M – matroid, ground set $[n]$, E

Definition

$$\operatorname{trop}(M) := \left\{ w \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \forall\, C \in \mathcal{C}(M) \;\; \exists\, i, j \in C,\ i \neq j \\ \text{s.t. } w_i = w_j = \min\{w_\ell \mid \ell \in C\} \end{array} \right\}$$

"minimum is attained twice"

Theorem $\boxed{\operatorname{trop}(M) = \widetilde{\mathcal{B}}(M)}$

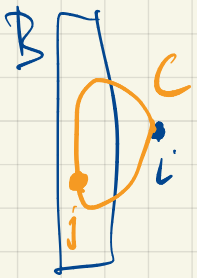
Bases = bases of M of max. w-weight

Pf: Must prove $w \in \operatorname{trop}(M) \iff M_w$ loopless

Note: $w \notin \operatorname{trop}(M) \iff \exists\, C \in \mathcal{C}(M), \exists\, i \in C : w_i < w_j \quad \forall j \in C \setminus \{i\}$

- If M_w has a loop, say i, then no basis of M_w contains i

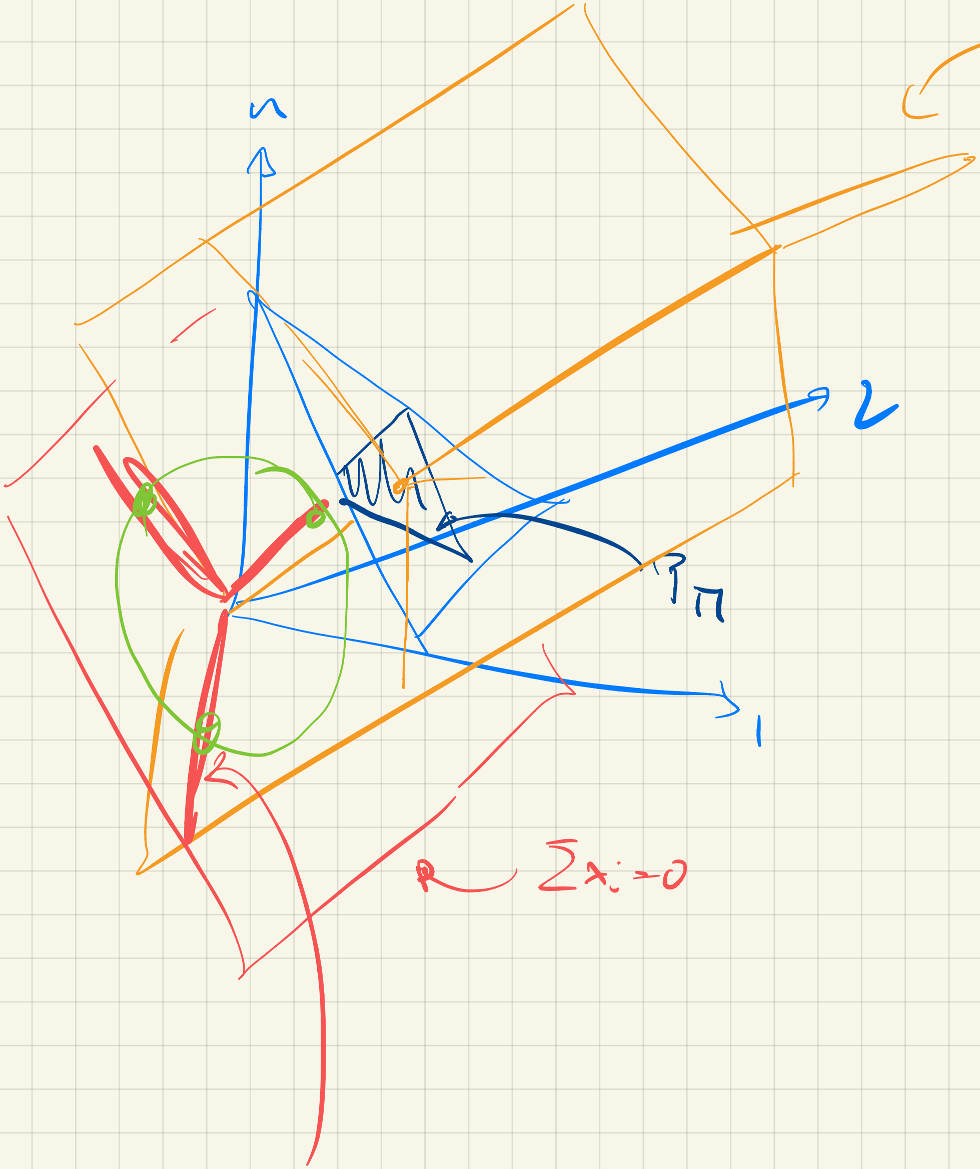
Take such a Basis. B. Then:



$$B \cup \{i\} \supseteq C \quad \text{for some circuit, } i \in C$$

Claim: $w_i < w_j \;\; \forall j \in C \setminus \{i\}$. In fact: if not, can exchange i for j and get basis of higher w-weight. ∎

