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NOTES FOR LECTURE 3

Note: A detailed list of sources and references can be found at the end of the document.

RECALL (E.G., FROM WARM-UP):
Given any polytope P in $\mathbb{R}^n$ and any vector $w \in \mathbb{R}^n$, the set

$$[P\uparrow w] := \arg\max_{x\in P}\langle w \mid x\rangle$$

of points of P where the linear form $x \mapsto \langle w|x\rangle$ attains its maximum is a face of P, and every face of P has this form.
The collection $\mathscr{N}(P) := \{N_Q\}$ where Q ranges over all faces of P and

$$N_Q := \{w \in \mathbb{R}^n \mid Q \subseteq [P\uparrow w]\}, \tag{1}$$

is a fan, called the *normal fan* of P.

1. The Bergman fan

In the proof of Proposition 2.13 from Lecture Notes 2 we associate to every flacet F the matroid M_F corresponding to the facet of P_M determined by the valid inequality $\sum_{i\in F} x_i \leq \mathrm{rk}(F)$. Explicitly, this face is $[P_M\uparrow e_F]$, the set of points of P_M where the linear form $\langle e_F|x\rangle = \sum_{i\in F} x_i$ takes its maximum value (this maximal value is $\mathrm{rk}(F)$).

We want to generalize this construction.

Definition 1.1. Let M be a matroid on $[n]$ and let $w \in \mathbb{R}^n$. Call M_w the matroid with

$$P_{M_w} = [P_M\uparrow w].$$

This is well-defined because from Lecture 2 we know that every face of a matroid polytope is again a matroid polytope, and that the matroid polytope completely determines the matroid.

Digression-Remark 1.2. Let F be a flacet of a connected matroid M. Then $M_{e_F} = M_F$, as defined in Proposition 2.13 of Lecture 2, where the matroids that appear are, in particular, loopless. We aim at generalizing this description to every face of P_M that intersects the relative interior of the simplex $r \cdot \Delta^{(n)}$.

Lemma 1.3. *Let M be a matroid of rank r on $[n]$ and let $w \in \mathbb{R}^n$. The following are equivalent.*

(1) M_w has no loops.

(2) Every element of the ground set $[n]$ appears in some basis of M_w.

(3) The face $[P_M\uparrow w]$ intersects the (relative) interior of $r \cdot \Delta^{(n)}$.

Proof. For the equivalence of (1) and (2) notice that by definition bases are the inclusion-maximal independent sets: thus, for an element e to be contained in some basis is equivalent to the set $\{e\}$ to have rank 1. Since e is a loop if and only if $\{e\}$ has rank 0, we conclude that e is contained in some basis if and only if e is not a loop. The equivalence of (1) and (2) follows.

Now for the equivalence of (2) and (3). The relative interior of $r \cdot \Delta^{(n)}$ consists of the points $x \in T$ with $x_i > 0$ for all $i \in [n]$. On the other hand, every point $p \in [P_M\uparrow w]$ is in the convex hull of the vertices of $[P_M\uparrow w]$, i.e., of all vectors e_B where B runs over all bases of M_w – this means $p = \sum_{B\in\mathcal{B}M_w} \lambda_B e_B$ for some $\lambda_B \geq 0$ with $\sum \lambda_B = 1$. On the one hand, this shows that for every nonzero coordinate i of P there must be a basis of M_w with $i \in B$. Thus, if there is $p \in [P_M \uparrow w] \cap \operatorname{relint} r \cdot \Delta^{(n)}$ then all coordinates of p are nonzero and so every element of $[n]$ appears in some basis of M_w. Conversely, if $\bigcup_{B\in\mathcal{B}(M_w)} = [n]$ then $p := \sum_{B\in\mathcal{B}(M_w)} \frac{1}{|\mathcal{B}(M_w)|} e_B$ is a point of $[P_M \uparrow w]$ that lies in the relative interior of $r \cdot \Delta^{(n)}$. □

Definition 1.4. Let M be a matroid on $[n]$ and let $w \in \mathbb{R}^n$. The set

$$\widetilde{\mathscr{B}}(M) := \{w \in \mathbb{R}^n \mid M_w \text{ has no loops}\}$$

is called the *Bergman fan* of M (see the following Proposition for a justification of this name).

Proposition 1.5. *We have*

$$\widetilde{\mathscr{B}}(M) = \bigcup_{\substack{F \text{ face of } P_M \\ F\cap \operatorname{relint}(r\cdot\Delta^{(n)})\neq\emptyset}} N_F \tag{$\dagger$}$$

and the set of cones on the right-hand side is a subfan of the normal fan to P_M.

