NOTES FOR LECTURE 4

1. The tropical space associated to a matroid

Definition 1.1. Let M be a matroid on the ground set $[n]$. The associated tropical space is the set

$$\mathrm{trop}(M) := \left\{ w \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \text{For every circuit } C \in \mathcal{C}(M) \text{ there are } i, j \in C, i \neq j \\ \text{such that } w_i = w_j = \min\{w_i \mid i \in C\} \end{array} \right\}$$

One commonly says that $\mathrm{trop}(M)$ is the set of vectors for which "the minimum is attained twice on every circuit".

Theorem 1.2. *Let M be a matroid on the ground set $[n]$. Then,*

$$\mathrm{trop}(M) = \widetilde{\mathscr{B}}(M)$$

Proof. For all $w \in \mathbb{R}^n$, $w \notin \mathrm{trop}(M)$ if and only if there is a circuit C of M and an element $i \in [n]$ with

$$w_i < w_j \text{ for all } j \in C \setminus \{i\}. \tag{1}$$

If this is the case, then for every basis B of M with $i \in B$ we can find $j \in C \setminus \{i\}$ such that $B' := B \setminus \{i\} \cup \{j\}$ is a basis of M. (Otherwise we would have $j \in \mathrm{cl}(B \setminus \{i\})$ for all $j \in C \setminus \{i\}$, hence $C \setminus \{i\} \subseteq \mathrm{cl}(B \setminus \{i\})$. Independence of B implies $i \notin \mathrm{cl}(B \setminus \{i\})$, hence we would have $i \notin \mathrm{cl}(C \setminus \{i\})$, a contradiction to C being a circuit.) Then B' has higher w-weight than B hence B cannot be a basis of M_w. We conclude that i would not be in any basis of M_w, hence it is a loop of M_w, witnessing $w \notin \widetilde{\mathscr{B}}(M)$.

Conversely, if i is not in any basis of M_w, then for every basis B of M_w we have that the circuit C contained in $B \cup \{i\}$ must satisfy (1), otherwise we could exchange i for another element of $C \cap B$ and obtain a basis of higher w-weight than B. □

2. Arrangements of hyperplanes and geometric lattices

Definition 2.1. Let V be vectorspace of finite dimension d. An *arrangement of hyperplanes* in V is a finite set

$$\mathscr{A} := \{H_1, \ldots, H_n\}$$

of codimension 1 linear subspaces of V. The poset of intersections of $\mathscr{A}$ is the set

$$\mathcal{L}(\mathscr{A}) := \left\{ \bigcap_{i \in I} H_i \;\middle|\; I \subseteq [n] \right\}$$

ordered by reverse inclusion: $X \leq Y$ if and only if $X \supseteq Y$.

Example 2.2. Let $\mathscr{A}$ be the arrangement in $\mathbb{R}^3$ consisting of the four planes

$$\alpha : \{x = 0\}, \quad \beta : \{y = 0\}, \quad \gamma : \{x = y\}, \quad \delta : \{z = 0\},$$

depicted in Figure 1. Then $\mathcal{L}(\mathscr{A})$ is the poset represented on the r.h.s. of Figure 1.

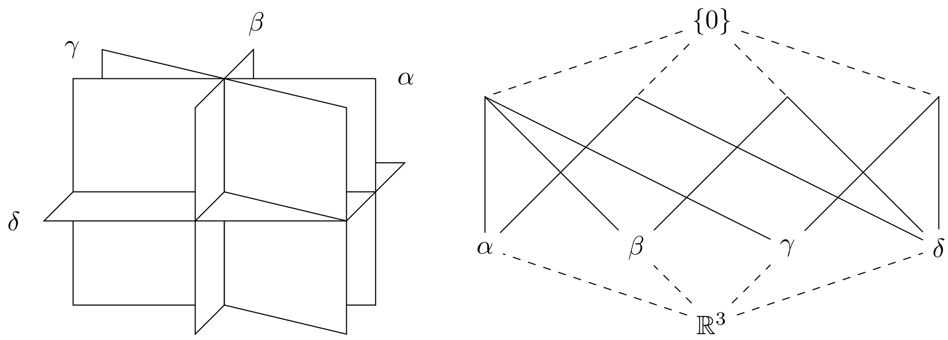


Figure 1

Definition 2.3. A partially ordered set is a pair $(P, \leq)$ where P is a set and $\leq$ is a *partial order*, i.e., an antisymmetric, transitive and reflexive relation on P. Often the relation $\leq$ is clear from the context and we speak simply of "the poset P".

Definition 2.4. A partially ordered set P is a *lattice* if, for any two elements $p, q \in P$,

- the subposet $P_{\geq p} \cap P_{\geq q}$ of all upper bounds to p and q has a unique minimal element - called *join* of p and q and denoted $p \vee q$, and
- the subposet $P_{\leq p} \cap P_{\leq q}$ of all lower bounds to p and q has a unique maximal element - called *meet* of p and q and denoted $p \wedge q$.

Notice that every finite lattice must have a unique minimal element (denoted by $\hat{0}$) and a unique maximal element (written $\hat{1}$).

Definition 2.5. Let P be a poset with a unique minimal element $\widehat{0}$ (we call such a P "bounded below"). Then the *atoms* of P are the elements of the set

$$A(P) := \{p \in P \mid p \gtrdot \widehat{0}\},$$

where here and in the following we write $x \gtrdot y$ when, for all z, $x \leq z < y$ implies $x = z$.