$B \setminus \{j\} \cup \{i\}$ basis of M



$\widetilde{B}(\pi)$ = w's with Π_w loopless

$U_{3,5}$

"$B(\pi) \subseteq \overline{B}(\pi) \subseteq \widetilde{B}(\pi)$"

cone over

$\mathbb{R}_+ *$

here: $B(\pi)$

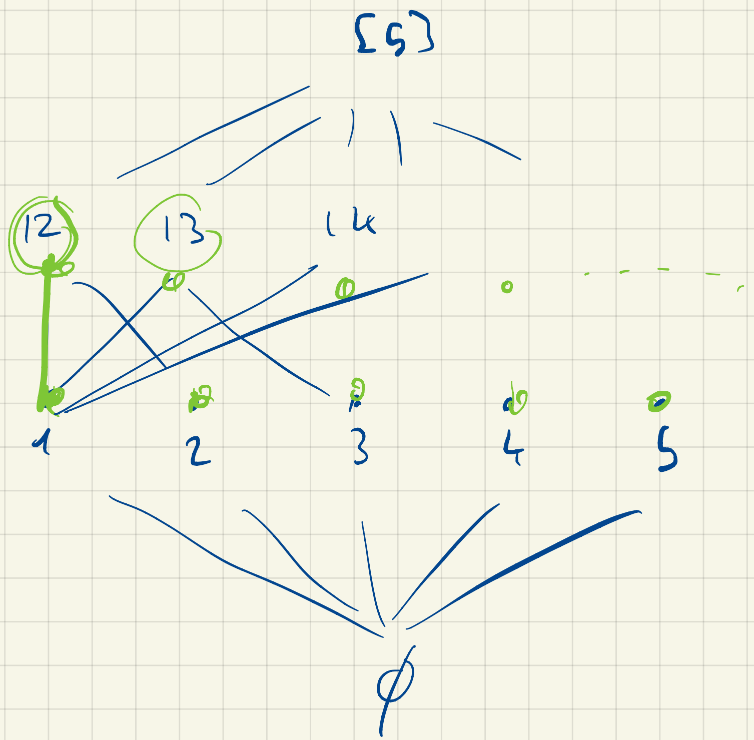
$\overline{B}(\pi)$

$B(\pi) = \overline{B}(\pi) \cap$ unit sphere

For "computations":

flats

Example: $M = U_{3,5}$



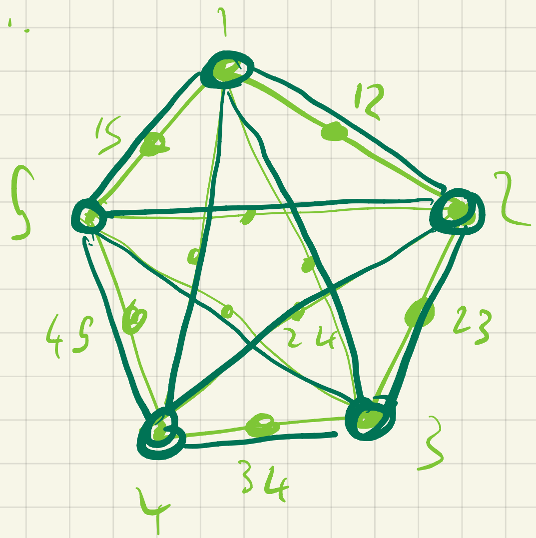
Last time:

$\mathcal{B}(M)$ "is subdivided by" the complex of chains of proper flats

$\neq$ minimal.
$\neq$ maximal

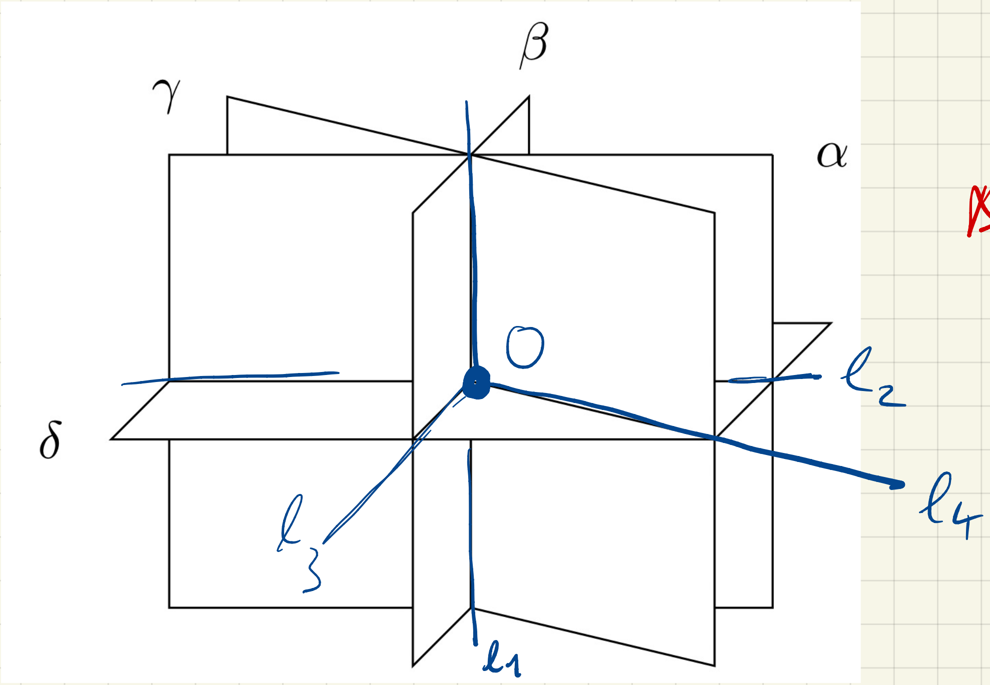
Here:

$\mathcal{B}(M)$:



complex of chains of flats.

$\mathcal{A}$:



"Arrangement of hyperplanes"

Let V a $\mathbb{K}$-vectorspace.