Proof. We start with proving the following claim.

Claim. If $w \in \widetilde{\mathscr{B}}(M)$, then $N_{[P_M \uparrow w]} \subseteq \widetilde{\mathscr{B}}(M)$.

Proof. It is enough to prove that if M_w is loop-less, then so is $M_{w'}$ for every $w' \in N_{[P_M \uparrow w]}$. Now, by Equation (1), $w' \in N_{[P_M \uparrow w]}$ implies $[P_M \uparrow w'] \supseteq [P_M \uparrow w]$ and thus

$$[P_M \uparrow w'] \cap \operatorname{relint}(r \cdot \Delta^{(n)}) \supseteq [P_M \uparrow w] \cap \operatorname{relint}(r \cdot \Delta^{(n)}) \neq \emptyset.$$

The last inequality holds by Lemma 1.3, and the same Lemma applied to w' now proves the claim.

The claim implies immediately the set-theoretic equality (†). For the second assertion we have to prove that the set of cones on the right-hand side of (†) contains all faces of each of its elements. But faces of N_F are of the form N_Q for faces $Q \supseteq F$, and if F meets the relative interior of $r \cdot \Delta^{(n)}$, so does every $Q \supseteq F$. □

2. An explicit description....

2.1. ... of M_w.

Definition 2.1. For any given $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ let us partition the set $[n]$ into blocks, so that elements of the same block index coordinates of w with the same value.

Precisely, consider the equivalence relation $\sim_w$ on $[n]$ with $i \sim_w j$ if and only if $w_i = w_j$. Let $\pi_1, \ldots, \pi_s$ be the equivalence classes of $\sim_w$, numbered in order of decreasing value – i.e., for $\pi_k = [w_i]$ and $\pi_l = [w_j]$, we have $k < l$ if and only if $w_i > w_j$.

Let us now define a chain of subsets of $[n]$ as follows:

$$\Phi(w) := \{F_i^w\}_{i=0,\ldots,s}, \quad \text{with} \quad F_0^w := \emptyset,\ F_i^w := \pi_1 \cup \ldots \cup \pi_i \ \text{ for all } i > 0.$$

Our next goal is to prove the following theorem.

Theorem 2.2. *A vector $w \in \mathbb{R}^n$ is contained in $\widetilde{\mathscr{B}}(M)$ if and only if all F_i^w are flats of M.*

We start with an explicit expression of M_w in terms of the family $\Phi(w)$.

Proposition 2.3. *Let M be a matroid on $[n]$ and $w \in \mathbb{R}^n$. Then*

$$M_w = \bigoplus_{i=1}^{s} M[F_i^w]/F_{i-1}^w,$$

where $\{F_i^w\}_{i=1,\ldots,s}$ is as above.

Proof. We start with an auxiliary claim that will unlock a recursive proof.

Claim. $M_w = M[F^w_{s-1}]_w \oplus M/F^w_{s-1}$.

Proof. Let W_{s-1} denote the value of $\langle x|w\rangle$ on any basis of $M[F^w_{s-1}]_w$, and let d be the difference between the rank of M and that of $M[F^w_{s-1}]$.

For every basis B of M we can complete $B \cap F^w_{s-1}$ to a basis $(B \cap F^w_{s-1} \uplus A)$ of $M[F^w_{s-1}]$ and we have

$$\begin{aligned}\langle w|e_B\rangle = \langle w|e_{(B\cap F_{s-1})}\rangle + (|A| + d)w_s &\le \langle w|e_{(B'\cap F_{s-1})}\rangle + |A|w_{s-1} + dw_s \\ &\le \langle w|e_{(B\cap F_{s-1})\cup A}\rangle + dw_s \le W_{s-1} + dw_s\end{aligned}$$

(where in the first equality we used that $|A| + d = |B \setminus F_{s-1}|$). In this series of inequalities, equality holds throughout if and only if $|A| = 0$ and $\langle w|e_{(B\cap F_{s-1})\cup A}\rangle = W_{s-1}$, i.e., if and only if $B \cap F_{s-1}$ is a basis of $M[F^w_{s-1}]_w$.