Recall that every finite lattice has a unique minimal element.

Definition 2.6. Let L be a finite lattice. We call L *geometric* if, for all $x, y \in L$:

(G) $x \lessdot y$ if and only if there is $p \in A(L)$, $p \not\leq x$, such that $y = x \vee p$.

Example 2.7. Unique least upper bounds exist in $\mathcal{L}(\mathscr{A})$ (for $X, Y \in \mathcal{L}(\mathscr{A})$ take $X \vee Y := X \cap Y$). Moreover, since $\mathcal{L}(\mathscr{A})$ is finite, this implies that unique greater lower bounds also exist (take $X \wedge Y := \vee\{Z \in \mathcal{L}(\mathscr{A}) \mid Z \leq X, Z \leq Y\}$). Thus, $\mathcal{L}(\mathscr{A})$ is a finite lattice.

Now, the atoms of $\mathcal{L}(\mathscr{A})$ are exactly the elements of $\mathscr{A}$, i.e., the hyperplanes. The other nontrivial elements of $\mathcal{L}(\mathscr{A})$ are subspaces of V obtained as intersections of the hyperplanes. Notice here that if W is any linear subspace and H is any hyperplane, the codimension of $H \cap W$ either equals that of W (namely if $H \supseteq W$) or else it surpasses it by one. Therefore, for $W_1, W_2 \in \mathcal{L}(\mathscr{A})$, we have $W_1 \lessdot W_2$ if and only if $W_2 = W_1 \cap H$ for some $H \not\supseteq W_1$ (i.e., $W_2 = W_1 \vee H$ for some $H \in A(\mathcal{L}(\mathscr{A}))$, $H \not\leq W_1$). In summary, we see that if $\mathscr{A}$ is an arrangement of hyperplanes, then $\mathcal{L}(\mathscr{A})$ is a geometric lattice.

3. Matroids "are" geometric lattices

3.1. **Matroids from geometric lattices.** In what follows we will derive from the definition some properties of a geometric lattice that are "intuitively evident" for intersection posets of hyperplane arrangements. One of these properties is that intersection posets come with a function that assigns to every intersection its codimension as a subspace of V, and this function increases exactly by one along every covering relation. We say that intersection posets are *ranked*. More generally, we have the following definition.

Definition 3.1. Let P be a poset. A *rank function* for P is a function $\rho : P \to \mathbb{N}$ such that

(i) $\rho(x) = 0$ if x is a minimal element in P,

(ii) $\rho(x) + 1 = \rho(y)$ if $x \lessdot y$ in P.

Remark 3.2. Notice that, if a bounded-below poset admits a rank function, then this function is unique.

Before going forward, let us establish that the *length* of a chain $\omega = \{x_0 < \ldots < x_k\}$ in a partially ordered set P is $\ell(\omega) = |\omega| - 1 = k$. The length of the poset $\ell(P)$ then is the maximum length of any chain in P.

Lemma 3.3. *In a geometric lattice any two maximal chains between the same elements have the same length.*

Proof. Let L be a geometric lattice. We prove by induction the following statement (note that in this proof, given $a, b \in L$, an (a, b)-chain is any chain in L of the form $a = x_0 < x_1 < \ldots < x_k = b$).

$(*_t)$ *For all $a, b \in L$, if one maximal (a, b)-chain has length t, then all of them do.*

The premise of $(*_1)$ can only be satisfied if $a \lessdot b$. In this case there is only one maximal (a, b)-chain, hence $(*_1)$ holds.

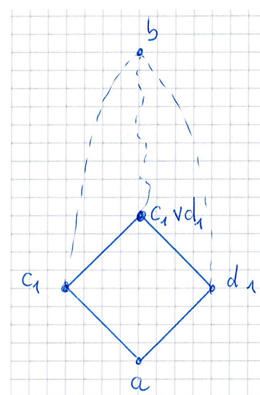
Then let $t \geq 2$ and suppose that $(*_r)$ holds for all $r < t$. Consider two maximal (a, b)-chains

$$a = c_0 \lessdot c_1 \lessdot \ldots \lessdot c_t = b \qquad\qquad a = d_0 \lessdot d_1 \lessdot \ldots \lessdot d_s = b.$$

Now, if $c_1 = d_1$, then by induction hypothesis all maximal (c_1, b)-chains have $t - 1$ elements, hence $s = t$ and we are done.

Suppose then $c_1 \neq d_1$. By property (G) we can find $x, y \in A(L)$ with $c_1 = a \vee x$, $d_1 = a \vee y$. If $x \leq d_1$ (resp. $y \leq c_1$) we would have $c_1 \leq d_1$ (resp. $d_1 \leq c_1$), reaching a contradiction; hence, $x \not\leq d_1$ (resp. $y \not\leq c_1$). Again by (G), we compute $c_1 \vee d_1 = a \vee x \vee y \gtrdot d_1, c_1$.

Now, by induction hypothesis applied to (c_1, b), every maximal (c_1, b)-chain has length $t - 1$, and in particular every maximal $(c_1 \vee d_1, b)$-chain has length $t - 2$. In the same way, induction hypothesis applied to (d_1, b) gives that every $(c_1 \vee d_1, b)$-chain has length $s - 2$. We conclude $s = t$, and $(*_t)$ holds.



