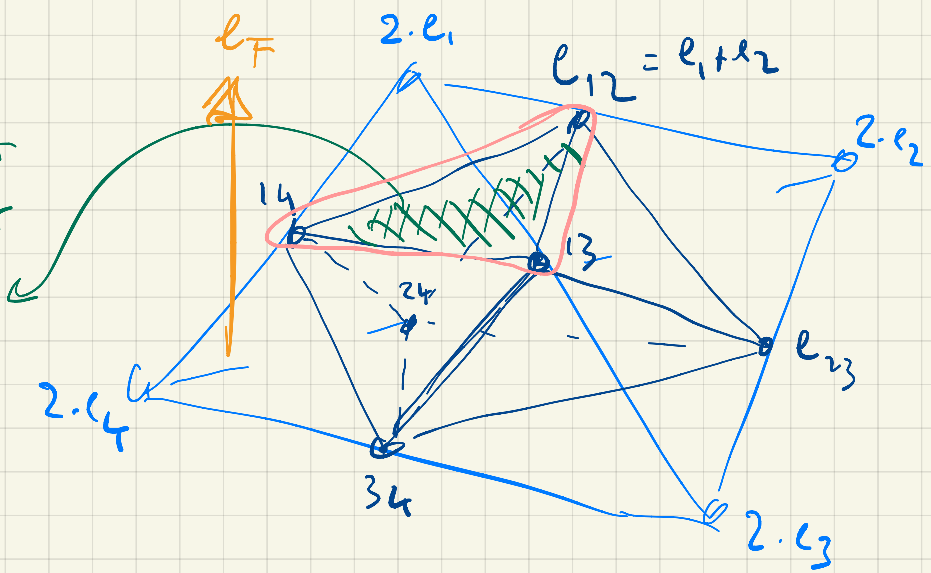
"$\mathcal{F}$" Flacet: "indexes"
facets that meet
interior of simplex

Defining ineq:

$$\underbrace{\sum_{i\in\mathcal{F}} x_i}_{\langle x \mid e_{\mathcal{F}}\rangle} \leq \mathrm{rk}(\mathcal{F})$$

$=$



$\to$ Flacet $\mathcal{F}$ corresponds to face

$$[P_M \uparrow e_{\mathcal{F}}] := \underset{x\in P}{\arg\max}\ \langle x \mid e_{\mathcal{F}}\rangle$$

warm-up

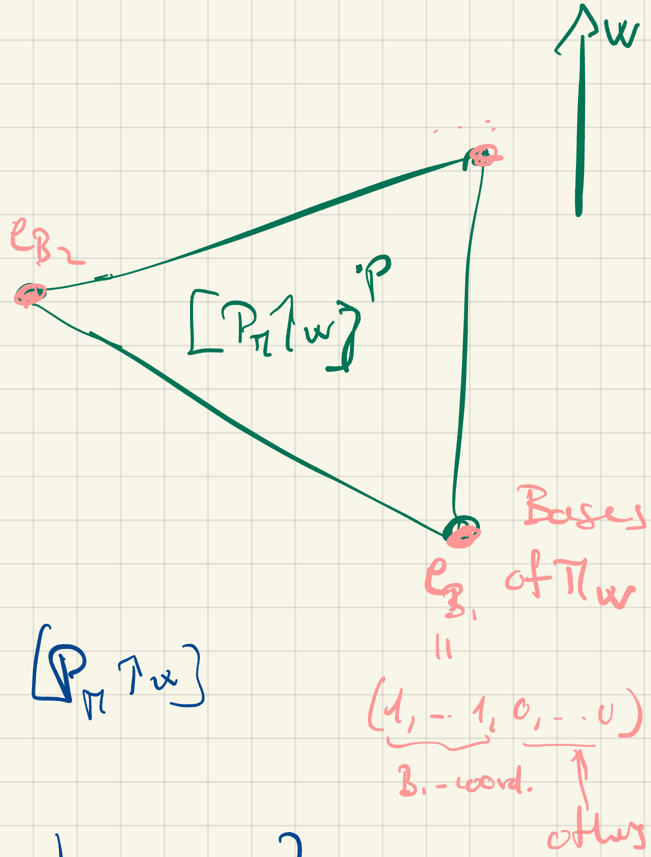
Matroid $M_{\mathcal{F}}$: matroid whose mat. polytope is $[P_M \uparrow e_{\mathcal{F}}]$

Lecture 2:
This is special

$$M_{\mathcal{F}} = M[\mathcal{F}] \oplus M/\mathcal{F}$$

Def: M matroid on $[n]$, let $w \in \mathbb{R}^n$ arbitrary. Let M_w be the matroid with.

$$P_{M_w} := [P_M \uparrow w]$$



Lemma 1.3. *Let M be a matroid of rank r on $[n]$ and let $w \in \mathbb{R}^n$. The following are equivalent.*

(1) M_w has no loops.

(2) Every element of the ground set $[n]$ appears in some basis of M_w.

(3) The face $[P_M \uparrow w]$ intersects the (relative) interior of $r \cdot \Delta^{(n)}$.

(1) $\Leftrightarrow$ (2) Loops $\Leftrightarrow$ elements in no bases

no loops $\Leftrightarrow$ every element in some basis.

(2) $\Leftrightarrow$ (3) For every $i \in [n]$ there is e_{B_i} vertex of $[P_M \uparrow w]$ with nonzero i-th coordinate.

$$[P_M \uparrow w] = \operatorname{conv}\{e_B \mid B \text{ basis of } M_w\} = \left\{ \sum_{B \in \mathcal{B}(M_w)} \lambda_B e_B \;\middle|\; \begin{array}{l} \lambda_B \geq 0 \\ \sum \lambda_B = 1 \end{array} \right\}$$

relint $r\Delta^{(n)} \Leftrightarrow$ points x with $x_i > 0 \quad \forall i$

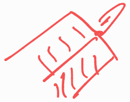
Definition 1.4. Let M be a matroid on $[n]$ ~~and let $w \in \mathbb{R}^n$~~. The set

$$\widetilde{\mathscr{B}}(M) := \{w \in \mathbb{R}^n \mid M_w \text{ has no loops}\}$$

is called the *Bergman fan* of M (see the following Proposition for a justification of this name).

Bases of M_w are bases of M with "maximal w-weight"

Proposition 1.5. *We have*



$$\widetilde{\mathscr{B}}(M) = \bigcup_{\substack{F \text{ face of } P_M \\ F \cap \operatorname{relint}(r \cdot \Delta^{(n)}) \neq \emptyset}} N_F \qquad (\dagger)$$

and the set of cones on the right-hand side is a subfan of the normal fan to P_M.