Ar. of H. is a finite

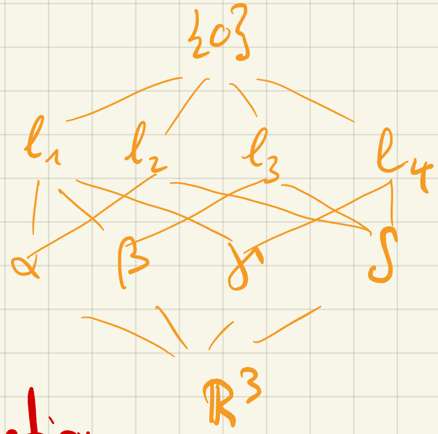
$$\mathcal{A} = \{H_1, \dots, H_n\}$$

H_i is linear subspace of codim. 1, i.e.

$$H_i = \{x \mid a_i^T x = 0\}$$

for some $a_i \in V$

$\mathcal{L}(\mathcal{A})$:

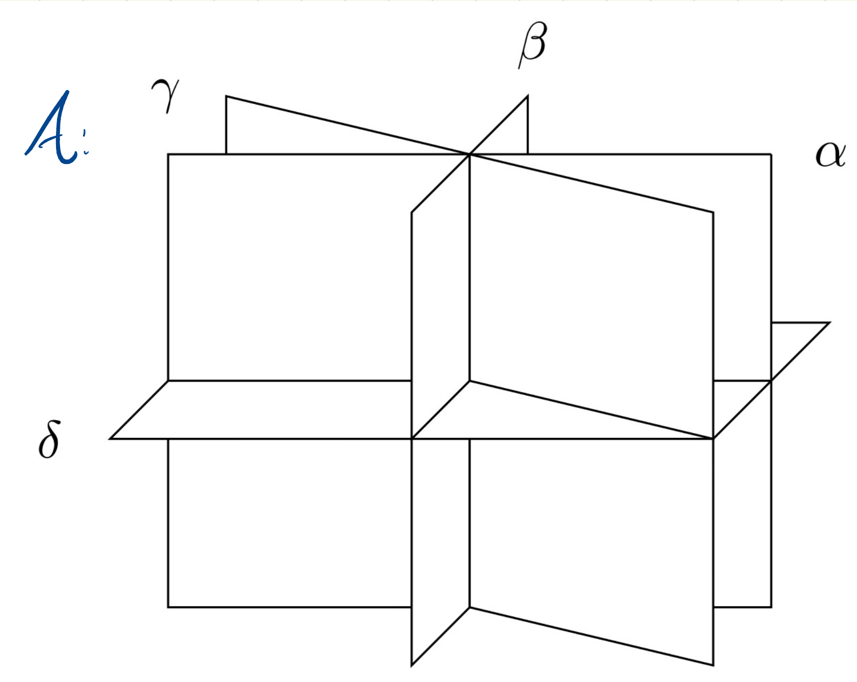


Poset of intersections:

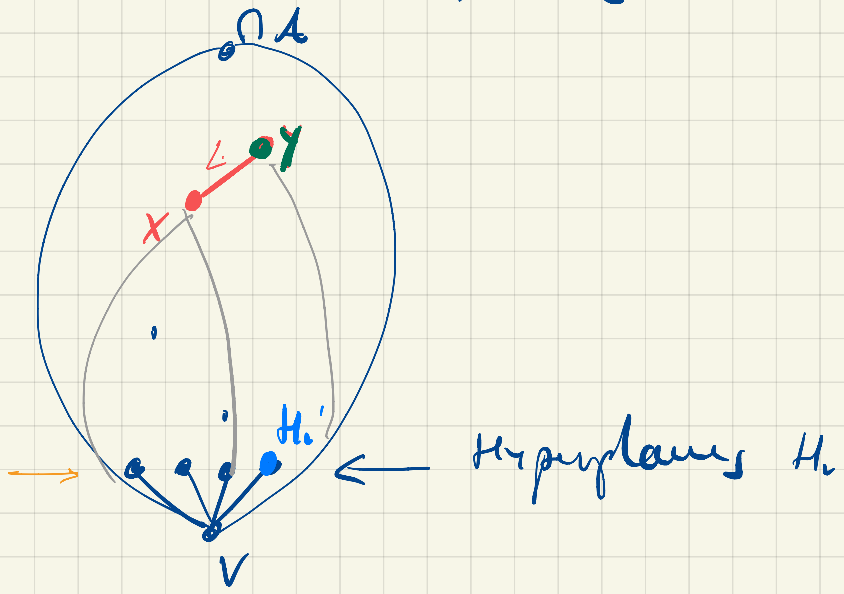
$$\mathcal{L}(\mathcal{A}) = \left\{ \bigcap_{i \in I} H_i \;\middle|\; I \subseteq [n] \right\}, \quad X \leq Y \iff X \supseteq Y$$

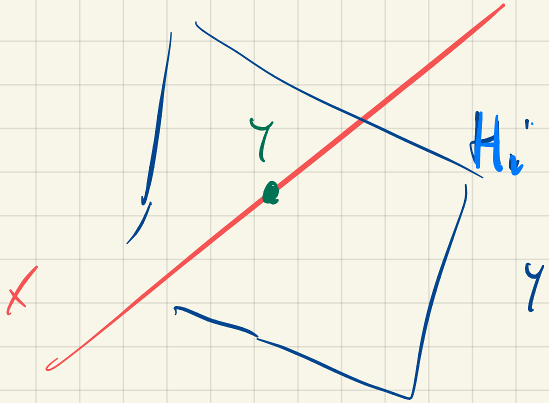
$\mathcal{A}$ – arrgt. of hyperplanes,

$L(\mathcal{A})$: poset of intersections



$L(\mathcal{A})$:





