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# NOTES FOR LECTURE 2

This Lecture Notes follow the treatment of this subject given in the papers by Gel'fand-Goresky-MacPherson-Serganova and by Feichtner-Sturmfels that are includer in the literature folder.

## 1. Matroid polytopes

1.1. **Basic definitions.** Let $n \in \mathbb{N}$. The *standard-simplex* $\Delta$ in $\mathbb{R}^n$ is the convex hull of the standard basis vectors,

$$\Delta := \operatorname{conv}\{e_i \mid i = 1, \dots, n\} \subseteq \mathbb{R}^n.$$

We write $\Delta^{(n)}$ if specification is needed. Notice that every edge of $\Delta$ is parallel to some vector of the form $e_i - e_j$ for $i, j \in [n]$.

Recall that we write $[n] := \{1, \dots, n\}$ and, for $r, n \in \mathbb{N}$, $\binom{n}{r} := \{A \subseteq [n] \mid |A| = r\}$.

Moreover, for any subset $A \subseteq [n]$ we write

$$e_A := \sum_{i \in A} e_i \in \mathbb{R}^n$$

for the sum of the basis vectors indexed by elements of $A$.

**Definition 1.1.** Given any family $\mathcal{X} \subseteq 2^{[n]}$, we consider then the polytope

$$P_{\mathcal{X}} := \operatorname{conv}\{e_X \mid X \in \mathcal{X}\}.$$

1.2. **The main theorem.**

**Theorem 1.2** (Gel'fand-Goresky-Macpherson-Serganova)**.** *A nonempty family $\mathcal{B} \subseteq \binom{n}{r}$ is the set of bases of a rank $r$ matroid on the ground set $[n]$ if and only if all edges of $P_{\mathcal{B}}$ are parallel to edges of $\Delta$.*

**Remark 1.3.** Often the statement of Theorem 1.2 is formulated in terms of a subset of vertices of the "hypersimplex" $P_{\binom{n}{r}}$ rather than "a nonempty family $\mathcal{B} \subseteq \binom{n}{r}$". The equivalence is obvious.

We comment on one aspect of the alternative formulation that is not immediatly apparent in ours: if $\mathcal{B} \subseteq \binom{n}{r}$, then the set of vertices of $P_{\mathcal{B}}$ is exactly $\{e_B \mid B \in \mathcal{B}\}$. In order to see this, given $X \in \binom{n}{r}$ consider the linear functional $\ell_X : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \langle e_X \mid x \rangle$. Then, $\ell_X(e_Y) \leq |X|$ for all $Y \in \binom{n}{r}$, with equality if and only if $X = Y$. In particular, $e_X$ cannot be a convex combination of any subset of the $e_Y$, $Y \in \binom{n}{r} \setminus \{X\}$.

$P_M$ (Matroid polytope)

**Definition 1.4.** Given a matroid $M$ we will write $P_M$ for $P_{\mathcal{B}(M)}$. Every such polytope is called a *matroid polytope.*

*Proof of Theorem 1.2.* We prove only one direction, and leave the other as a reading assignment, from the original paper (§4.3 of the paper by Gel'fand, Goresky, MacPherson and Serganova, available in the literature folder).

Assume then that every edge of the nonempty polytope $P_{\mathcal{B}}$ is parallel to a vector of the form $e_i - e_j$ for some $i, j \in [n]$. We have to prove that $\mathcal{B}$ satisfies axiom ($\mathcal{B}2$). Let then $B_1, B_2 \in \mathcal{B}$. Up to re-ordering the coordinates we can assume that

$$e_{B_1} = (1,\dots,1,0,\dots,0,1,\dots,1,0,\dots,0)$$
$$e_{B_2} = (\underbrace{0,\dots,0}_{A\text{-coord.}},\underbrace{1,\dots,1}_{B\text{-coord.}},\underbrace{1,\dots,1}_{C\text{-coord.}},\underbrace{0,\dots,0}_{D\text{-coord.}})$$

For some partition $[n] = A \uplus B \uplus C \uplus D$.