Proof: Key: Prove that if $N_F \cap \widetilde{\mathcal{B}}(M) \neq \emptyset$, then $N_F \subseteq \widetilde{\mathcal{B}}(M)$

For this: Prove $w \in \widetilde{\mathcal{B}}(M) \Rightarrow \underbrace{N_{[P_M \uparrow w]}}_{} \subseteq \widetilde{\mathcal{B}}(M)$

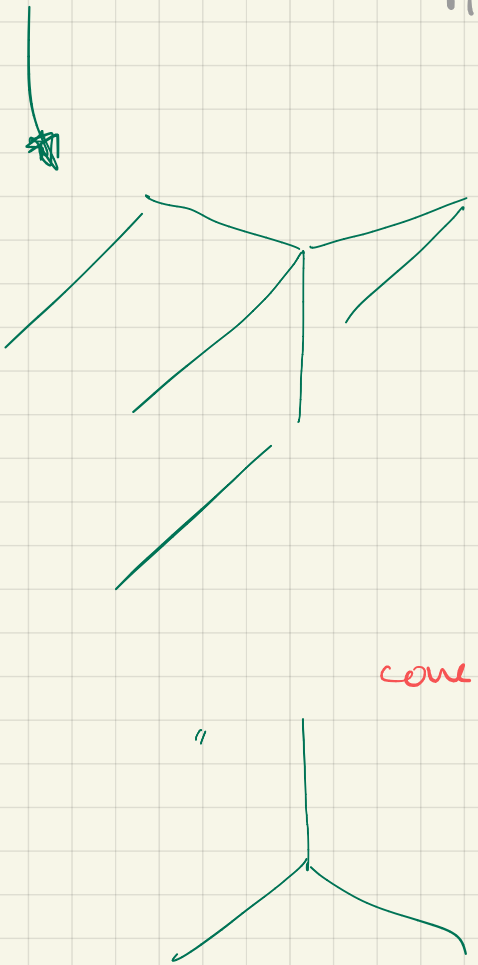
faces of this are normal cones to faces of P_M that contain $[P_M \uparrow w]$

EXAMPLE: $U_{2,3}$ matroid on [3]

Bases: 12, 23, 13

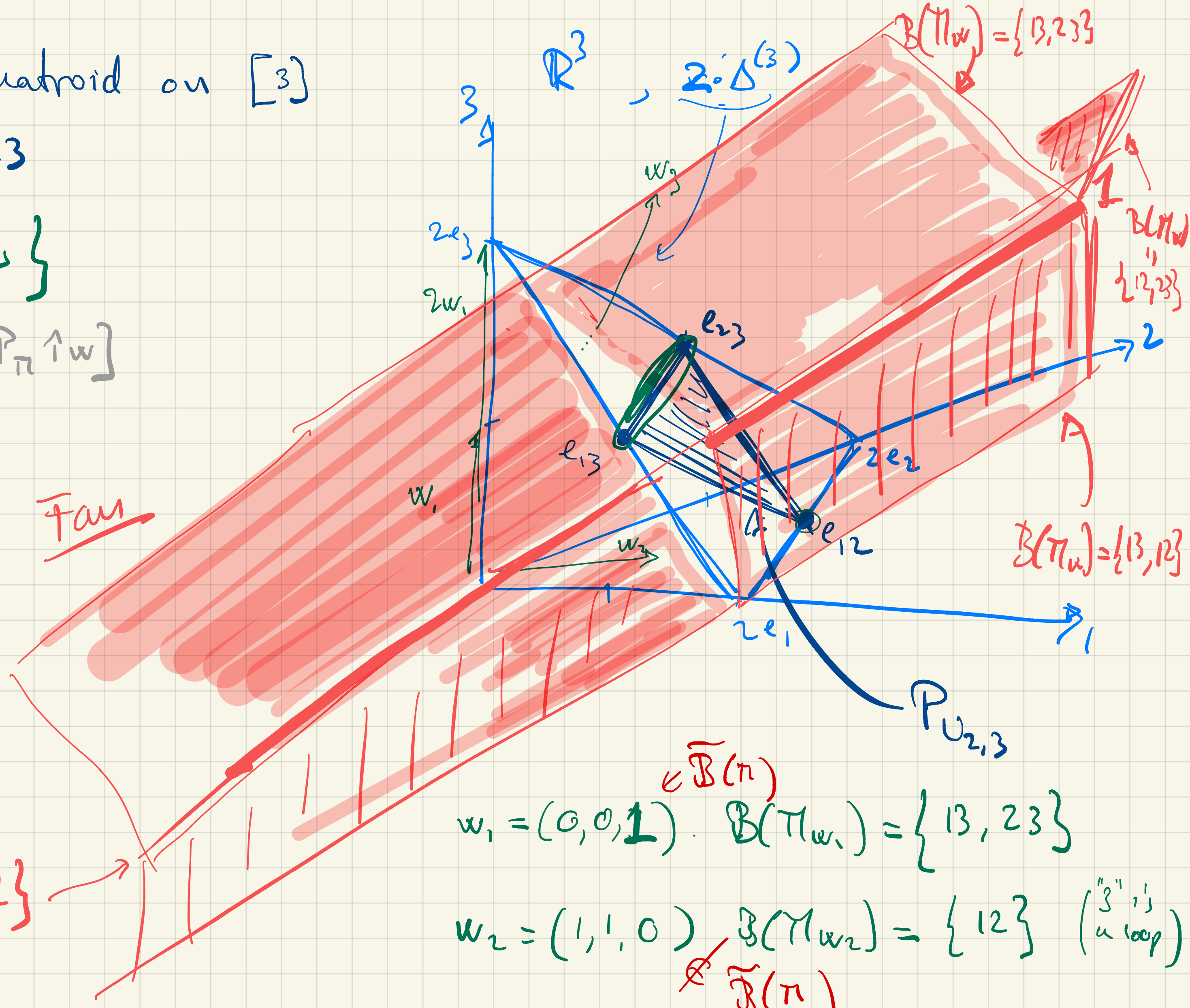
$\widetilde{\mathcal{B}}(U_{2,3}) = \{ w \mid \underbrace{M_w}$ loop-less $\}$

↳ $P_{M_w} = [P_M \uparrow w]$



cone$\{\mathbb{1}, -\mathbb{1}\}$

$\|$



$w_1 = (0,0,1)$. $\mathcal{B}(M_{w_1}) = \{13, 23\}$ $\in \widetilde{\mathcal{B}}(M)$

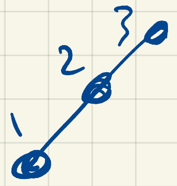
$w_2 = (1,1,0)$ $\mathcal{B}(M_{w_2}) = \{12\}$ ("3" is a loop) $\notin \widetilde{\mathcal{B}}(M)$

$2\cdot w_1 \in \widetilde{\mathcal{B}}(M)$

$w_3 = (1,1,2)$: $M_{w_3} = M_{w_1}$

Note: adding $\mathbb{1} = (1,1\cdots1)$ irrelevant.