□

Corollary 3.4. *Every geometric lattice admits a rank function.*

Proof. Given a geometric lattice L a rank function is given by choosing, for every $x \in L$,

$$\rho(x) := \text{ length of any maximal chain from } \widehat{0} \text{ to } x. \tag{2}$$

Lemma 3.3 ensures that this is well-defined, and one readily checks that the conditions of Definition 3.1 are satisfied. □

Corollary 3.5. *Let L be a geometric lattice with rank function ρ. For every $X \subseteq A(L)$ we have $\rho(\vee X) \leq \#X$.*

Proof. First notice that by uniqueness of the rank function we know that ρ can be expressed as in Equation (2). Induction on the cardinality of X. If $X = \emptyset$, $\rho(\vee X) = \rho(\widehat{0}) = 0$ and the claim holds.

If $\# X > 0$, choose $x \in X$ and notice that either $\vee(X \setminus \{x\}) = \vee X$ (when $x \leq \vee(X \setminus \{x\})$) or, by (G), $\vee(X \setminus \{x\}) \lessdot \vee X$. In any case, a maximal chain from $\widehat{0}$ to $\vee X$ can be obtained by adding at most one new element to a maximal chain from $\widehat{0}$ to $\vee(X \setminus \{x\})$. Therefore, $\rho(\vee X) \leq \rho(\vee(X \setminus \{x\})) + 1$ and by induction hypothesis this is at most $\# X$. □

Lemma 3.6. *Let L be a geometric lattice and ρ its*[1] *rank function. Then, for all $x, y \in L$,*

$$\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y).$$

Proof. Consider $z := x \wedge y$ and any saturated chain $z = z_0 \lessdot z_1, \lessdot z_2, \lessdot \cdots \lessdot z_k = y$. Then,

$$k = \rho(y) - \rho(x \wedge y). \tag{3}$$

By (G) we can choose atoms $a_1, \ldots, a_k$ so that $a_i \leq z_i$, $a_i \not\leq z_{i-1}$ and $z_i = z_{i-1} \vee a_i$ for all $i = 1, \ldots, k$.

Define now elements $w_0, \ldots, w_k$ by setting $w_0 = x$ and $w_i := w_{i-1} \vee a_i$ for all $i \geq 1$. Notice that $w_k = x \vee a_1 \vee \ldots \vee a_k = x \vee z \vee a_1 \vee \ldots \vee a_k = x \vee y$.

Then, by (G) we have either $w_i = w_{i-1}$ or $w_{i-1} \lessdot w_i$ for all i, so that $k \geq \rho(w_k) - \rho(w_0) = \rho(x \vee y) - \rho(x)$ and the claim follows by recalling Equation (3). □

We have proved the following.

Proposition 3.7. *Let E be a finite set and let $\mathcal{L} \subseteq 2^E$ a family of subsets of E, partially ordered by inclusion and such that $E \in \mathcal{L}$. Suppose further that $\mathcal{L}$ is a geometric lattice with rank function ρ,with meet operation given by set intersection, and such that the union of the atoms of $\mathcal{L}$ equals E. Then, for every $X \subseteq E$ there is a unique minimal X' in $\mathcal{L}$ such that $X \subseteq X'$, and the extension r of ρ on 2^E given by $r(X) := \rho(X')$ is a matroid rank function.*

Proof. The set X' exists for any given X because meets exist in $\mathcal{L}$ and are given by set intersection.

Axiom (R2) is trivially satisfied. For Axiom (R1) notice first that ρ is never negative by definition. Moreover, given $X \subseteq E$ we can consider a minimal family $A_1, \ldots, A_k$ of all atoms of $\mathcal{L}$ such that $X \subseteq \bigcup_i A_i$ (this is possible since $E = \bigcup_i A_i$). Then surely $k \leq |X|$ and $X \subseteq \bigvee_i A_i$. Thus, $X' \leq \bigvee_i A_i$ in $\mathcal{L}$, and by Corollary 3.5 $\rho(\bigvee_i A_i) \leq k$. Thus $r(X) = \rho(X') \leq k \leq |X|$ as desired.

We now turn to Axiom (R3). First notice that, trivially, $X' \wedge Y' \geq (X \cap Y)'$. By definition, $X' \vee Y'$ is the minimal element of $\mathcal{L}$ containing X' and Y', while $(X \cup Y)'$ is the minimal element of $\mathcal{L}$ containing X and Y. Since $X \subseteq X'$ and $Y \subseteq Y'$, we have $X' \vee Y' \geq (X \cup Y)'$. With the trivial inequality $X' \vee Y' \leq (X \cup Y)'$ we obtain $X' \vee Y' = (X \cup Y)'$.