Now, by convexity of $P_{\mathcal{B}}$ the segment between $e_{B_1}$ and $e_{B_2}$ is fully contained in $P_{\mathcal{B}}$, which is contained in the vertex cone $C(P_{\mathcal{B}}, v)$ of $P_{\mathcal{B}}$ at $v := e_{B_1}$. The vertex cone was treated in the Warm-up to this lecture, where it was also explained that this cone is generated by the directions of the edges emanating from $v$, and is based at $v$. In particular,

$$e_{B_2} - e_{B_1} = (-1,\dots,-1;1,\dots,1;0,\dots,0;0,\dots,0) = \sum_{\substack{\text{conv}\{v,w\}\text{ is}\\ \text{edge of } P_{\mathcal{B}}}} \lambda_w(w-v) \qquad (\dagger)$$

with all $\lambda_w \in \mathbb{R}_{\geq 0}$. Now by assumption for every vertex $w$ of $P_{\mathcal{B}}$ appearing in ($\dagger$) the vector $w - v$ is a positive multiple of $e_i - e_j$ for some $i, j \in [n]$, i.e., in coordinates $i$ and $j$ we have $w_i > v_i$ and $w_j < v_j$[1]. Now both $v$ and $w$ are $0-1$ vectors (as is the case for every vertex of $P_{\mathcal{B}}$), thus necessarily $v_i = (e_{B_1})_i = 0$ and $v_j = (e_{B_1})_j = 1$. This means that $i \in B \cup D$ and $j \in A \cup C$.

In particular, on the r.h.s of ($\dagger$) there is never a negative contribution to coordinates in $D$. This implies that, if $\lambda_w > 0$, then $i \notin D$, since otherwise we'd have a positive contribution of $e_i$ to a $D$-coordinate that can't be cancelled by other terms. Analogously, $j \notin C$ whenever $\lambda_w > 0$.

In summary, for every vertex $w$ adjacent to $v$ such that $\lambda_w > 0$ in ($\dagger$), if $e_i - e_j$ is the direction of the edge between $w$ and $v$ then $i \in B$ and $j \in A$.

Now we return to checking ($\mathcal{B}2$). Let $b \in B_1 \setminus B_2$ and recall that in fact $B_1 \setminus B_2 = A$. Then $(e_{B_1} - e_{B_2})_b = -1$, and so there must be $w$ with $\lambda_w > 0$ in ($\dagger$) such that $w - v$ is a positive multiple of $e_{b'} - e_b$ for some $b' \in B = B_2 \setminus B_1$. Since $w$ and $v$ are vertices of $P_{\mathcal{B}} \subseteq \{x \mid 0 \leq |x_k| \leq 1 \text{ for all } k\}$, in fact $w - v = e_{b'} - e_b$. Thus, for the $B' \in \mathcal{B}$ such that $w = e_{B'}$ we have

$$B' = (B_1 \setminus \{b\}) \cup \{b'\},$$

and the proof is complete. □

Some corollaries follow directly from Theorem 1.2.

**Corollary 1.5.** *Every face of a matroid polytope is a matroid polytope.*

**Corollary 1.6.** *Every matroid polytope is a subset of the simplex $r \cdot \Delta$, where $r$ is the rank of the matroid and $\Delta = \Delta^{(n)}$, with $n$ the cardinality of the ground set. Notice that $r \cdot \Delta$ is described by $x_1 \geq 0, \dots, x_n \geq 0$ and $\sum_i x_i = r$.*

1.3. **Flats and flacets.** Our next goal is to determine which inequalities determine a matroid polytope *as a subset of* $r \cdot \Delta$. To this end we need a definition.

(Flats)

**Definition 1.7.** Let $M$ be a matroid with rank function rk on the ground set $E$. A subset $F \subseteq E$ is called a *flat* of $M$ if $\text{rk}(F \cup \{e\}) > \text{rk}(F)$ for all $e \in E \setminus F$.

**Lemma 1.8.** *Let $F_1, F_2$ be flats of $M$. Then, $F_1 \cap F_2$ is a flat of $M$.*

*Proof.* Let $e \in E \setminus (F_1 \cap F_2)$. In particular, there is $j \in \{1, 2\}$ for which $e \notin F_j$. Then,

$$\text{rk}(F_j) + \text{rk}((F_1 \cap F_2) \cup \{e\}) \geq \text{rk}(F_1 \cap F_2) + \text{rk}(F_j \cup \{e\}).$$

Since $F_j$ is a flat, $\text{rk}(F_j) < \text{rk}(F_j \cup \{e\})$, and the inequality above implies $\text{rk}((F_1 \cap F_2) \cup \{e\}) > \text{rk}(F_1 \cap F_2)$ as desired. □

[1] I owe this formulation to Tim - thanks!