Now, if B is any basis or the r-h.s. of the claim, then B is a basis of $M[F^w_{s-1}] \oplus M/F^w_{s-1}$ and, by definition of contraction, is a basis of M. Moreover, in this case equality is attained because $B \cap F^w_{s-1}$ is a basis of $M[F^w_{s-1}]$ by definition of B. Hence, B is a basis of M_w. In general, the bases of M_w are exactly those bases B of M for which $\langle w|e_B\rangle = W_{s-1} + dw_s$. These are precisely those with $B \cap F_{s-1} \in \mathcal{B}(M[F^w_{s-1}]_w)$, i.e., those that are bases of $M[F^w_{s-1}]_w \oplus M/F^w_{s-1}$.

We can apply the claim repeatedly, obtaining

$$M_w = M[F^w_{s-1}]_w \oplus M/F^w_{s-1} = \big(M[F^w_{s-2}]_w \oplus M[F^w_{s-1}]/F^w_{s-2}\big) \oplus M/F^w_{s-1} = \ldots$$

reaching the decomposition

$$M_w = M[F^w_1] \oplus \cdots \oplus M[F^w_{s-2}]/F^w_{s-3} \oplus M[F^w_{s-1}]_w/F^w_{s-2} \oplus M/F^w_{s-1}.$$

Since $M[F^w_1]/F^w_0 = M[F^w_1]/\emptyset = M[F^w_1]$ and $M[F^w_s]/F^w_{s-1} = M[[n]]/F^w_{s-1} = M/F^w_{s-1}$, the claim follows. □

Proof of Theorem 2.2. Let M be a matroid on $[n]$ and $w \in \mathbb{R}^n$. By Proposition 2.3, it is enough to prove that $\bigoplus_{i=1}^s M[F^w_i]/F^w_{i-1}$ is loop-less if and only if all F^w_i are flats. To this end, notice that for every loop e in the direct sum there is an index i such that e is a loop in the summand $M[F^w_i]/F^w_{i-1}$, and the loops of $M[F^w_i]/F^w_{i-1}$ are exactly the elements of $\mathrm{cl}(F^w_{i-1}) \setminus F^w_{i-1}$. By definition, the latter set is empty if and only if F^w_{i-1} is a flat. □

2.2. ... of $\widetilde{\mathscr{B}}(M)$.

We are led to consider chains of flats.

Definition 2.4. Given a matroid M, let $\mathcal{L}(M)$ denote the set of all flats of M. Moreover, let $\overline{\mathcal{L}}(M) := \mathcal{L}(M) \setminus \{\mathrm{cl}(\emptyset), [n]\}$ denote the set of all flats with the smallest and the biggest removed.

By a *chain* in either $\mathcal{L}(M)$ or $\overline{\mathcal{L}}(M)$ we mean a set $\Phi = \{F_1 \subseteq F_2, \ldots\}$ of increasing elements of $\mathcal{L}(M)$ or $\overline{\mathcal{L}}(M)$, respectively. The set of all chains in $\mathcal{L}(M)$, resp. $\overline{\mathcal{L}}(M)$ is commonly denoted by $\Delta(\mathcal{L}(M))$, resp. $\Delta(\overline{\mathcal{L}}(M))$.

The next definition associates two polyhedra to a family of subsets of a ground set.

Definition 2.5. For a given family $\Phi \subseteq 2^{[n]}$ of subsets of $[n]$ let

$$\Gamma^\Phi := \mathrm{cone}\{e_F \mid F \in \Phi\}$$

and for every matroid M let

$$\Gamma(M) := \{\Gamma^\Phi\}_{\Phi\in\Delta(\overline{\mathcal{L}}(M))}.$$

Moreover, write $\mathbf{1} := e_{[n]} \in \mathbb{R}^n$ for the all-one vector.

Lemma 2.6. *If Φ is an increasing family of subsets of $[n]$ and $w \in \Gamma^\Phi$, then $\Phi(w) \subseteq \Phi$.*

Proof. By definition, $w \in \Gamma^\Phi$ implies that $w = \sum_{F\in\Phi} \lambda_F e_F$ with $\lambda_F \ge 0$ for all F. Then, $\Phi(w) = \{F \in \Phi \mid \lambda_F > 0\}$. □

Lemma 2.7. *Let M be a matroid and $\Phi \in \Delta(\overline{\mathcal{L}}(M))$. Then Γ^Φ is a simplicial cone, whose faces are all Γ^Ψ with $\Psi \subseteq \Phi$ (we set $\Gamma^\emptyset = \{0\}$). Moreover, given $\Phi, \Phi' \in \Delta(\overline{\mathcal{L}}(M))$ we have $\Gamma^\Phi \cap \Gamma^{\Phi'} = \Gamma^{\Phi\cap\Phi'}$. In particular, $\Gamma(M)$ is a (simplicial) fan.*

Proof. The set $\{e_F \mid F \in \Phi\}$ is linearly independent (because of the strict containment relation among the F_i), thus Γ^Φ is simplicial, and in particular its faces are the cones generated by all subsets of $\{e_F \mid F \in \Phi\}$, i.e., the cones of the form Γ^Ψ with $\Psi \subseteq \Phi$.

