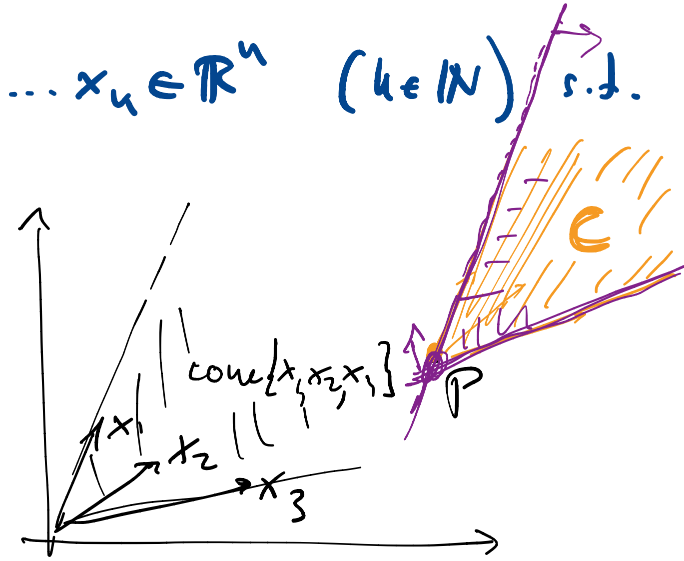
Note: P is the intersection of all valid halfspaces.

A polyhedron is any intersection of finitely many halfspaces

"Main theorem" for polytopes: $P \subseteq \mathbb{R}^n$ is a convex polytope if and only if it is a bounded polyhedron.

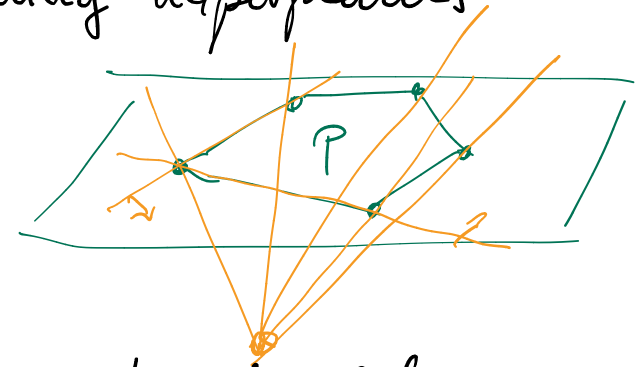
$C \subseteq \mathbb{R}^n$ is a **polyhedral cone** if there are $p, x_1 \dots x_k \in \mathbb{R}^n$ ($k \in \mathbb{N}$) s.t.

$$C = p + \underbrace{\left\{ \sum_{i=1}^{k} \lambda_i x_i \;\middle|\; \lambda_i \geq 0 \ \ \forall i \right\}}_{\text{cone}\{x_1 \dots x_k\}}$$



"Main theorem" for cones (implies the one for polytopes)

$C \subseteq \mathbb{R}^n$ is a polyhedral cone if and only if it is the intersection of finitely many halfspaces, all of whose bounding hyperplanes contain a common point p.



Proof: see Ziegler's book §1.3

Structure theorem: $Q \subseteq \mathbb{R}^n$ is a polyhedron if and only if

$$Q = P + C$$

convex polytope

polyhedral cone