$U_{2,3}$ - Bases: $\{12, 23, 13\}$



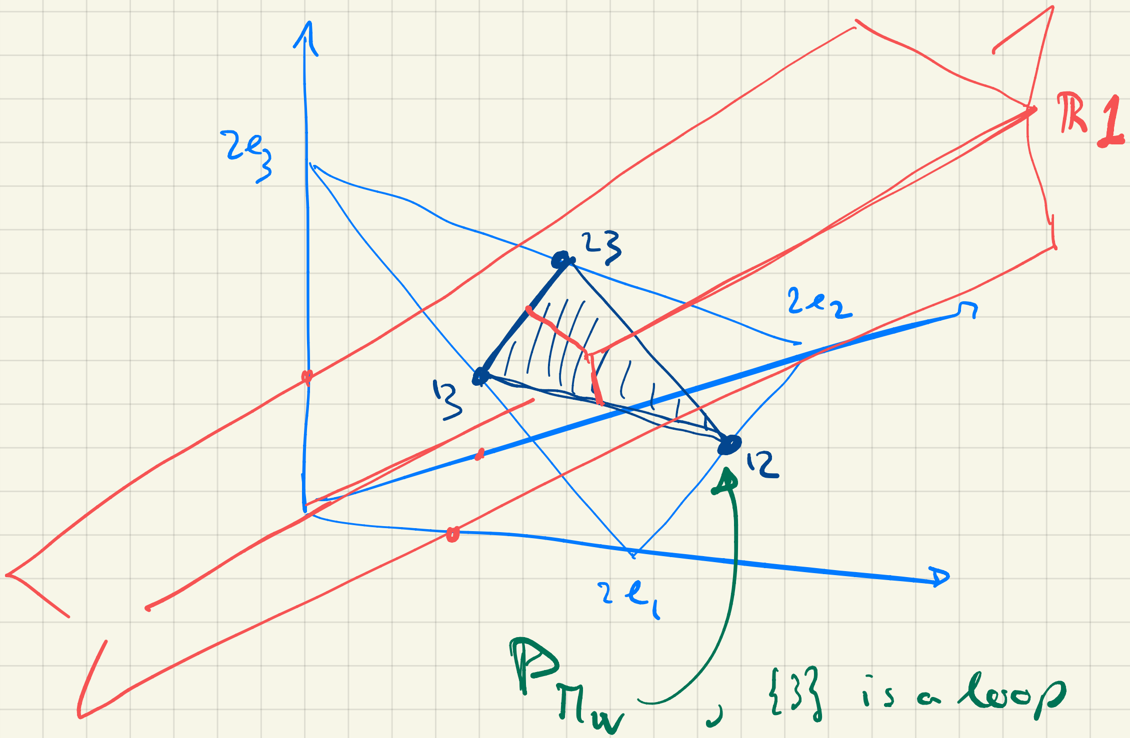
Flats: $\{1,2,3\}$

$\{1\} \quad \{2\} \quad \{3\}$

$\emptyset$

$$w = \begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ (5, 5, 0) \end{array}$$

$$\{\emptyset \subsetneq \underbrace{\{1,2\}}_{\text{not a flat}} \subsetneq \{1,2,3\}\} = \Phi(w)$$



P_{M_w}, $\{3\}$ is a loop

$$\mathcal{B}(M_w) = \{\{1,2\}\}$$

Proposition's claim:

$$M_w = M[\{1,2\}]/\emptyset \oplus M[\{1,2,3\}]/\{1,2\}$$

$$= M[\{1,2\}] \oplus M/\{1,2\}$$

$$\mathcal{B}(M_w) = \{\{1,2\}\} \oplus \{\emptyset\}$$

3 is a loop

2.1. ... of M_w.

Definition 2.1. For any given $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ let us partition the set $[n]$ into blocks, so that elements of the same block index coordinates of w with the same value.

Precisely, consider the equivalence relation $\sim_w$ on $[n]$ with $i \sim_w j$ if and only if $w_i = w_j$. Let $\pi_1, \dots, \pi_s$ be the equivalence classes of $\sim_w$, numbered in order of decreasing value – i.e., for $\pi_k = [w_i]$ and $\pi_l = [w_j]$, we have $k < l$ if and only if $w_i > w_j$.

Let us now define a chain of subsets of $[n]$ as follows:

$$\Phi(w) := \{F_i^w\}_{i=0,\dots,s}, \qquad \text{with} \qquad F_0^w := \emptyset,\ F_i^w := \pi_1 \cup \ldots \cup \pi_i \ \text{ for all } i > 0.$$

Our next goal is to prove the following theorem.

Theorem 2.2. *A vector $w \in \mathbb{R}^n$ is contained in $\widetilde{\mathscr{B}}(M)$ if and only if all F_i^w are flats of M.*

We start with an explicit expression of M_w in terms of the family $\Phi(w)$.

Proposition 2.3. *Let M be a matroid on $[n]$ and $w \in \mathbb{R}^n$. Then*

$$M_w = \bigoplus_{i=1}^{s} M[F_i^w]/F_{i-1}^w,$$

where $\{F_i^w\}_{i=1,\dots,s}$ is as above.

weights

Say $w = (3, 1, 0, -7, 14, 1, 0) \in \mathbb{R}^7$

Elements: 1 2 3 4 5 6 7

$\pi_1 = \{5\}$ – weight 14

$\pi_2 = \{1\}$ " 3

$\pi_3 = \{2, 6\}$ " 1

$\pi_4 = \{3, 7\}$ " 0

$\pi_5 = \{4\}$ " -7

$$\Phi(w) := \{ \underset{\emptyset}{\underset{\|}{F_0^w}} \subsetneq \underset{\pi_1}{\underset{\|}{F_1^w}} \subsetneq \underset{\pi_1 \cup \pi_2}{\underset{\|}{F_2^w}} \dots \subsetneq \underset{\pi_1 \cup \dots \cup \pi_s}{\underset{\|}{F_s^w}} = [n] \}$$

Here: $\emptyset \subsetneq \{5\} \subsetneq \{1,5\} \subsetneq \{1,2,5,6\} \subsetneq \{1,2,3,5,6,7\} \subsetneq [7]$

Theorem 2.2. *A vector $w \in \mathbb{R}^n$ is contained in $\widetilde{\mathscr{B}}(M)$ if and only if all F_i^w are flats of M.*

We start with an explicit expression of M_w in terms of the family $\Phi(w)$.