For the statement about intersections, let $\Phi_1, \Phi_2 \in \Delta(\overline{\mathcal{L}}(M))$. Now, by Lemma 2.6 any $w \in \Gamma^{\Phi_1} \cap \Gamma^{\Phi_2}$ must have $\Phi(w) \subseteq \Phi_1 \cap \Phi_2$, from which $w \in \Gamma^{\Phi_1\cap\Phi_2}$. The inclusion $\Gamma^{\Phi_1\cap\Phi_2} \subseteq \Gamma^{\Phi_1} \cap \Gamma^{\Phi_2}$ is evident, and the claim follows. $\square$

With this, we can draw some more consequences from Theorem 2.2.

Proposition 2.8. *Let M be a matroid on the ground set $[n]$. Then*

$$\widetilde{\mathscr{B}}(M) = \bigcup_{\Phi \in \Delta(\overline{\mathcal{L}}(M))} \Gamma^\Phi + \mathbb{R}\mathbf{1}, \tag{‡}$$

and the right-hand side defines a polyhedral fan that is combinatorially isomorphic to $\Gamma(M)$.

Proof. Observe that, for all $w \in \mathbb{R}^n$,

$$\Phi(w) = \Phi(w + t\mathbf{1}) \text{ for all } t \in \mathbb{R}. \tag{2}$$

Now, elements of the right-hand side of (‡) are $\mathbf{1}$-translates of elements of (some face of) some Γ^Φ for $\Phi \in \Delta(\overline{\mathcal{L}}(M))$; i.e., $w \in \mathbb{R}^n$ is in the r.-h.s. if and only if there is $t \in \mathbb{R}$ and $\Psi \in \Delta(\overline{\mathcal{L}}(M))$ such that $\Phi(w + t\mathbf{1}) = \Psi$. By (2), this is equivalent to requiring $\Phi(w) \subseteq \Delta(\overline{\mathcal{L}}(M))$, i.e., (by Theorem 2.2) $w \in \widetilde{\mathscr{B}}(M)$. This proves the set-theoretic equality.

In order to prove that the collection $\mathscr{G}_M$ of all cones appearing in the r.-h.s of (‡). is a fan, notice first that

$$\Gamma^\Phi + \mathbf{1}\mathbb{R} = \operatorname{cone}(\{e_F \mid F \in \Phi\} \cup \{\mathbf{1}, -\mathbf{1}\})$$

In particular, every member of $\mathscr{G}_M$ is a cone, whose faces are of the form $\operatorname{cone}(A)$ for some $A \subseteq \{e_F \mid F \in \Phi\} \cup \{\mathbf{1}, -\mathbf{1}\}$. Now, if $\operatorname{cone}(A)$ is a face, say with defining inequality $\langle a|x\rangle \leq b$, then $\langle a|x + \mathbb{R}\mathbf{1}\rangle \leq b$, hence $a \perp \mathbf{1}$ and in particular $A \supset \{\pm\mathbf{1}\}$. Therefore, every face of the cone $\Gamma^\Phi + \mathbf{1}\mathbb{R}$ must be of the form $\Gamma^\Psi + \mathbf{1}\mathbb{R}$ for some $\Psi \subseteq \Phi$, thus $\mathscr{G}_M$ contains every face of each of its members. Conversely, given any $\Psi = \{F^w_{i_1}, F^w_{i_2}, \ldots\} \subsetneq \Phi$,

consider the vector

$$a_\Psi := \Bigg(\underbrace{\underbrace{\frac{1}{1-|F^w_{i_1}|}, \frac{1}{1-|F^w_{i_1}|}, \ldots, \frac{1}{1-|F^w_{i_1}|}, 1,}_{F^w_{i_1}\text{ - coordinates}}\ \frac{1}{1-|F^w_{i_2}\setminus F^w_{i_1}|}, \frac{1}{1-|F^w_{i_2}\setminus F^w_{i_1}|}, \ldots, \frac{1}{1-|F^w_{i_2}\setminus F^w_{i_1}|}, 1,}_{F^w_{i_2}\text{ - coordinates}}\ \ldots \Bigg), \tag{3}$$