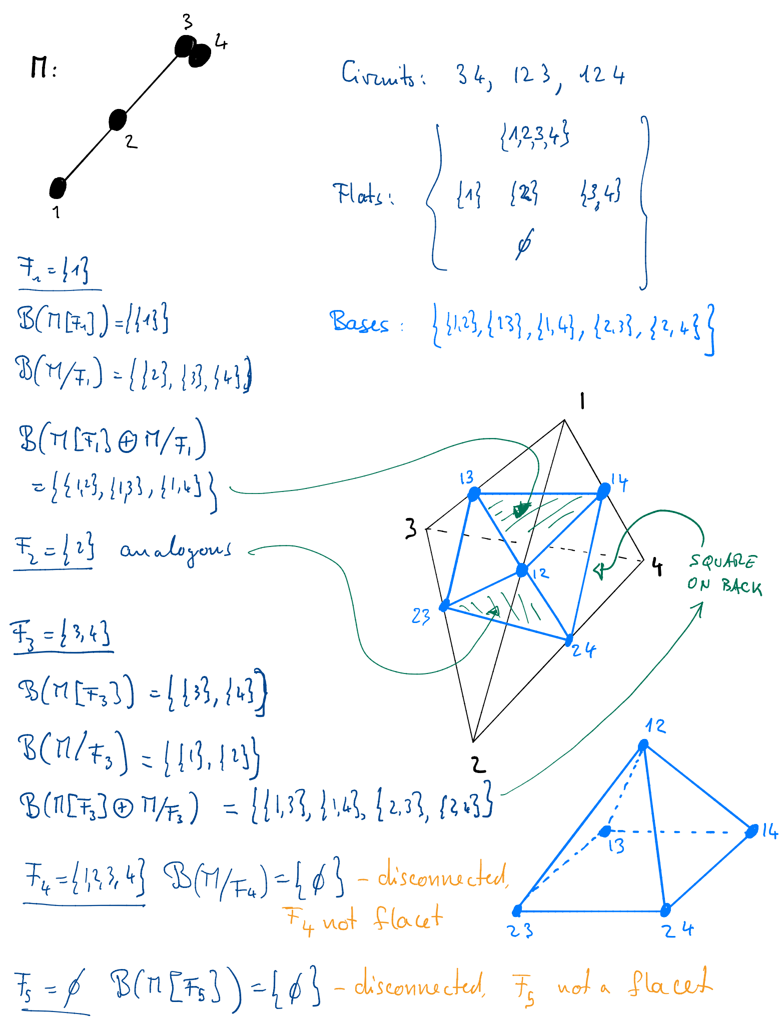



FIGURE 1. Running example for this lecture

(Closure)

**Corollary 1.9.** *For every $A \subseteq E$ there is a unique minimal flat of $M$, called $\mathrm{cl}(A)$ ("closure" of $A$), that is inclusion-minimal among all flats containing $A$. The closure satisfies $\mathrm{rk}(A) = \mathrm{rk}(\mathrm{cl}(A))$.*

*Proof.* Set

$$\mathrm{cl}(A) := \bigcap_{\substack{F \text{ flat of } M \\ A \subseteq F}} F$$

and apply Lemma 1.8 to obtain the first claim.

For the second claim, consider $\overline{A} := \max\{X \supseteq A \mid \mathrm{rk}(X) = \mathrm{rk}(A)\}$. This definition is well-posed: indeed, given $X', X''$ in the set on the r.h.s., (R3) applied to $X'$ and $X''$ implies that $\mathrm{rk}(X' \cup X'') = \mathrm{rk}(A)$, hence $X' \cup X''$ is of the set on the r.-h. s. as well, and so the maximum exists. Now, clearly $\overline{A}$ is a flat and $\mathrm{rk}(\overline{A}) = \mathrm{rk}(A)$. Thus $\mathrm{cl}(A) \subseteq \overline{A}$, which implies the middle inequality of the following expression

$$\mathrm{rk}(A) = \mathrm{rk}(\overline{A}) \geq \mathrm{rk}(\mathrm{cl}(A)) \geq \mathrm{rk}(A)$$

where the last inequality holds by monotonicity of the rank function.