Proposition 2.3. *Let M be a matroid on $[n]$ and $w \in \mathbb{R}^n$. Then*

$$M_w = \bigoplus_{i=1}^{s} M[F_i^w]/F_{i-1}^w,$$

where $\{F_i^w\}_{i=1,\dots,s}$ is as above.

$\Rightarrow$ loops in this matroid

$\Leftrightarrow$ loops in some $M[F_i^w]/F_{i-1}^w$

warning

$\Leftrightarrow$ $cl(F_{i-1}^w) \setminus F_{i-1}^w$

empty iff F_{i-1}^w flat

Proof: recursive (se lecture notes)

Let $\mathcal{M}$ matroid on $[n]$

$\mathcal{L}(M) = \{\text{set of all flats}\}$

$\overline{\mathcal{L}}(M) = \mathcal{L}(M) \setminus \{cl(\emptyset), [n]\}$

want to consider chains $\Phi = \{F_1 \subsetneq F_2 \subsetneq \cdots\}$

The set of all chains in $\mathcal{L}(M)$ or $\overline{\mathcal{L}}(M)$ is $\Delta(\mathcal{L}(M))$, resp. $\Delta(\overline{\mathcal{L}}(M))$

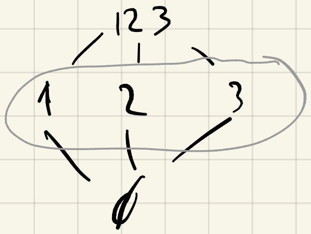
Def: For any $\Phi \subseteq 2^{[n]}$ let