Now using Lemma 3.6 and the monotony of ρ we can write

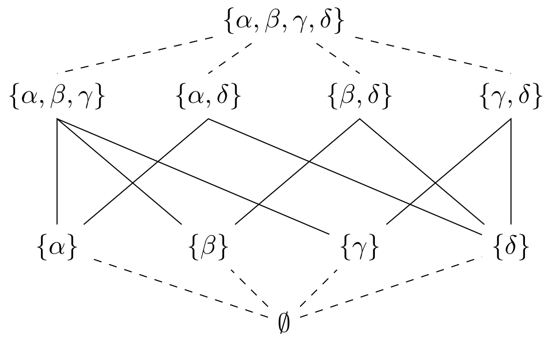
$$\begin{aligned} r(X) + r(Y) &\overset{df}{=} \rho(X') + \rho(Y') \\ &\geq \rho(X' \wedge Y') + \rho(X' \vee Y') \geq \rho((X \cap Y)') + \rho((X \cup Y)') \\ &\overset{df}{=} r(X \cap Y) + r(X \cup Y) \end{aligned}$$

□

Corollary 3.8. *Given any (abstract) geometric lattice $\mathcal{L}$, we can associate to every $x \in \mathcal{L}$ the set $A(x)$ of all atoms of $\mathcal{L}$ below x. Then, $\mathcal{L}$ is isomorphic to the set $\mathcal{L}' := \{A(x) \mid x \in \mathcal{L}\}$ ordered by inclusion (since $x < y$ if and only if $A(x) \subset A(y)$). The matroid constructed from the proposition, then, has the set $A(\mathcal{L})$ of all atoms as a ground set and rank function given by $r(X) = \rho(\vee X)$ for all $X \subseteq A(\mathcal{L})$. This matroid has no loops, and it is referred to as the "simple matroid associated to $\mathcal{L}$.*

[1] Unique by Remark 3.2

Example 3.9. Let us consider the geometric lattice from Figure 1. The set of atoms is $\{\alpha, \beta, \gamma, \delta\}$, and the associated geometric lattice $\mathcal{L}'$ in Corollary 3.8 is as follows.



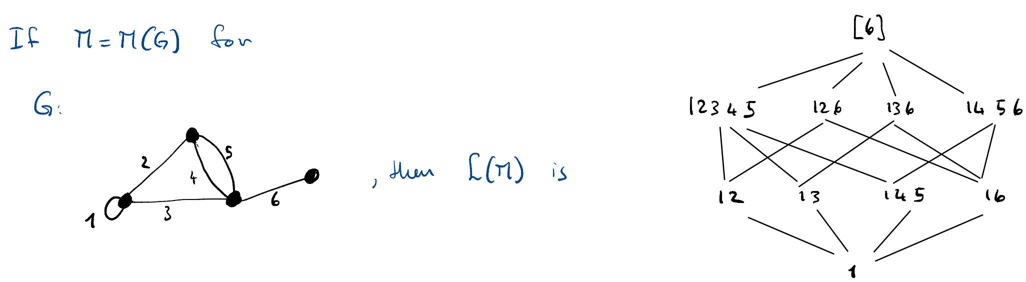
The claim of Corollary 3.8 is then that this is the lattice of flats of a matroid on $E = \{\alpha, \beta, \gamma, \delta\}$ with rank function given by $r(A) = |A|$ if $|A| \leq 2$, $r(\{\alpha, \beta, \gamma\}) = 2$, and $r(A) = 3$ for all A with $A \neq \{\alpha, \beta, \gamma\}$ and $|A| \geq 3$.

3.2. **Geometric lattices from matroids.** We aim at a "converse" of Proposition 3.7, constructing a geometric lattice for every given matroid.

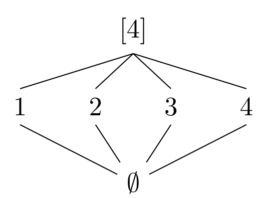
Let E be a finite set and $\mathrm{rk} : 2^E \to \mathbb{N}$ a matroid rank function. Recall from Lecture 2 the notion of *flats* and of *closure operator* associated to a matroid.

Definition 3.10. Let $\mathcal{L}_{\mathrm{rk}}$ be the poset of all closed sets ordered by inclusion (i.e., for $F, F' \in \mathcal{L}_{\mathrm{rk}}$ we have $F \leq F'$ if $F \subseteq F'$).

Example 3.11.



Example 3.12. Consider the rank function $\mathrm{rk} : 2^{[4]} \to \mathbb{N}$ defined by $\mathrm{rk}(X) = 1$ if $|X| \leq 1$ and $\mathrm{rk}(X) = 2$ otherwise. This is the rank function of the uniform matroid $U_{2,4}$. The associated poset of flats is depicted below.



Our next goal is to prove that, in general, $\mathcal{L}_{\mathrm{rk}}$ is a geometric lattice.

Lemma 3.13. *Let* rk *be a matroid rank function. Then, meet and join of every* $F_1, F_2 \in \mathcal{L}_{\mathrm{rk}}$ *exist. In fact,*

(1) $F_1 \vee F_2 = \mathrm{cl}(F_1 \cup F_2)$

(2) $F_1 \wedge F_2 = F_1 \cap F_2$

In particular, $\mathcal{L}_{\mathrm{rk}}$ *is a lattice.*

Proof.

(1) By definition of the ordering, every element of $(\mathcal{L}_{\mathrm{rk}})_{\geq F_1} \cap (\mathcal{L}_{\mathrm{rk}})_{\geq F_2}$ must contain $F_1 \cup F_2$. But, e.g. by Corollary 1.9 in Lecture 2, $\mathrm{cl}(F_1 \cup F_2)$ is the (unique) smallest closed set containing $F_1 \cup F_2$.