$y = H_i \cap x$

$H_i \not\leq x$ (otherwise $x \subseteq H_i$)

$H_i \leq y$

$$x \lessdot y \iff \exists \text{ ~~hyperpl.~~ atom } H_i,\ H_i \not\leq x, \text{ s.t. } y = \underbrace{H_i \cap x}_{H_i \vee x}$$

"$x \leq z < y \Rightarrow x = z$"

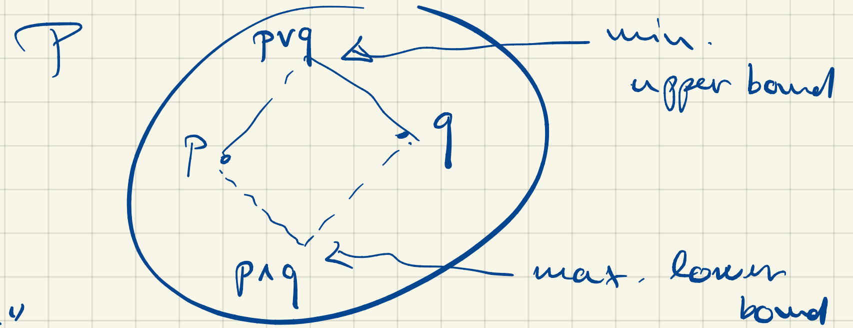
"Partially ordered set" (poset) : $(P, \leq)$

set

(reflexive, antisymmetric, transitive relation)

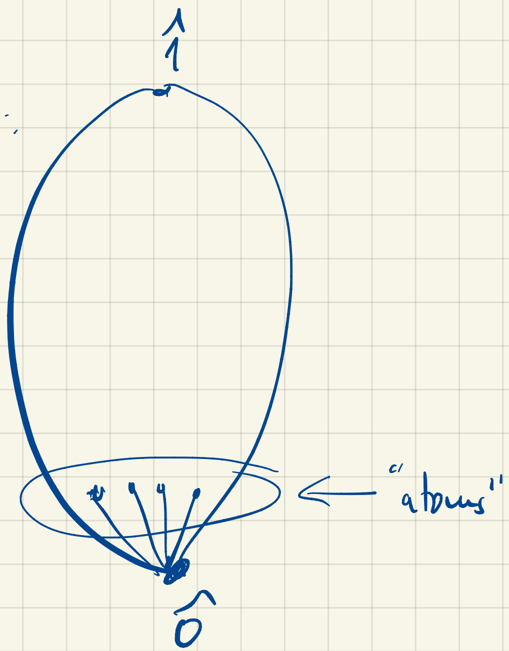
<u>Def</u>: Call P a <u>lattice</u> if, $\forall p, q \in P$,

- $P_{\geq p} \cap P_{\geq q}$ has unique min. element, $p \vee q$ "join"
- $P_{\leq p} \cap P_{\leq q}$ " " max. " $p \wedge q$ "meet"



<u>Note</u>: Every <u>finite</u> lattice has a unique maximal ($\hat{1}$) & a unique minimal ($\hat{0}$) element

<u>Lattice</u>:



<u>Example</u>: $\mathcal{L}(\mathcal{A})$ is a lattice with $X \vee Y = X \cap Y$,

$\hat{0} = V$, $\hat{1} = \bigcap \mathcal{A}$.

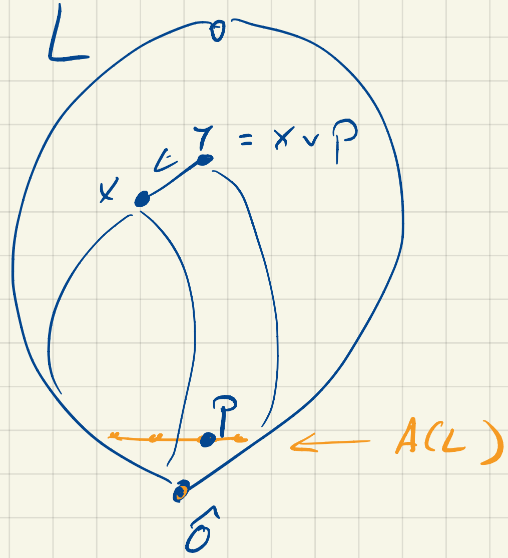
<u>Def</u>. If a poset P has $\hat{0}$, call <u>atoms</u> of P the elements of

$$A(P) := \{ p \in P \mid p \gtrdot \hat{0} \}$$

Definition Let L be a finite lattice.

Call L a *geometric lattice* if, $\forall x, y \in L$

(G) $x \lessdot y \iff \exists\, p \in A(L),\ p \not\leq x$, s.t. $y = x \vee p$



Example: $\mathcal{L}(A)$.

$\to$ Matroids "are" geometric lattices

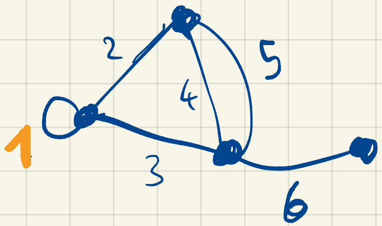
Matroid M, rk $\xrightarrow{\text{take flats}}$ $\mathcal{L}(M) \subseteq 2^{[n]}$, st., ...