$$\Gamma^{\Phi} := \mathrm{cone}\{ e_F \mid F \in \Phi \}.$$

Given matroid M on $[n]$, let

$$\Gamma(M) = \{\Gamma^{\Phi}\}_{\Phi \in \Delta(\overline{\mathcal{L}}(M))}$$

Example: $U_{2,3}$

$\mathcal{L}(M)$ 

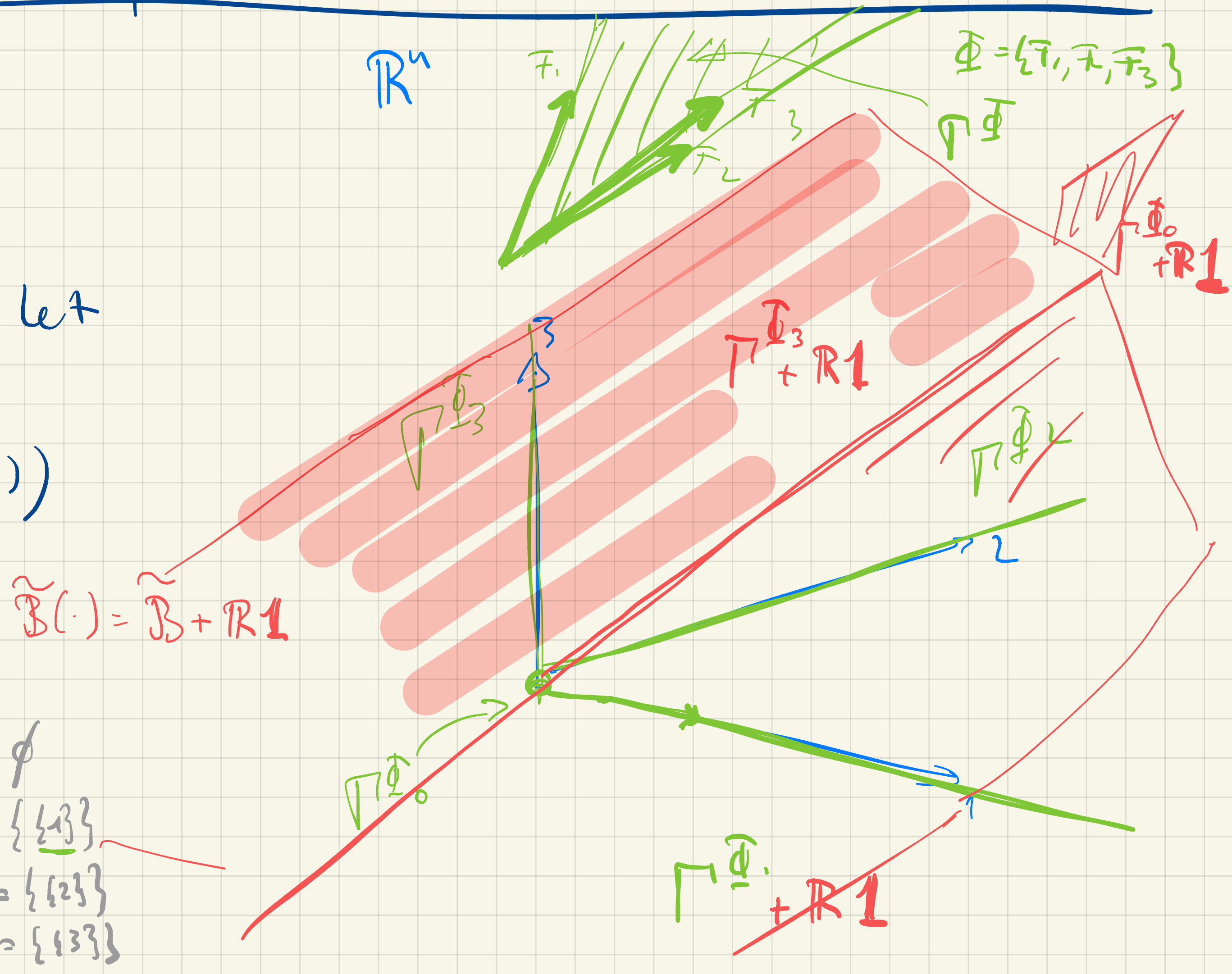
$\overline{\mathcal{L}}(M)$

↳ chains: $\Phi_0 = \emptyset$

$\Phi_1 = \{\{1\}\}$

$\Phi_2 = \{\{2\}\}$

$\Phi_3 = \{\{3\}\}$

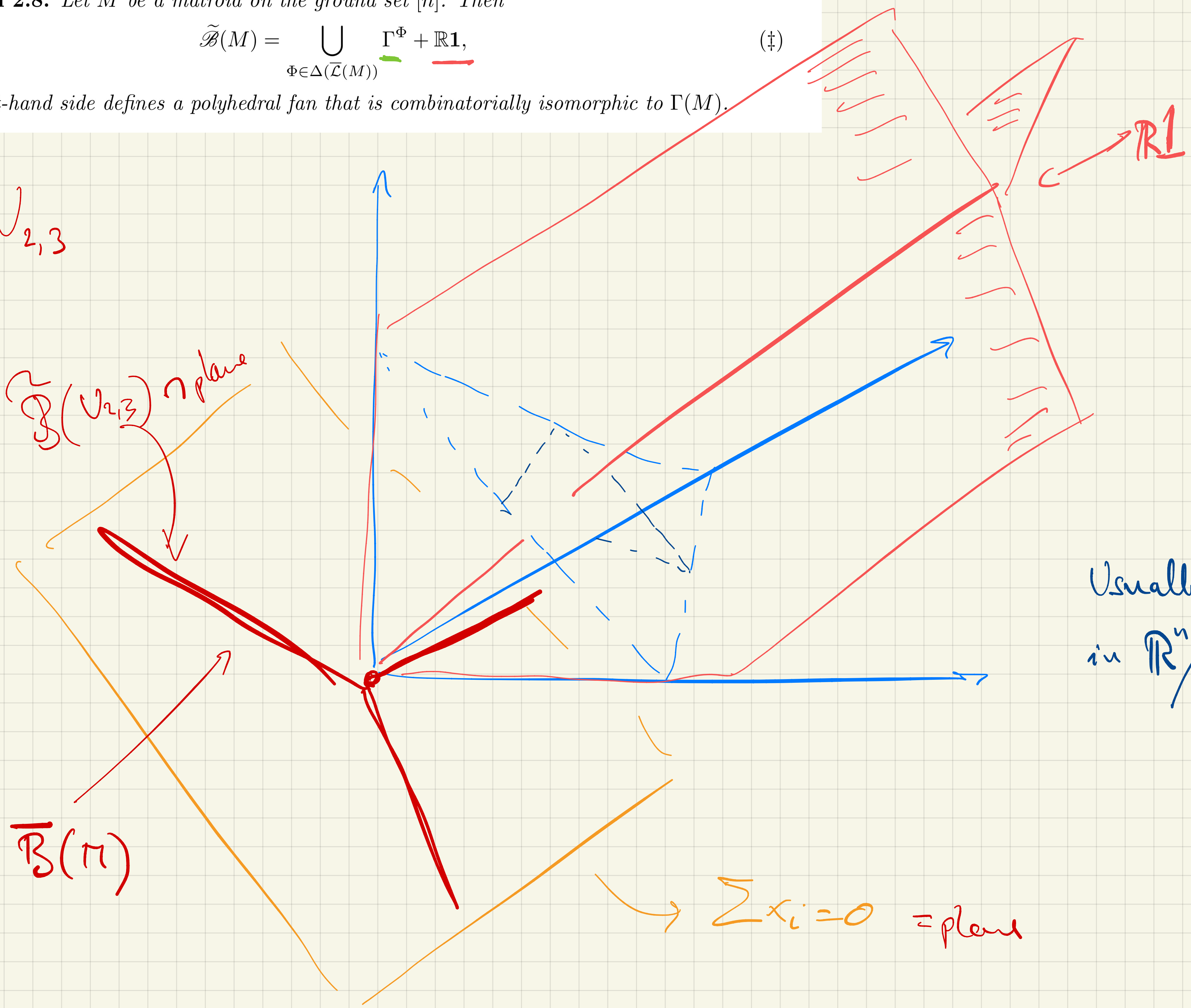


Proposition 2.8. *Let M be a matroid on the ground set $[n]$. Then*

$$\widetilde{\mathscr{B}}(M) = \bigcup_{\Phi\in\Delta(\overline{\mathcal{L}}(M))} \Gamma^{\Phi} + \mathbb{R}\mathbf{1}, \qquad (‡)$$

and the right-hand side defines a polyhedral fan that is combinatorially isomorphic to $\Gamma(M)$.

$U_{2,3}$



Usually: in $\mathbb{R}^n / \mathbb{R}\mathbb{1}$

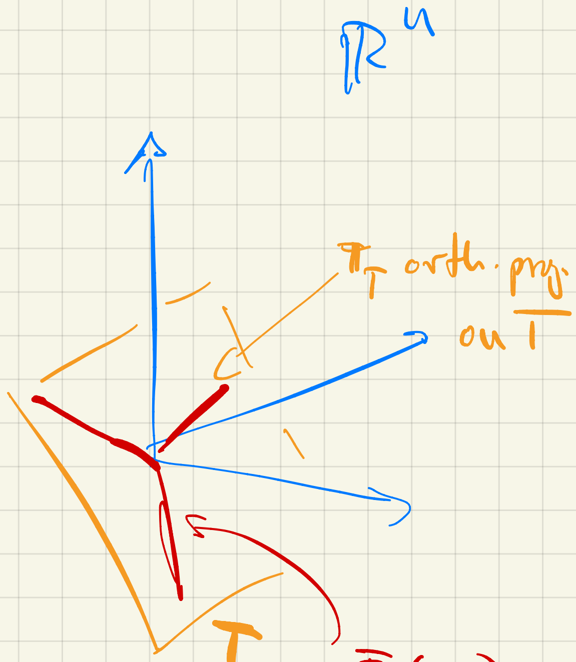
Remark-Definition 2.10. From Proposition 2.8 we have that translation by $\mathbf{1}$ preserves $\widetilde{\mathscr{B}}(M)$ and its fan structure. Thus there is no loss of information in considering, as one often does in tropical geometry, the Bergman fan as a subset of the quotient $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. In order to study this situation let π_T denote the orthogonal projection onto the hyperplane $T = \mathbf{1}^\perp$ (with equation $\sum_{i\in[n]} x_i = 0$), and let

$$\overline{\mathscr{B}}(M) := \pi_T(\widetilde{\mathscr{B}}(M)). \quad = T \cap \widetilde{\mathscr{B}}(M)$$

Lemma 2.11. *We have*

$$\overline{\mathscr{B}}(M) = \bigcup_{\Phi\in\Delta(\overline{\mathcal{L}}(M))} \pi_T(\Gamma^\Phi),$$

where the union on the right-hand side defines a (simplicial) fan structure that refines the coarse structure given by $\{\pi_T(N_{[P_M\uparrow w]})\}_{w\in T}$ *(see Equation (†)).*