(2) It is enough to prove that $F_1 \cap F_2$ is closed, which was done in Lemma 1.8 of Lecture 1.

□

Recall (e.g., from Corollary 1.9 in Lecture 2) that the closure operator cl is monotone ($X \subseteq Y$ implies $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$) and increasing ($X \subseteq \mathrm{cl}(X)$).

Remark 3.14. If $X < Y$ in $\mathcal{L}_{\mathrm{rk}}$, then $\mathrm{rk}(X) < \mathrm{rk}(Y)$. Otherwise, by (R2) we would have $\mathrm{rk}(X) = \mathrm{rk}(Y)$ and so, since $Y \subseteq X$ by assumption, $X = Y$ – a contradiction.

Proposition 3.15. *For any* $F_1, F_2 \in \mathcal{L}_{\mathrm{rk}}$*, (G) holds. I.e.,*

$$F_1 \lessdot F_2 \Leftrightarrow \exists P \in A(\mathcal{L}_{\mathrm{rk}}),\ P \not\leq F_1, \ \ \textit{s.t.}\ F_2 = F_1 \vee P.$$

Proof.

⇐ Let P be as in the claim. Since P is an atom, $P = \mathrm{cl}(\{e\})$ for some element $e \in E$ and, since $P \not\leq F_1$, it must be $e \in E \setminus F_1$ Now we can write F_2 as

$$F_2 = F_1 \vee P = \mathrm{cl}(F_1 \cup P) = \mathrm{cl}(F_1 \cup \{e\})$$

(we have used Lemma 3.13 in the middle equality) and we have

$$\mathrm{rk}(F_1) + \mathrm{rk}(\{e\}) \geq \mathrm{rk}(\emptyset) + \mathrm{rk}(F_1 \cup \{e\}).$$

Now, $\mathrm{rk}(\{e\}) = \mathrm{rk}(\mathrm{cl}(e))$ and since $P = \mathrm{cl}(\{e\})$ has rank 1, $\mathrm{rk}(\{e\}) = 1$. Thus

$$\mathrm{rk}(F_2) = \mathrm{rk}(F \cup \{e\}) \leq \mathrm{rk}(F_1) + 1.$$

Moreover, since F_1 is closed and $e \notin F_1$ we have $\mathrm{rk}(F_1 \cup \{e\}) > \mathrm{rk}(F_1)$, and we conclude that $\mathrm{rk}(F_2) = \mathrm{rk}(F_1) + 1$.

Now by Remark 3.14 any $Z \in \mathcal{L}_{\mathrm{rk}}$, $F_1 < Z < F_2$, would force $\mathrm{rk}(F_2) \geq \mathrm{rk}(F_1) + 2$, hence a contradiction. We conclude $F_1 \lessdot F_2$.

⇒ $F_1 \lessdot F_2$ implies $F_1 \subsetneq F_2$ and so we can choose $e \in F_2 \setminus F_1$. Then $\mathrm{rk}(\{e\}) = 1$ since otherwise e is in the closure of every flat, in particular we would have $e \in F_1$. It follows that $P := \mathrm{cl}(\{e\})$ is an atom of $\mathcal{L}_{\mathrm{rk}}$, and $P \leq F_2$ by monotonicity of the closure operator. Now define

$$F := F_1 \vee P = \mathrm{cl}(F_1 \cup \{e\}).$$

Then the following claim concludes the proof.

Claim. $F_2 = F$.

Proof. We have

$$\mathrm{rk}(F) \geq \mathrm{rk}(F_1) + 1 = \mathrm{rk}(F_2). \tag{4}$$

The inequality holds since $F \supseteq F_1 \cup \{e\}$, F_1 is closed and $e \notin F_1$, the equality is immediate since $F_1 \lessdot F_2$.

Now since $F_1 \cup \{e\} \subseteq F_2$, monotonicity of cl implies $F \subseteq F_2$. Together with Equation (4) this shows $F = F_2$.

□

Theorem 3.16. *Let* rk *be any matroid rank function. Then the poset* $\mathcal{L}_{\mathrm{rk}}$ *is a geometric lattice whose rank function* ρ *satisfies* $\rho(F) = \mathrm{rk}(F)$ *for every* $F \in \mathcal{L}_{\mathrm{rk}}$.

Proof. That $\mathcal{L}_{\mathrm{rk}}$ is a geometric lattice follows from Lemma 3.13 and Proposition 3.15. For the claim about rank consider any $F \in \mathcal{L}_{\mathrm{rk}}$ and let $\hat{0} \lessdot F_1 \lessdot \ldots \lessdot F_k = F$ be a maximal chain below F. Then, $\rho(F) = k$.

Choose atoms $A_1, \ldots, A_k$ with $F_i = F_{i-1} \vee A_i$ for all i. Since every F_{i-1} is closed and $A_i \not\subseteq F_{i-1}$, we must have

$$\mathrm{rk}(F_{i-1}) > \mathrm{rk}(F_{i-1} \cup A_i) = \mathrm{rk}(F_i) \tag{5}$$

(the last equality by 3.13.(1)). On the other hand, (R2) implies

$$\mathrm{rk}(F_{i-1}) + \mathrm{rk}(A_i) \geq \mathrm{rk}(\underbrace{F_{i-1} \cap A_i}_{=\widehat{0}}) + \mathrm{rk}(F_{i-1} \cup A_i) = \mathrm{rk}(F_i) \tag{6}$$

and since $\mathrm{rk}(A_i) = 1$ because of Equations (5) and (6), we conclude $\mathrm{rk}(F_i) = \mathrm{rk}(F_{i-1}) + 1$, thus $r(X) = \mathrm{rk}(F_k) = k = \rho(X)$.