$F_1 \leq F_2 \iff F_1 \subseteq F_2$

$\xleftarrow{\ ?\ }$

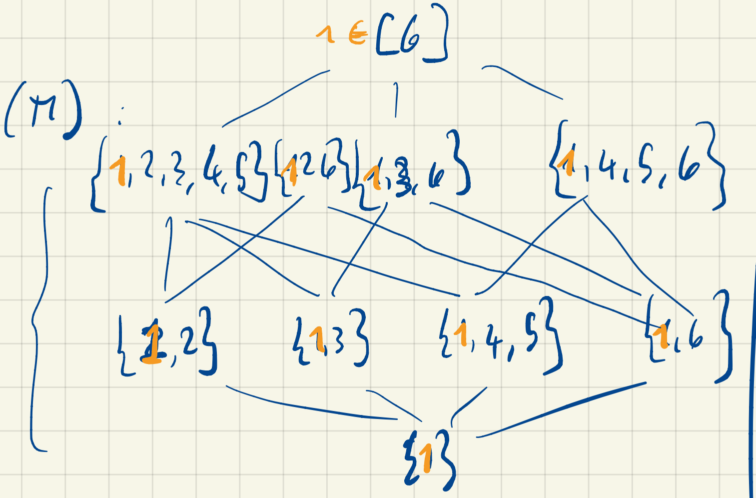
Example: Let $M = M(G)$

for G:



$rk(\{3,4\})$

$\mathcal{L}(M)$:



rk = 3

rk = 2

rk = 1

rk = 0

$\mathcal{L}(M[2,3,\dots,6])$ poset-isom. to $\mathcal{L}(\pi)$

$$rk(\{3,4\}) = rk(cl(\{3,4\})) = 2$$

Say M is <u>simple</u> (no loops & no parallel elements, like $\{4,5\}$),

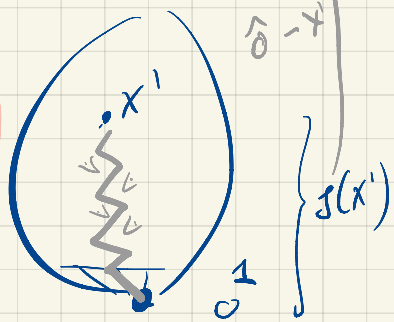
then atoms of $\mathcal{L}(M)$ are single-element sets.

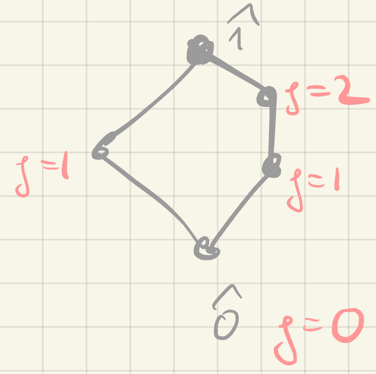
Then, given $X \subseteq [n]$ can read off $rk(X)$ by:

$X \longrightarrow$ smallest $X' \in \mathcal{L}(M)$ with $X \subseteq X'$ $\longrightarrow$ look "on what floor" X' lies

"$X' = cl(X)$"

length of any max. chain $\hat{0} - X'$





Def: Let P be poset. a <u>rank function</u> on P is $\rho: P \to \mathbb{N}$ s.t. (i) $\rho(x) = 0$ if x is a minimal element

(ii) $\rho(x)+1 = \rho(y)$ if $x \lessdot y$

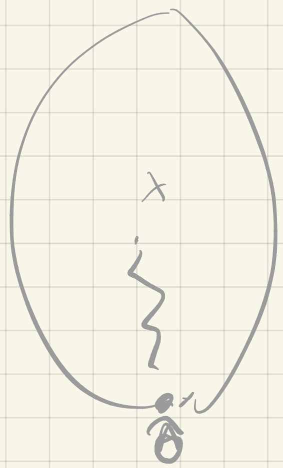
if a rank fn. exists, call P <u>ranked</u>

(Notice: if P has $\hat{0}$ & a rank function ρ, then ρ is unique)

Def A <u>chain</u> in P is any subset $\omega = \{x_0 < x_1 < \dots < x_n\}$. Its <u>length</u> is $\ell(\omega) = |\omega| - 1$

<u>Lemma 3.3</u> In a geometric lattice any two max. chains between the same elements have same length

(so: Geom. l. are ranked &c....)



$\rho(x) :=$ length of any maximal $\hat{0}$ - x chain

Lemma.

Let L be a geometric lattice with rank function ρ $\left(\begin{array}{l}\rho(x) = \text{length} \\ \text{of max. } \hat{0}-x \text{ chain}\end{array}\right)$

Then, for all $x, y \in L$

$$\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y)$$

Proof: let $\overset{x \wedge y}{\overset{\shortparallel}{z_0}} < z_1 < \cdots < \overset{y}{\overset{\shortparallel}{z_k}}$ saturated chain,

then $\quad k = \rho(y) - \rho(x \wedge y)$.

