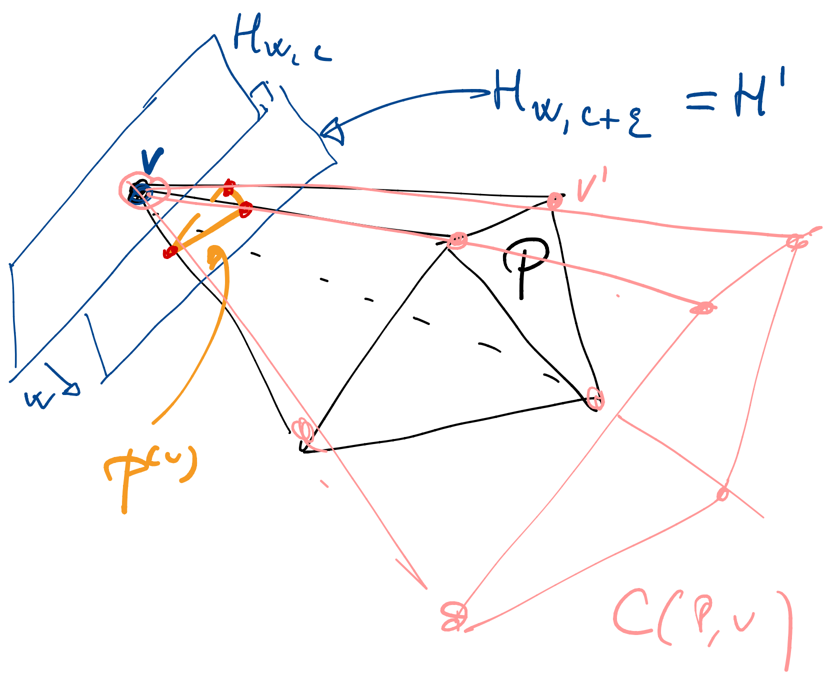
VERTEX CONES



Let $P = \mathrm{conv}\{v_1 \cdots v_k\} = \bigcap_{H^+ \in \mathcal{H}} H^+$ be a polytope,

$\mathcal{H}$ ← finite set of halfspaces

v a vertex of P; $H_{w,c}$ supporting v.

Let $H' := H_{w,c+\varepsilon}$

with $\varepsilon > 0$ s.t. $\left(\{v_1 \cdots v_k\} \setminus \{v\}\right) \subseteq (H')^+$

Set: $P^{(v)} := P \cap H'$ "vertex figure" of P at v.

1) $P^{(v)}$ is a polytope $\left(\text{bounded polyhedron } \bigcap_{H^+ \in \mathcal{H}} (H' \cap H^+)\right)$

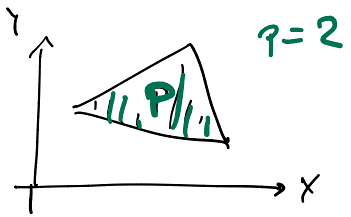
2) The $\overset{\text{facets}}{\text{vertices}}$ of $P^{(v)}$ are exactly the intersections of H' with $\overset{\text{facets}}{\text{edges}}$ of P that contain v.

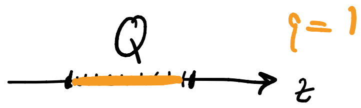
Then: P is contained in the "vertex cone" $C(P,v) := v + \mathrm{cone}\left\{ v' - v \;\middle|\; \begin{array}{l} v' \text{ vertex of } P \\ \overline{vv'} \text{ edge of } P \end{array} \right\}$

PRODUCTS

Let $P = \operatorname{conv}\{p_1,\dots,p_k\} \subseteq \mathbb{R}^p$

$Q = \operatorname{conv}\{q_1 \dots q_\ell\} \subseteq \mathbb{R}^q$

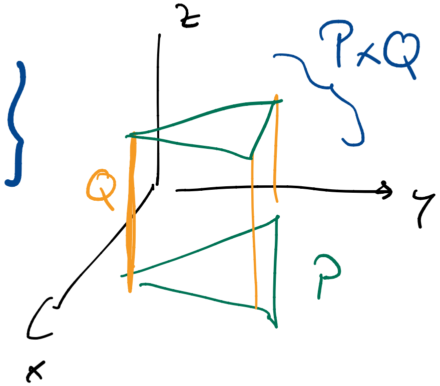




For $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ let $(x,y) \in \mathbb{R}^p \times \mathbb{R}^q$ be the point projecting to x, resp. y.

Then:

$$P \times Q = \operatorname{conv}\left\{ (p_i, q_j) \;\middle|\; i \in [k],\, j \in [\ell] \right\}$$



<u>Proof</u>: $\supseteq$ easy

$\subseteq$ Let $x := \left(\sum_{i=1}^{k} \lambda_i p_i, \sum_{j=1}^{\ell} \mu_j q_j \right) \in P \times Q$,

then $x = \sum_{i,j} \lambda_i \mu_j (p_i, q_j)$ $\quad (*)$

with: $\lambda_i \mu_j \geq 0 \;\; \forall i,j$, and $\sum_{i,j} \lambda_i \mu_j = \sum_i \lambda_i \Big(\underbrace{\sum_j \mu_j}_{1} \Big) = 1$

hence $(*)$ is convex comb. of (p_i, q_j) $\boxtimes$