□

4. Back to business: Arrangements

4.1. **The one and only rank function.** At this stage we have two, a priori different, rank functions associated to an arrangement $\mathscr{A} = \{H_1, \ldots, H_m\}$ of hyperplanes:

- The rank function r_{lat} of the simple matroid associated to the geometric lattice $\mathcal{L}(\mathscr{A})$ as in Theorem 3.8:
$$r_{\mathrm{lat}} : 2^{[m]} \to \mathbb{N}, \quad I \mapsto \rho\left(\bigvee_{i \in I} H_i\right)$$
- The rank function r_{dep} of the matroid of linear dependencies of the $[n]$-tuple of vectors $\{n_1, \ldots, n_m\}$, where n_i is any choice of normal vector for the hyperplane H_i:
$$r_{\mathrm{dep}} : 2^{[m]} \to \mathbb{N}, \quad I \mapsto \dim \operatorname{span}\{v_i \mid i \in I\}.$$

Our next goal is to show that they are the same.

Lemma 4.1. *For every intersection $X \in \mathcal{L}(\mathscr{A})$ we have $\rho(X) = \operatorname{codim} X$*

Proof. By definition $\rho(X) = k$ means that k is the length of a maximal chain $\widehat{0} \lessdot X_1 \lessdot \cdots \lessdot X_k = X$. Now consider the subspaces X_i. By property (G), every X_i is of the form $X_{i-1} \cap H_i$ for some atom H_i of $\mathcal{L}(\mathscr{A})$ (i.e., hyperplane in $\mathscr{A}$) with $H_i \not\leq X_{i-1}$ (i.e., $H_i \not\supseteq X_{i-1}$). Notice that the latter implies that $X_{i-1} + H_i = \mathbb{R}^d$, the ambient space. Now, elementary linear algebra tells us that

$$\dim(\underbrace{X_{i-1} \cap H_i}_{=X_i}) + \underbrace{\dim(X_{i-1} + H_i)}_{=d} = \dim(X_{i-1}) + \underbrace{\dim(H_i)}_{d-1}$$

and thus $\dim(X_i) = \dim(X_{i-1}) - 1$. Therefore, X has dimension k less than $\widehat{0} = \mathbb{R}^d$, and the proof is complete. □

Proposition 4.2.

$$r_{\mathrm{lat}} \equiv r_{\mathrm{dep}}.$$

Proof. Let $I \subseteq [m]$ and write $X := \bigvee_{i \in I} H_i = \bigcap_{i \in I} H_i$ Then, with 4.1 we know that $r_{\mathrm{lat}}(I) = \rho(X) = \operatorname{codim}(X)$. On the other hand, $r_{\mathrm{dep}}(I)$ equals the rank of the $d \times |I|$ matrix M whose columns are v_i for i in I. Now, X is the subspace of all points that are orthogonal to each v_i, $i \in I$, and therefore $X = \ker M$. Now, again by elementary linear algebra we know that $\dim \ker M = d - \operatorname{rank} M$. We summarize and conclude

$$r_{\mathrm{dep}}(I) = \operatorname{rank} M = d - \dim \ker M = d - \dim X = \operatorname{codim}(X) = r_{\mathrm{lat}}(I)$$

□

4.2. **Arrangements' complements.**

Definition 4.3. Let $\mathscr{A}$ be an arrangement of hyperplanes in V. The *complement* of $\mathscr{A}$ is the space

$$M(\mathscr{A}) := V \setminus \cup\mathscr{A}$$

The space $M(\mathscr{A})$ is a *fascinating* object, especially in the case where V is a complex vectorspace – more on this later.

Notation 4.4. Let $\mathbb{K}$ denote a field and let $\mathscr{A} = \{H_1, \ldots, H_n\}$ denote an arrangement of hyperplanes in $\mathbb{K}^d$.

Suppose for simplicity that $\cap\mathscr{A} = \{0\}$.

We want to find a special "parametrization" of $M(\mathscr{A})$. For every $i = 1, \ldots, n$ let a_i denote a (arbitrary) normal vector to H_i, and let $A := [a_1 \cdots a_n]$ be the $d \times n$ matrix whose columns are the vectors a_i.

Let $b_1, \ldots, b_d$ denote the rows of A. Since $\cap\mathscr{A} = \{0\}$, we know that A has full rank and hence the b_i are linearly independent, so that the transpose

$$A^t : \mathbb{K}^d \to \mathbb{K}^n, \quad x \mapsto A^t x$$

is an injective linear map, whose image is the rowspace V of A (i.e., $V = \operatorname{im} A^t$).

Notice that $x \in H_i$ if and only if $\langle x|a_i\rangle = 0$, if and only if $(A^t x)_i = 0$. If we call $f : \mathbb{K}^d \to V$ the restriction to the map A^t to V, and letting $E_i := \{x_i = 0\}$ be the i-th coordinate hyperplane in $\mathbb{K}^n$, we have:

The function f is an invertible linear function between $\mathbb{K}^d$ and V that maps $M(\mathscr{A})$ to $V \cap (\mathbb{K}^*)^n$.