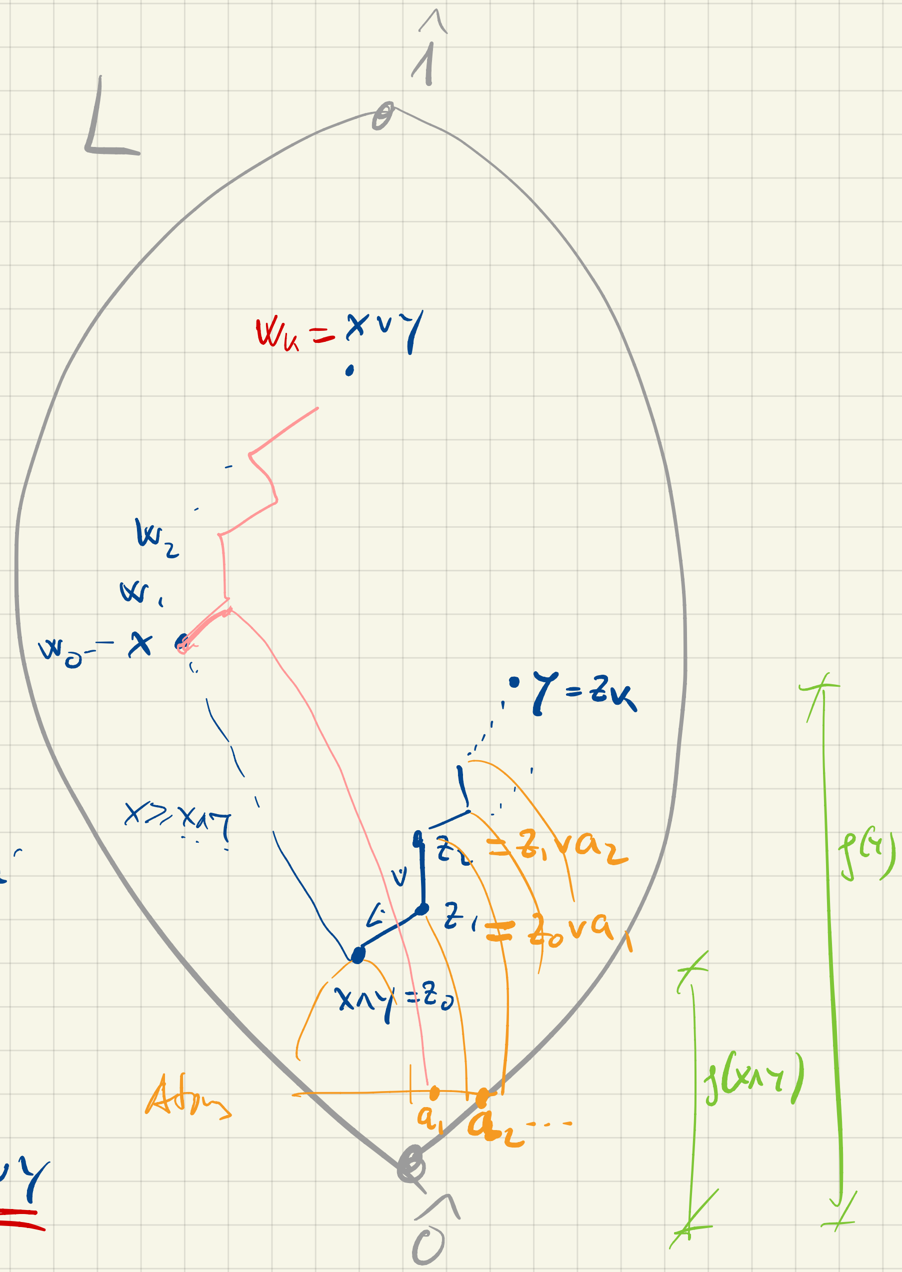
By (G) there are atoms $a_1, a_2 \ldots$ s.t. $z_i = z_{i-1} \vee a_i \;\; \forall i$

Define new $w_0 \leq w_1 \leq \cdots w_k$ by $\begin{array}{l} w_0 = x \\ w_i = w_{i-1} \vee a_i \end{array}$

$$\Rightarrow \quad w_k = x \vee a_1 \vee a_2 \cdots \vee a_k = x \vee \underbrace{(x \wedge y) \vee a_1 \vee \cdots \vee a_k}_{z_k = y} = x \vee y$$

By (G) for all i either $w_i = w_{i-1}$ or $w_i \gtrdot w_{i-1}$

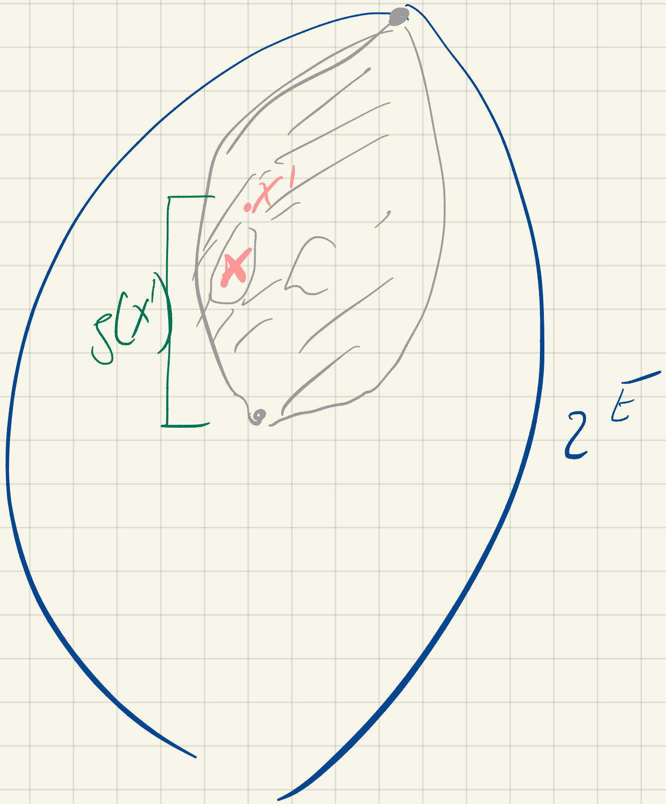
Thus $\rho(x \vee y) - \rho(x) = \text{length}\{w_0 \leq w_1 \leq \cdots \leq w_k\} \leq k \stackrel{\text{Ass.}}{=} \rho(y) - \rho(x \wedge y)$ ☒