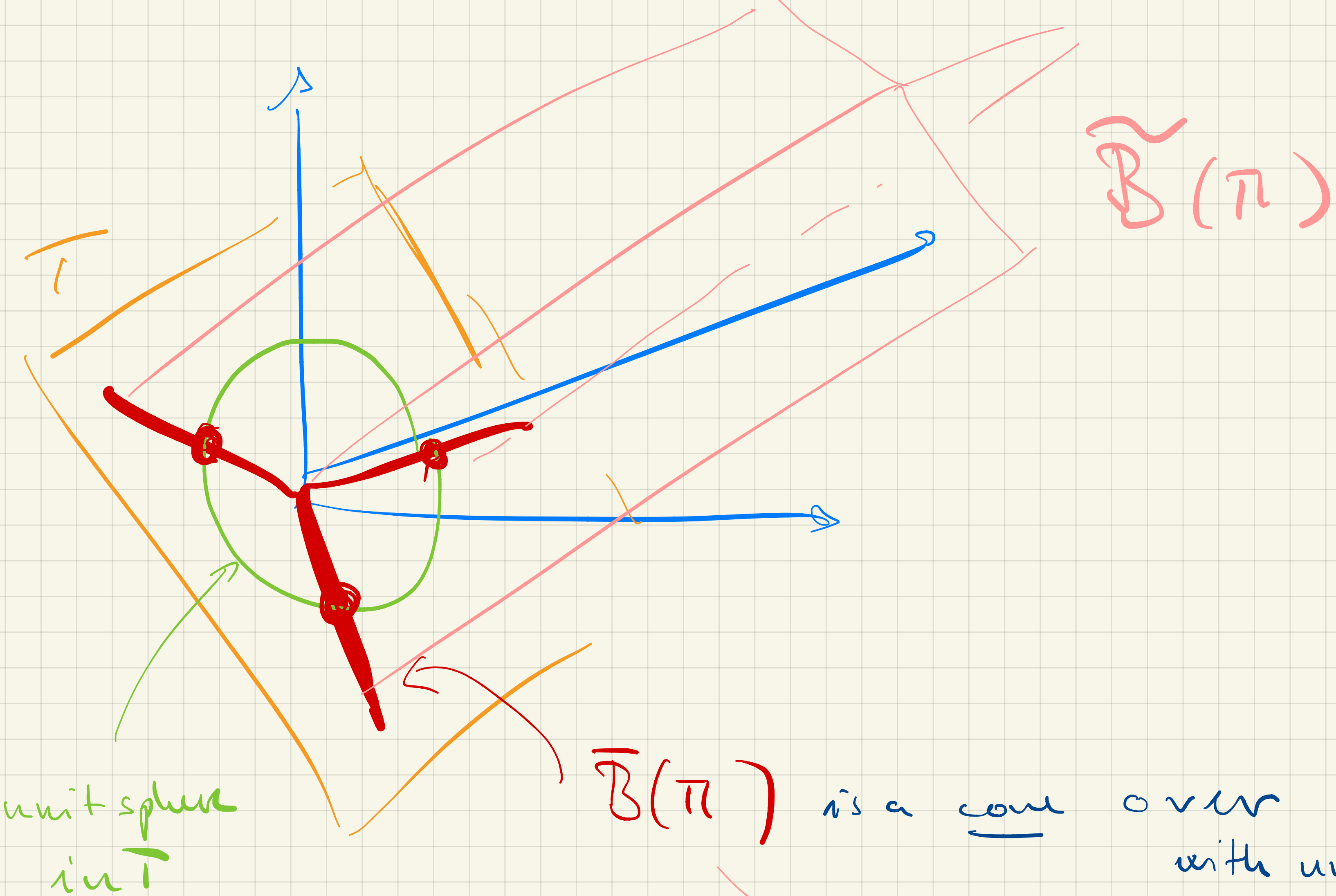


$\overline{\mathscr{B}}(M)$ is a fan in T.

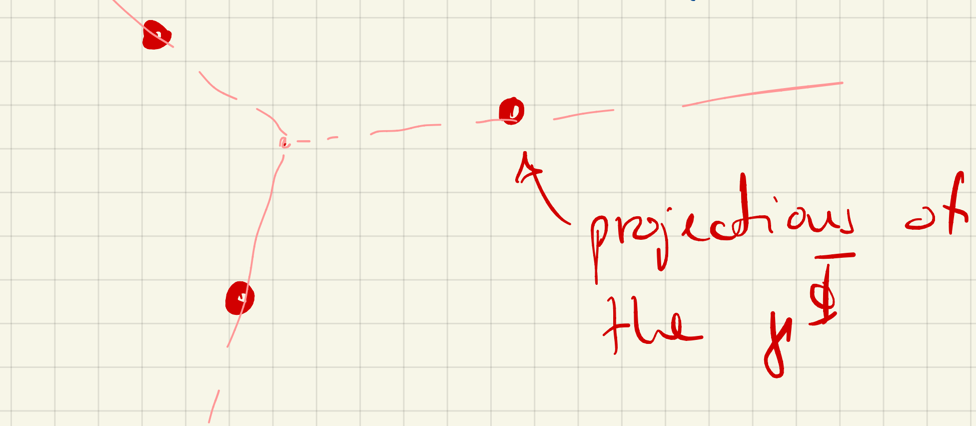
Recall proof of prop. 2.8 (just read.)

Bounding half. of $\Gamma^\phi + \mathbb{R}\mathbf{1}$ are $\langle x \mid a_\psi \rangle \leq 0$
$\in T$

$\rightarrow$ the <u>same</u> vectors a_ψ define bounding halfspaces for the cones $\pi_T(\Gamma^\phi) \subseteq T$.



$\bar{B}(\pi)$ is a cone over its inters. with unit sphere in T.



Remark-Definition 2.12. The fan $\overline{\mathscr{B}}(M)$ is the cone over a cell complex denoted by $\mathscr{B}(M)$ and called the *Bergman complex* of M (one way to see this is to think about $\mathscr{B}(M)$ as the intersection of $\widetilde{\mathscr{B}}(M)$ with the unit sphere in T). In order to express this complex, for a given family $\Phi \subseteq 2^{[n]}$ of subsets of $[n]$ let

$$\gamma^{\Phi} := \operatorname{conv}\{e_F \mid F \in \Phi\}, \qquad \gamma(M) := \bigcup_{\Phi \in \Delta(\overline{\mathcal{L}}(M))} \gamma^{\Phi}.$$