In particular, the study of either of those spaces is equivalent. In order to characterise the latter space by polynomial equalities, notice that for every $y \in \mathbb{K}^n$ we have $y^t V = 0$ if and only if $y^t b_i = 0$ for all i or equivalently, since the b_i are the rows of A, $y \in \ker A$.

Now, $y \in \ker A$ if and only if the coordinates of y are the coefficients of a linear dependency among the a_i. We are led to consider the matroid $M(\{a_1, \ldots, a_n\})$ that we call henceforth just M.

Lemma 4.5. *For every $C \in \mathcal{C}(M)$ there is $v(C) \in \ker A$ such that $v(C)_i = 0$ for all $i \notin C$. The vector $v(C)$ is uniquely determined up to a nonzero scalar multiple.*

Proof. The existence of $v(C)$ follows from the very definition of a circuit of the matroid $M(A)$. For the uniqueness part suppose that there is a $v'(C)$ with the same support as $v(C)$ and with $i, j \in C$ with $v(C)_i / v'(C)_i \neq v(C)_j / v'(C)_j$. Then the difference $v(C) - \frac{v_i(C)}{v'_i(C)} v'(C)$ is again in $\ker A$, hence it defines a linear dependency among the a_i, but its support is nonempty (since it contains j) and strictly smaller than C (since it does not contain i), a contradiction to C being a circuit. □

Lemma 4.6. *The set $\{v(C) \mid C \in \mathcal{C}(M)\}$ spans $\ker A$.*

Proof. Let $v \in \ker A$. By definition, $D := \{i \in [n] \mid v_i \neq 0\}$ is a dependent set in $M(A)$, hence it contains a circuit C. If $C = D$ then by Lemma 4.5 v is a multiple of $v(C)$ and we are done. Otherwise, choose $i \in C$ and let $\lambda \in \mathbb{K}$ be such that $\lambda v_i = v(C)_i$. Then, $v' = \lambda v - v(C)$ is an element of $\ker A$ with set of nonzero coordinates strictly contained in D (since $v'_i = 0$). Repeat then the argument with v' and, since D is finite, eventually we will terminate with a $v'^{\cdots\prime}$ whose support is a circuit. □

We conclude that V is defined as the locus of common solutions of the set of equations

$$\left\{ \sum_i v(C)_i x_i = 0, \qquad \text{one for each } C \in \mathcal{C}(M) \right\} \tag{7}$$

In particular, if $\mathbb{K} = \mathbb{C}$, then $M(\mathscr{A})$ is equivalent to the subvariety of the complex torus $(\mathbb{C}^*)^n$ defined by the Equations (7). In this **very special** case, the tropicalization of $M(\mathscr{A})$ is the locus of all $w \in \mathbb{R}^n/\mathbf{1}\mathbb{R}$ such that

$$\min_{i \in C}\{w_1, \ldots, w_n\} \text{ is attained twice.,} \qquad \text{for each } C \in \mathcal{C}(M).$$

Therefore:

The tropicalization of $M(\mathscr{A})$ is the Bergman fan $\widetilde{\mathscr{B}}(M(A))$, where A is any matrix whose columns are a set of normals for the hyperplanes in $\mathscr{A}$.

5. More on flats and abstract simplicial complexes

5.1. Direct sums, again.

We have seen the notion of direct sum of matroids in terms of bases. Our goal is to prove the following

Theorem 5.1. *Let M_1, M_2 be matroids on disjoint ground sets. Then,*

$$\mathcal{L}(M_1 \oplus M_2) = \mathcal{L}(M_1) \times \mathcal{L}(M_2).$$

First of all, let us explain the expression on the right-hand side.

Definition 5.2. Let $(P, \leq_P), (Q, \leq_Q)$ be partially ordered sets. Their *cartesian product* is the poset $P \times Q, \leq_{P\times Q})$, i.e., the cartesian product of the sets P and Q with a partial order defined by

$$(p_1, q_2) \leq_{P\times Q} (p_2, q_2) \text{ iff } p_1 \leq_P p_2,\ q_1 \leq_Q q_2.$$

Example 5.3. The poset of all (nonnegative) divisors of 36 is the product of the posets of (nonnegative) divisors of 4 and of 9.

We continue our way towards Theorem 5.1 by exploring the notion of direct sum of matroids in terms of cryptomorphisms other than via bases.

Lemma 5.4. *Let M_1, M_2 be matroids on disjoint ground sets E_1, E_2.*

(1) For $X \subseteq E_1 \cup E_2$, $\mathrm{rk}_{M_1\oplus M_2}(X) = \mathrm{rk}_{M_1}(X \cap E_1) + \mathrm{rk}_{M_2}(X \cap E_2)$.
(2) A set $F \subseteq E_1 \cup E_2$ is a flat of $M_1 \oplus M_2$ if and only if $F \cap E_i$ is a flat of M_i for $i = 1, 2$.

Proof.