where we ordered entries so that the last coordinate of $F^w_{i_j}$ is not contained in $F^w_{i_j-1}$ (hence also not contained in any F^w_i with $i_{j-1} < i < i_j$). One checks that $\langle a_\Psi|e_F\rangle = 0$ for $F \in \Psi \cup \{\pm\mathbf{1}\}$ and $\langle a_\Psi|e_F\rangle < 0$ for $F \in \Phi \setminus \Psi$. Thus, every $\Gamma^\Psi + \mathbb{R}\mathbf{1}$ is a face of $\Gamma^\Psi + \mathbb{R}\mathbf{1}$. In particular, given $\Phi_1, \Phi_2 \in \Delta(\overline{\mathcal{L}}(M))$ the set

$$(\Gamma^{\Phi_1} + \mathbf{1}\mathbb{R}) \cap (\Gamma^{\Phi_2} + \mathbf{1}\mathbb{R}) = (\Gamma^{\Phi_1} \cap \Gamma^{\Phi_2}) + \mathbf{1}\mathbb{R} = \Gamma^{\Phi_1\cap\Phi_2} + \mathbf{1}\mathbb{R}$$

is a face of both cones $\Gamma^{\Phi_i} + \mathbf{1}\mathbb{R}$, $i = 1, 2$ (the displayed equalitites follow by Lemma 2.7). This proves that the intersection of any two members of $\mathscr{G}_M$ is a face of both, thus $\mathscr{G}_M$ is a fan. Moreover, the same observation shows that the correspondence $\Gamma^\Phi \mapsto \Gamma^\Phi + \mathbf{1}\mathbb{R}$ defines the desired combinatorial isomorphism. $\square$

Remark 2.9. Notice that the fan structures in (†) and (‡) are different: the latter has, in general, more cones. One usually refers to (‡) as the *fine* subdivision, and to (†) as the *coarse* subdivision of the Bergman fan. If M is connected, the rays of the fine subdivision that are also rays of the coarse subdivision are exactly the $\Gamma^{\{F\}}$ where F is a flacet.

Remark-Definition 2.10. From Proposition 2.8 we have that translation by $\mathbf{1}$ preserves $\widetilde{\mathscr{B}}(M)$ and its fan structure. Thus there is no loss of information in considering, as one often does in tropical geometry, the Bergman fan as a subset of the quotient $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. In order to study this situation let π_T denote the orthogonal projection onto the hyperplane $T = \mathbf{1}^\perp$ (with equation $\sum_{i\in[n]} x_i = 0$), and let

$$\overline{\mathscr{B}}(M) := \pi_T(\widetilde{\mathscr{B}}(M)).$$

Lemma 2.11. *We have*

$$\overline{\mathscr{B}}(M) = \bigcup_{\Phi\in\Delta(\overline{\mathcal{L}}(M))} \pi_T(\Gamma^\Phi),$$

where the union on the right-hand side defines a (simplicial) fan structure that refines the coarse structure given by $\{\pi_T(N_{[P_M\upharpoonright w]})\}_{w\in T}$ (see Equation (†)).

Proof. The set-theoretic union follows from Proposition 2.8 by definition of $\overline{\mathscr{B}}(M)$. We have to prove that $\{\pi_T(\Gamma^\Phi)\}_{\Phi\in\Delta(\overline{\mathcal{L}}(M))}$ is a fan.

To this end, notice that since $\{\mathbf{1}\}\cup\{e_F \mid F\in\Phi\}$ is a linearly independent set, the set $\{\pi_T(e_F) \mid F\in\Phi\}$ is linearly independent in T. Therefore $\pi_T(\Gamma^\Phi) = \mathrm{cone}\{\pi_T(e_F) \mid F\in\Phi\}$, and this is a simplicial cone; in particular, its faces are the $\pi_T(\Gamma^\Psi)$ with $\Psi\subseteq\Phi$. We are left with considering intersections of cones. Let $\Phi_1,\Phi_2\in\Delta(\overline{\mathcal{L}}(M))$. Obviously $\pi_T(\Gamma^{\Phi_1})\cap\pi_T(\Gamma^{\Phi_2})$