Now obviously all above inequalities must be equalities, and the second part of the claim follows. □

**Theorem 1.10.** *Let $M$ be a matroid on the ground set $[n]$ with rank function* rk. *Then*

$$P_M = \{x \in r \cdot \Delta^{(n)} \mid \sum_F x_i \leq \mathrm{rk}(F) \text{ for all flats } F \text{ of } M\}$$

*Proof.*

Claim 1 For every $A \subseteq E$, $P_M$ satisfies the inequality

$$\sum_{i \in A} x_i \leq \mathrm{rk}(A). \tag{1}$$

Moreover, the inequality is satisfied with equality for at least one vertex of $P_M$.
*Proof.* We prove that every vertex of $P_M$ satisfies (1). Let $B$ be a basis of $M$ and consider the associated vertex of $P_M$ given by $v := e_B$. Then, $\sum_{i \in A} v_i = |B \cap A|$, and $B \cap A$ is independent. The maximum size of an independent subset of $A$ is, by definition, $\mathrm{rk}(A)$, whence $v$ satisfies (1). If we pick any maximal independent set $I \subseteq A$ and extend $I$ to a basis $B$ of $M$[2], we have $\sum_{i \in A}(e_B)_i = |B \cap A| = |I| = \mathrm{rk}(A)$, thus the vertex $e_B$ satisfies Equation 1 with equality.

Claim 2 Every facet $Q$ of $P_M$ that intersects the interior of $r \cdot \Delta$ can be defined by an inequality of the form

$$\sum_{i \in A} x_i \leq \mathrm{rk}(A)$$

for some $A \subseteq E$ .
*Proof.* The claim is void, hence trivially true, in case $\dim P_M \leq 1$. In that case the facets of $P_M$ are its vertices, and those all lie on the boundary of the simplex $r \cdot \Delta$. Assume then $\dim P_M \geq 2$.

First, we prove that we can find a normal vector to $Q$ with all entries either equal to 0 or 1. A vector is normal to $Q$ if and only if it is normal to the direction of every edge of $Q$. Now, the edges of $Q$ are edges of $P_M$ and so they have the direction $e_i - e_j$ for some $i, j \in [n]$. Consider now on $[n]$ the equivalence relation defined as the transitive closure of $i \sim_Q j$ if $Q$ has an edge in direction $\pm(e_i - e_j)$. Let $X$ be an equivalence class of $\sim_Q$ with $|X| > 1$. Such an $A$ exists because $Q$ has at least one edge (since $\dim Q = \dim P_M - 1 \geq 1$). Pick a vertex $v$ of $Q$. Since $\dim P_M > \dim Q$ and $P_M \subseteq C(P_m, v)$, there must be an edge of $P_M$ that does not lie in $\mathrm{aff}(Q)$. Letting $e_k - e_l$ be the direction of this edge, we must have $k \not\sim_Q l$ (since $i \sim_Q j$ implies that $\mathrm{aff}(Q)$ contains an affine line in direction $e_i - e_j$). Thus, up to relabeling $k$ and $l$ we can assume that the equivalence class $[l]_{\sim_Q}$ is disjoint from $X$. Let then

$$A := X \cup [k]_{\sim_Q}, \qquad a := e_A.$$

This vector is orthogonal to every edge of $Q$ (since $a_i = a_j$ whenever $i \sim_Q j$), but not to every edge of $P_M$ (because $a_k \neq a_l$). This means that the linear form $\sum_{i \in A} x_i$ is constant on $Q$ - say with value $b$ - but not on $P_M$. In particular, since $Q$ is a facet of the polytope $P_M$ we know that the facet $Q$ is defined either by $\sum_{i \in A} x_i \geq b$ or $\sum_{i \in A} x_i \leq b$. If the former case arises, we can use the fact that $P_M$ lies in the affine hyperplane $\sum_{i \in E} x_i = r$ and substract $r$ on both sides of the inequality, obtaining $-\sum_{i \in E \setminus A} x_i \geq b - r$, i.e., $\sum_{i \in E \setminus A} x_i \leq r - b$.