(1) The rank $\mathrm{rk}_{M_1\oplus M_2}(X)$ is the size of a maximal independent subset $I \subseteq X$. Fom the definition of direct sum we have that I is independent in $M_1 \oplus M_2$ if and only if $I \cap E_i$ is independent in M_i, for $i = 1, 2$. Now, if $I \cap E_1$ is not M_1- maximal independent in $X \cap E_1$, then it is contained in such a maximal independent I_1', and $I \cup I_1' \subseteq X$ is independent in the direct sum and strictly larger than I, a contradiction. Therefore $I \cap E_i$ is maximal independent in $X \cap E_i$ with respect to M_i for both i, and so $\mathrm{rk}_{M_1\oplus M_2}(X) = |I| = |I \cap E_1| + |I \cap E_2| = \mathrm{rk}_{M_1}(X \cap E_1) + \mathrm{rk}_{M_2}(X \cap E_2)$.

(2) An $F \subseteq E_1 \cup E_2$ is a flat if and only if $\mathrm{rk}_{M_1\oplus M_2}(F \cup \{e\}) > \mathrm{rk}_{M_1\oplus M_2}(F)$ for all $e \in (E_1 \cup E_2) \setminus F$. But for every such e, say with $e \in E_1$, we have

$$\mathrm{rk}_{M_1\oplus M_2}(F \cup \{e\}) = \mathrm{rk}_{M_1}((F \cap E_1) \cup \{e\}) + \mathrm{rk}_{M_2}(F \cap E_2)$$

and this is strictly greater than $\mathrm{rk}_{M_1\oplus M_2}(F)$ if and only if $\mathrm{rk}_{M_1}(F \cap E_1 \cup \{e\}) > \mathrm{rk}_{M_1}(F \cap E_1)$. The same computation goes if $e \in E_2$, and the claim follows.

□

5.2. **Abstract and geometric simplicial complexes.** In the warmup to Lecture 3 we introduced the notion of a simplicial complex as a collection $\mathscr{K}$ of simplices in Euclidean space such that (1) the collection contains all faces of each of its members (2) any two members of the collection intersect at a face of both.

Given a simplex S let $V(S)$ denote the set of vertices of S, and let V be the set of all vertices of simplices in $\mathscr{K}$. Since the convex hull of every subset of the vertices of a simplex is a (different) face of the simplex itself, the collection

$$\Sigma(\mathscr{K}) = \{V(S) \mid S \in \mathscr{K}\}$$

satisfies:

(ASC) $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ implies $\tau \in \Sigma$.

Definition 5.5. Every collection Σ of subsets of a given finite set that satisfies (ASC) is called an *abstract simplicial complex*.

Proposition 5.6 (Without proof). *For every abstract simplicial complex Σ on a finite set V there is a (geometric) simplicial complex $\mathscr{K}_\Sigma$ in $\mathbb{R}^n$ (for n big enough) such that, after identifying elements of V with the corresponding points in $\mathbb{R}^n$, we have $\Sigma = \Sigma(\mathscr{K}_\Sigma)$.*

We write

$$\|\Sigma\| := \bigcup_{S \in \mathscr{K}_\Sigma} S$$

for what we call the "geometric realization" of the abstract simplicial complex Σ. This is justified by the fact (also without proof) that any choice of $\mathscr{K}_\Sigma$ yields a homeomorphic space $\|\Sigma\|$.

Now, if P is a finite poset then we write $\Delta(P) := \{\{p_1, \ldots, p_n\} \subseteq P \mid p_1 < p_2 < \ldots < p_n\}$ for the family of all totally ordered subsets of P. Clearly this is an abstract simplicial complex, called the *order complex* of P. We write for short

$$\|P\| := \|\Delta(P)\|$$

for the geometric realization of the order complex.

In particular, we can say that the tropicalization of $M(\mathscr{A})$ is combinatorially isomorphic to the cone over (a realization of) the simplicial complex $\Delta(\mathcal{L}(\mathscr{A}) \setminus \{\hat{0}, \hat{1}\})$ – the latter we know to be isomorphic to the Bergman complex $\mathscr{B}(M)$.

Another proposition that we list without proof is the following.

Proposition 5.7. *Let P and Q be finite posets. Then $\|\Delta(P \times Q)\|$ is homeomorphic to $\|\Delta(P)\| \times \|\Delta(Q)\|$*

Notice that the cartesian product of simplicial complexes appearing in the proposition is properly a polyhedral complex (since products of simplices are not simplices, but just polytopes).

Corollary 5.8. *Let M be a matroid. Then the Bergman complex is the product of the Bergman complexes of the connected components of M.*

6. References and complementary literature

Section 1 is again based on [3]. The exposition in Section 3 follows only partially [4] and [1]. Proofs of the statements that appear without justification in the last part of Section 5 can be found in [2, Sections B.3, C.2, C.3].

References

[1] Martin Aigner; *Combinatorial theory.* Classics in Mathematics. Springer-Verlag, Berlin, 1997. viii+483 pp.
[2] Mark De Longueville; *A course in topological combinatorics.* Universitext. Springer, New York, 2013. xii+238 pp.
[3] Eva Maria Feichtner, Bernd Sturmfels; *Matroid polytopes, nested sets and Bergman fans.* Port. Math. (N.S.) 62 (2005), no. 4, 437-468.
[4] James Oxley; *Matroid theory.* Second edition. Oxford Graduate Texts in Mathematics, 21. Oxford University Press, Oxford, 2011. xiv+684 pp.