To this end, recall the proof of Proposition 2.8 and in particular that the facets of the cone $\Gamma^\Phi+\mathbb{R}\mathbf{1}$ are exactly the hyperplanes orthogonal to the vectors $a_{\Phi\setminus\{F\}}$ defined in (3), where F ranges in Φ. Now, obviously $\pi_T(\Gamma^\Phi) = \pi_T(\Gamma^\Phi+\mathbb{R}\mathbf{1}) = (\Gamma^\Phi+\mathbb{R}\mathbf{1})\cap T$, and since $a_\Psi\subseteq T$ for all Ψ, the set $\pi_T(\Gamma^\Phi)$ is defined, inside the vectorspace T, by the intersection of the halfspaces $\langle x|a_{\Phi\setminus\{F\}}\rangle\geq 0$, all bounded by hyperplanes $(a_{\Phi\setminus\{F\}})^\perp$ containing the origin. Therefore $\pi_T(\Gamma^\Phi)$ is a simplicial cone whose faces are all $\pi_T(\Gamma^\Psi)$ for $\Psi\subseteq\Phi$. Moreover, for any Φ_1,Φ_2 with Lemma 2.7 we have

$$\pi_T(\Gamma^{\Phi_1})\cap\pi_T(\Gamma^{\Phi_2}) = \cap\pi_T(\Gamma^{\Phi_1}\cap\Gamma^{\Phi_2}) = \pi_T(\Gamma^{\Phi_1\cap\Phi_2}),$$

proving that $\{\pi_T(\Gamma^\Phi)\}_{\Phi\in\Delta(\overline{\mathcal{L}}(M))}$ is a fan in T, combinatorially isomorphic to $\Gamma(M)$. □

Remark-Definition 2.12. The fan $\overline{\mathscr{B}}(M)$ is the cone over a cell complex denoted by $\mathscr{B}(M)$ and called the *Bergman complex* of M (one way to see this is to think about $\mathscr{B}(M)$ as the intersection of $\widetilde{\mathscr{B}}(M)$ with the unit sphere in T). In order to express this complex, for a given family $\Phi\subseteq 2^{[n]}$ of subsets of $[n]$ let

$$\gamma^\Phi := \mathrm{conv}\{e_F \mid F\in\Phi\},\qquad \gamma(M) := \bigcup_{\Phi\in\Delta(\overline{\mathcal{L}}(M))}\gamma^\Phi.$$

Proposition 2.13. *Let M be a matroid.*

(1) *The collection $\{\gamma^\Phi\}_{\Phi\in\Delta(\overline{\mathcal{L}}(M))}$ defines a structure of simplicial complex on the space $\gamma(M)$.*

(2) *The projection π_T induces a (linear) isomorphism of simplicial complexes between $\gamma(M)$ and $\pi_T(\gamma(M)) = \bigcup_{\Phi\in\Delta(\overline{\mathcal{L}}(M))}\pi_T(\gamma^\Phi)$.*

(3)
$$\overline{\mathscr{B}}(M) = \mathrm{cone}\,\pi_T(\gamma(M))$$

Proof. Exercise. □

Remark 2.14. The content of this proposition can be summarized as saying that the simplicial complex $\gamma(M)$ is isomorphic to a subdivision of the Bergman complex $\mathscr{B}(M)$. In fact, sometimes the name "Bergman complex" is used to refer to $\gamma(M)$.

Visualisation Method 2.15. In order to think about the Bergman fan $\widetilde{\mathscr{B}}(M)$ one usually ignores the lineality space $\mathbb{R}\mathbf{1}$ – essentially thinking about something like $\overline{\mathscr{B}}(M)$. In order to draw this simplicial complex, one exploits Proposition 2.13 and draws a representation of $\gamma(M)$, a simplicial complex whose vertices are the flats of M (except the minimal and the maximal one) and whose simplices correspond to increasing chains of flats.

If M is connected, we can recover the structure of $\widetilde{\mathscr{B}}(M)$ by noting that the vertices of the simplicial complex that correspond to rays of $\widetilde{\mathscr{B}}(M)$ are those e_F such that F is a flacet.

3. Sources and references

Section 1 follows [2], and the approach of Section 2 is inspired by [1]. An alternative treatment of a selection from the material of Lectures 2,3,4 is in [3], where one finds also a brief treatment of normal fans of polytopes. For more on the latter subject, see [4]. The notations follow the commonalities of [1, 2, 3].

References

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