Proposition 2.13. *Let M be a matroid.*

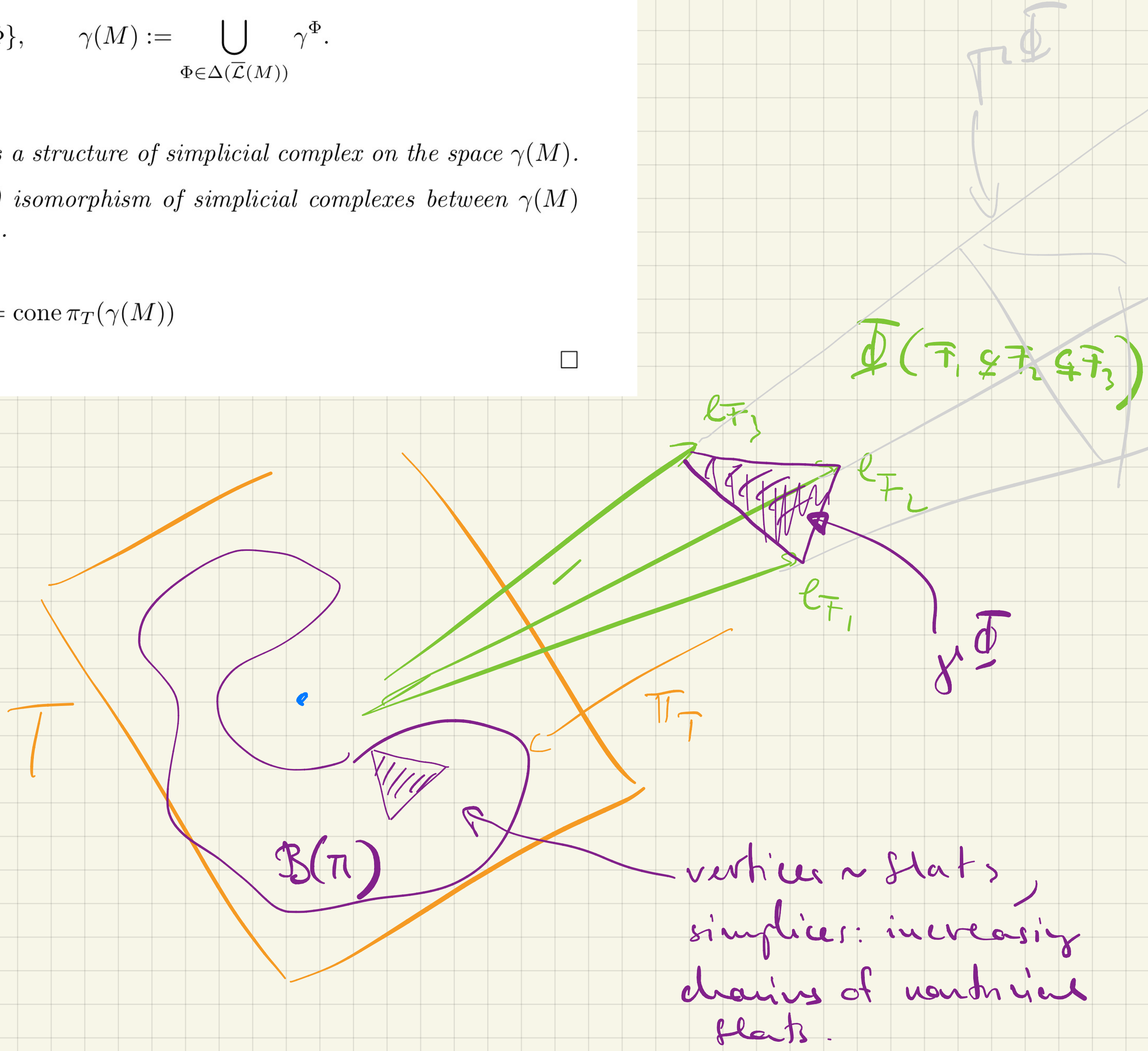
(1) The collection $\{\gamma^{\Phi}\}_{\Phi \in \Delta(\overline{\mathcal{L}}(M))}$ defines a structure of simplicial complex on the space $\gamma(M)$.

(2) The projection π_T induces a (linear) isomorphism of simplicial complexes between $\gamma(M)$ and $\pi_T(\gamma(M)) = \bigcup_{\Phi \in \Delta(\overline{\mathcal{L}}(M))} \pi_T(\gamma^{\Phi})$.

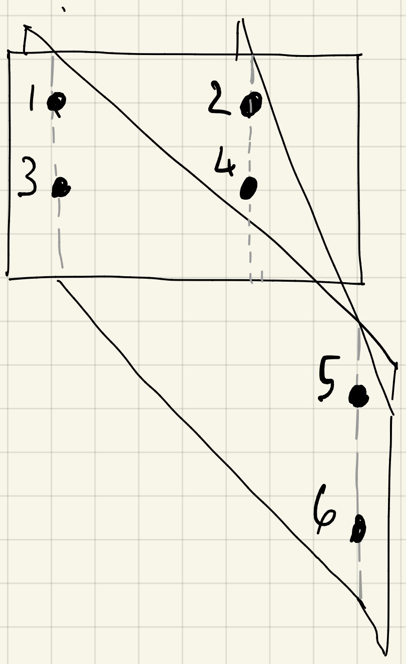
(3)

$$\overline{\mathscr{B}}(M) = \operatorname{cone} \pi_T(\gamma(M))$$

Proof. Exercise. □

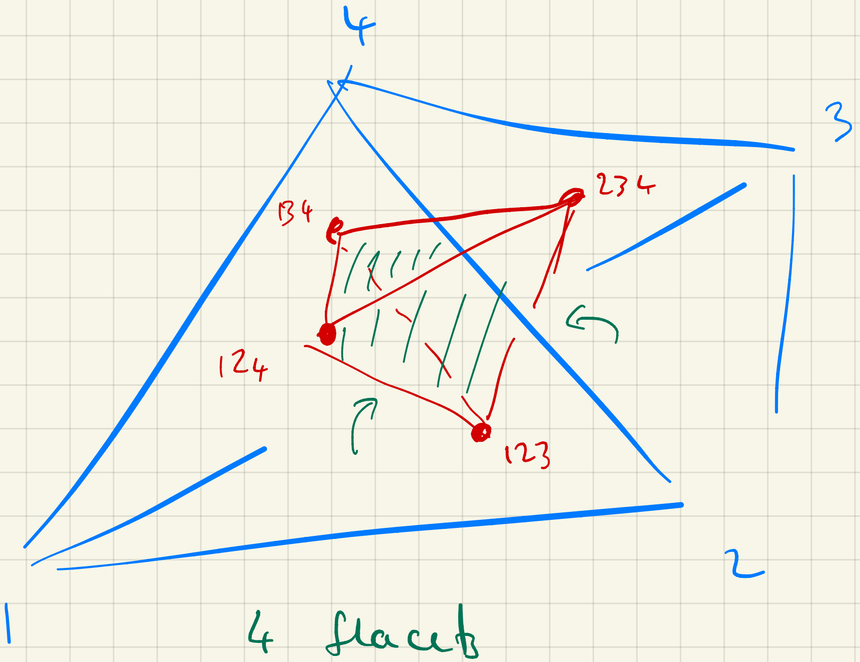


EXAMPLE



Ex: $U_{3,4}$

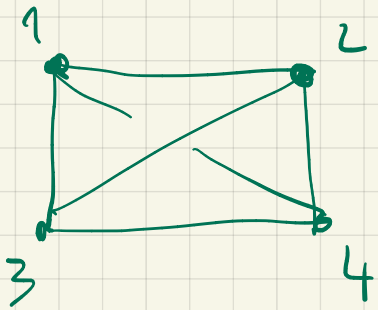
$\mathcal{B}(\pi)$: vertices = rays of $\overline{\mathcal{B}}(\pi)$,
= lowest-dim. cones in $\widetilde{\mathcal{B}}(\pi)$
~ Flacets.