$$\text{Matroid} \rightleftarrows \text{Geom. lattices}$$

Proposition 3.7. *Let E be a finite set and let $\mathcal{L} \subseteq 2^E$ a family of subsets of E, partially ordered by inclusion and such that $E \in \mathcal{L}$. Suppose further that $\mathcal{L}$ is a geometric lattice with rank function ρ,with meet operation given by set intersection, and such that the union of the atoms of $\mathcal{L}$ equals E. Then, for every $X \subseteq E$ there is a unique minimal X' in $\mathcal{L}$ such that $X \subseteq X'$, and the extension r of ρ on 2^E given by $r(X) := \rho(X')$ is a matroid rank function.*

$$rk(X) := \rho(X')$$



For "Matroid $\longrightarrow$ Geom. lattice",

Given matroid M let $\mathcal{L}(M)$ the set of flats, ordered by inclusion.

Then, for $F_1, F_2 \in \mathcal{L}(M)$:

$$F_1 \vee F_2 = cl(F_1 \cup F_2) \qquad F_1 \wedge F_2 = F_1 \cap F_2 \qquad \longleftarrow \text{Lecture 2}$$

So $\mathcal{L}(M)$ is a lattice, and it is geometric (Prop. 3.13)

$\mathcal{A} = \{H_1 \cdots H_n\}$ arrgt. of hyps., $\{a_1 \cdots a_n\}$

say $H_i = \{a_i^T x = 0\}$

today $\Big\downarrow$

$\mathcal{L}(\mathcal{A})$ geom. lattice, rank fn. ρ, induces a matroid on $[n]$, with rank function

r_{lat}

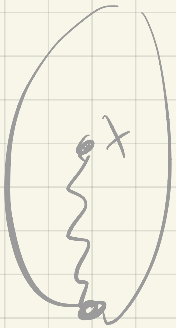
$\left(r_{lat}(X) = \operatorname{codim}(X) \right)$

long ago $\Big\downarrow$

Matroid of linear dependencies, with rank function

r_{dep}

$\left(r_{dep}(X) = \dim \operatorname{span}(X) \right)$



IN FACT: $\boxed{r_{lat} \equiv r_{dep}}$

"the one and only matroid"

Def Let $\mathcal{A}$ be arrangem. of hyp in V. Then $M(\mathcal{A}) := V \setminus \cup \mathcal{A}$

(– interesting e.g. over $\mathbb{C}$)

Remember. $\mathcal{A} = \{H_1 \dots H_n\}$, "normal vectors" $a_1 \dots a_n$, arrgt. in $\mathbb{K}^d$, say $\cap \mathcal{A} = \{0\}$

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \text{———} & b_1 & \text{———} \\ & \vdots & \\ \text{———} & b_d & \text{———} \end{bmatrix}$$

Then A is full-rank, and so is A^t

Now $A^t : \mathbb{K}^d \longrightarrow \mathbb{K}^n$ is injective, with image
$V := \operatorname{im} A^t = A^t \mathbb{K}^d$

$\mathcal{A}$

Notice: $x \in H_i$ iff $a_i^t x = 0$, iff $(A^t x)_i = 0 \iff A^t x \in V \cap \{y_i = 0\}$

ith coordinate plane

So: $x \in M(\mathcal{A})$ iff. $A^t x \in \left(V \setminus \bigcup_i \{y_i = 0\} \right) = (\mathbb{K}^*)^n$

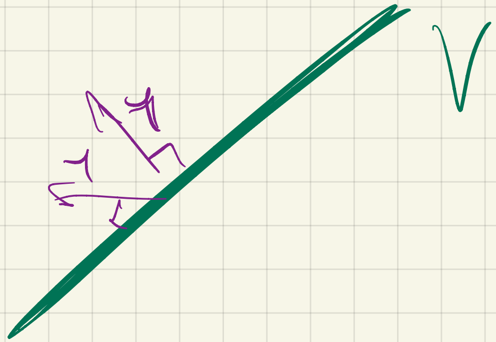
Can we characterize $V = \operatorname{im} A^t$ as solution of set of equations?

So look at $V = \operatorname{im} A^t = \operatorname{im} \begin{bmatrix} | & & | \\ b_1 & \cdots & b_d \\ | & & | \end{bmatrix}$

For $y \in \mathbb{R}^n$ we have $y^t V$ iff $y \in \ker(A)$

$$A = \begin{bmatrix} | & & | \\ a_1 & \text{———} & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \text{ so } y \in \ker(A) \text{ iff } \sum y_i a_i = 0$$

in particular, if $y \in \ker(A)$, then $\{i \mid y_i \neq 0\}$ is a dep. set in the matroid $M(A)$



Ultimately:

$$V = \left\{ \sum y_i x_i = 0 \;\middle|\; \text{~~for some } y \in \ker A \text{ (a basis suffices)~~} \right\}$$

a generating set for $\ker(A)$ is $\{\underline{\underline{v(C)}} \mid C \text{ circuit of the matroid}\}$?

<u>In fact</u> if C is circuit, then there is a vector $v(C) \in \ker(A)$ with $C = \{i \mid v(C)_i \neq 0\}$, and $v(C)$ unique up to scalar mult!