4 flacets

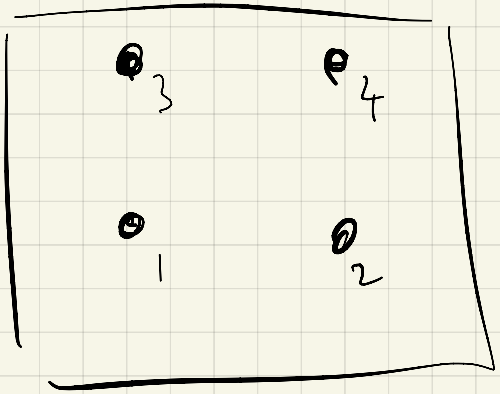
Higher dim. cells ⟷ lower dim. faces of P_π that intersect interior of P_π

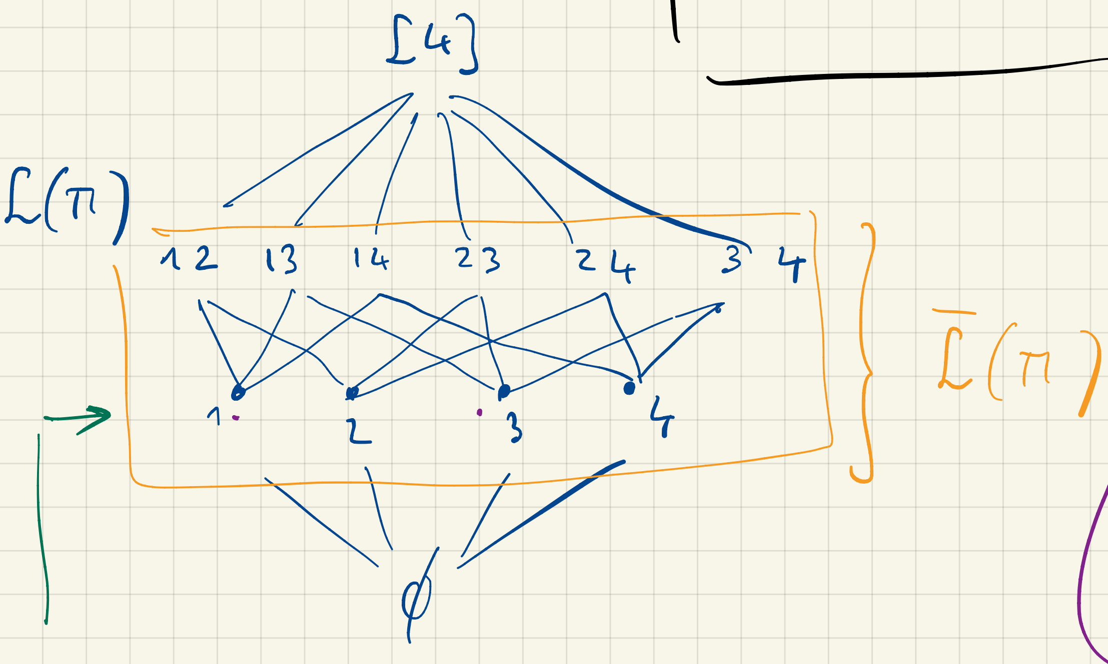
$\mathcal{B}(\pi)$:

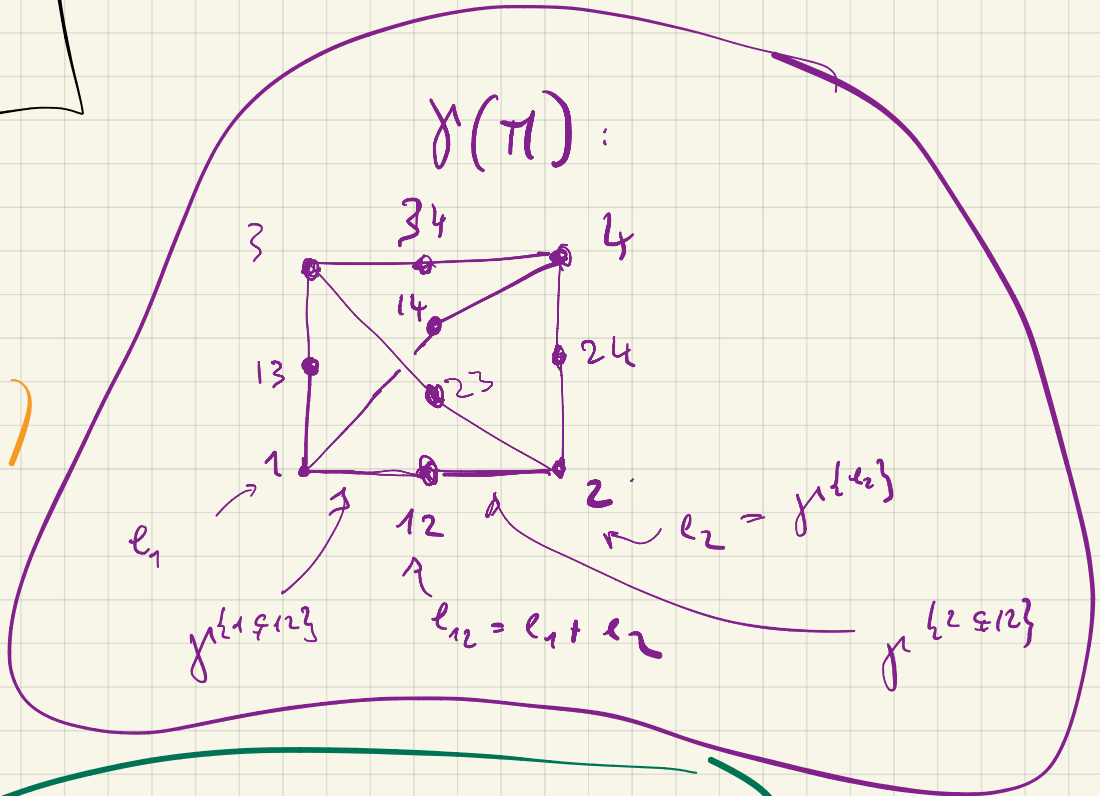


Flats of $U_{3,4}$

Exercise: do $U_{3,5}$







Flacets