Proof: let $\lambda_1 a_1 + \dots + \lambda_n a_n = 0$ minimal-support linear dependency.
$\lambda_i \neq 0$
(pertains to circuit $\{1, 2, \dots, n\}$)

Any other lin. dependency supported on $1, \dots, n$:

$$\mu_1 a_1 + \dots \mu_n a_n = 0 \qquad \mu_i \neq 0$$

Look at substraction $\lambda_1 a_1 + \dots + \lambda_n a_n = 0$

$$- \quad \frac{\lambda_1}{\mu_1} \mu_1 a_1 + \frac{\lambda_1}{\mu_1} \mu_2 a_2 + \dots \frac{\lambda_1}{\mu_1} \mu_n a_n = 0$$

$$0$$

must be all zero, otherwise this is lin. dependency "smaller" than C

$\Rightarrow$ μ is a scalar multiple of λ.

Summary:

$$M(A) \simeq V \cap (\mathbb{K}^*)^n$$

$\uparrow$ im A^t

and is defined inside $(\mathbb{K}^*)^n$ by

$$\sum_i v(C)_i x_i = 0 \quad \text{, one for each circuit } C$$

taking minimum $\qquad x_i \qquad$ "min. is attained twice"

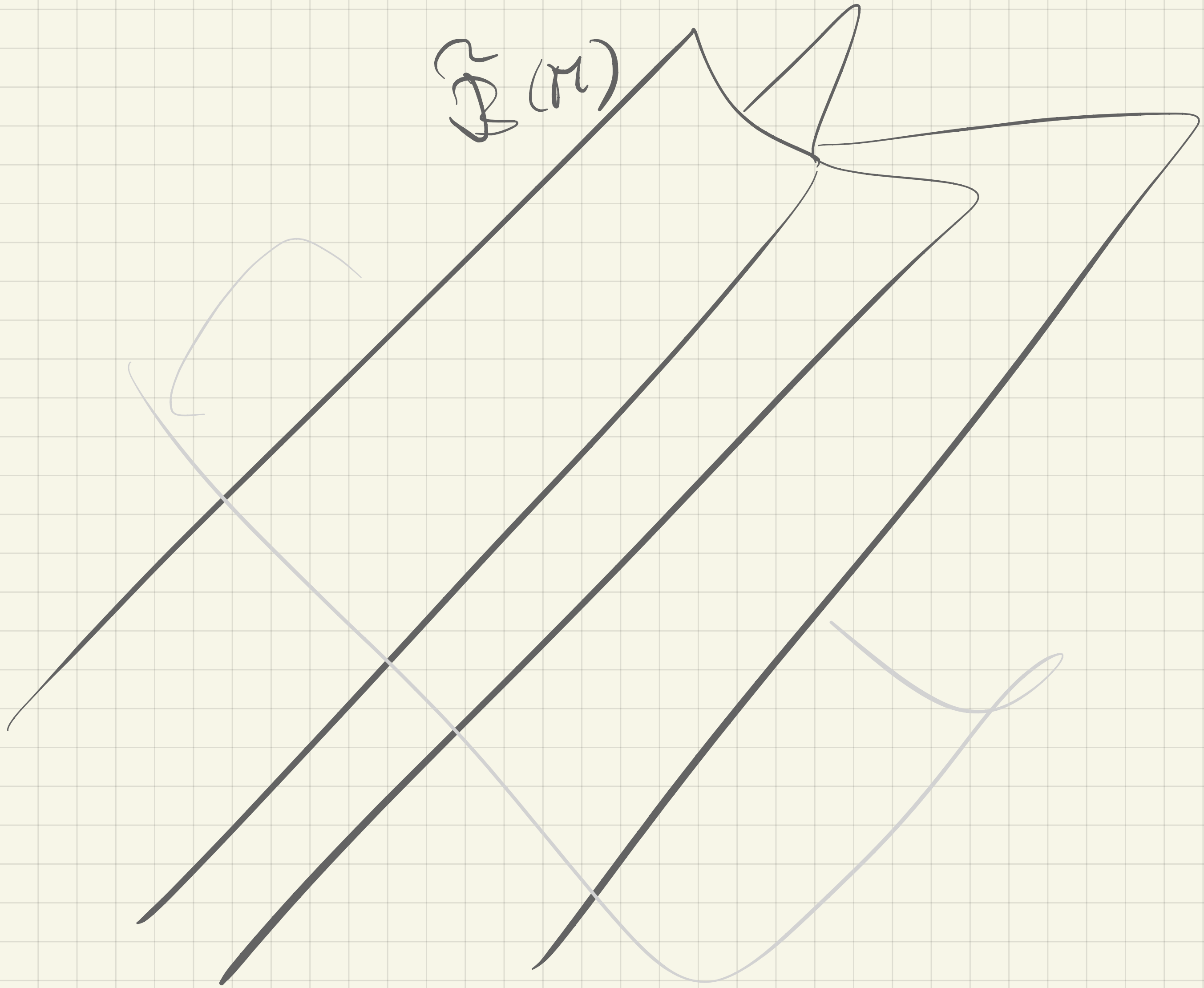
Now let $\mathbb{K} = \mathbb{C}$

$$\mathrm{Trop}(M(A)) = \mathrm{trop}(M) = \mathcal{B}(M).$$

anything like this, where M is any matroid, is a tropical linear space.

Those that arise from ($\mathbb{C}$-)representable matroids are only some of them (called "tropicalized linear spaces")


$\hat{1}$
$\hat{0}$


$\mathfrak{B}(M)$