In any case (i.e., up to switching $A$ with $E \setminus A$ and $b$ with $r - b$) there is a subset $A \subseteq E$ and a number $b \in \mathbb{R}$ such that $\sum_{i \in A} x_i \leq b$ is an inequality defining the facet $Q$. Now by Claim 1 we know that it must be $b = \mathrm{rk}(A)$.

[2] This is possible by repeated application of $(\mathcal{I}3)$.

**Illustrative example.** Consider the example in Figure 1 and let $Q$ be the "top horizontal" triangular facet (i.e., with vertices $13, 14, 12$). If we want to determine a defining inequality for $Q$ following the method of the proof of Claim 2, we first look at the equivalence relation $\sim_Q$ which, in this case, has equivalence classes $\{1\}, \{2, 3, 4\}$. We would choose $X = \{2, 3, 4\}$. We'd then look at the direction of an edge exiting $Q$, say the edge between 12 and 24 with direction $e_4 - e_1$, hence $k = 4$, $l = 1$ so that $[l]_{\sim_Q} = \{1\}$ is disjoint from $X$. Now, in this case $[k]_{\sim_Q} = [4]_{\sim_Q} = X$, therefore $a = e_X = e_A = (0, 1, 1, 1)$. Now $\sum_{i \in A} x_i = x_2 + x_3 + x_4$ has value 1 on every vertex of $Q$, and value 2 on other vertices of $P_M$, thus we obtain the facet-defining inequality is $x_2 + x_3 + x_4 \geq 1$. This is not yet of the desired form. We substract $x_1 + x_2 + x_3 + x_4 = 2 = r$ on both sides and we obtain $-x_1 \geq -1$, i.e., $x_1 \leq 1$, which is now of the desired form.

Claim 3 For every $A \subseteq E$ and every $x \in r \cdot \Delta$, the inequality $\sum_{i \in \mathrm{cl}(A)} x_i \leq \mathrm{rk}(\mathrm{cl}(A))$ implies $\sum_{i \in A} x_i \leq \mathrm{rk}(A)$.

*Proof.* In fact,

$$\sum_{i \in A} x_i \leq \sum_{i \in \mathrm{cl}(A)} x_i \leq \mathrm{rk}(\mathrm{cl}(A)) = \mathrm{rk}(A).$$

The first inequality because $A \subseteq \mathrm{cl}(A)$, the last equality by the second claim in Corollary 1.9.

Now, by Claims 1 and 2 the Theorem's claimed equality holds without restriction on $F$, and Claim 3 shows that it is possible to restrict $F$ to be a flat.

□

## 2. Minors, matroid connectivity and faces of $P_M$

2.1. **Connectivity and dimension.** Let $M$ be a matroid on the ground set $E$. Define a relation on $E$ via

$$e \sim_M f \Leftrightarrow \text{ either } e = f, \text{ or there are bases } B_1, B_2 \in \mathcal{B}(M) \text{ with } B_2 = (B_1 \setminus \{e\}) \cup \{f\}.$$

**Lemma 2.1.** *Let $M$ be a matroid on the ground set $E$. Then, for all $e, f \in E$:*

$$e \sim_M f \Leftrightarrow \text{ either } e = f, \text{ or } \{e, f\} \subseteq C \text{ for some circuit } C \in \mathcal{C}(M).$$

*In particular, $\sim_M$ is an equivalence relation.*

*Proof.* Consider two elements $e, f \in E$, $e \neq f$.

If $\{e, f\} \subseteq C$ for some circuit $C$, the set $C \setminus \{f\}$ is independent and can be completed to a basis $B_1$ of $M$ not containing $f$. Now let $B_2 := B_1 \setminus \{e\} \cup \{f\}$. If there is any circuit $C' \subseteq B_2$, then it must contain $f$ (otherwise $C' \subseteq B_1$, contradicting independence of $B_1$). Now $(\mathcal{C}3)$ applied to $C$, $C'$ and $f$ would give a circuit fully contained in $(C \cup C') \setminus \{f\} \subseteq B_1$, a contradiction. Thus, $B_2$ is independent and, since $|B_1| = |B_2|$, it is a basis.

On the other hand, if there are bases $B_1, B_2 \in \mathcal{B}(M)$ with $B_2 = B_1 \setminus \{e\} \cup \{f\}$, then in particular $B_1 \cup \{f\}$ is dependent, and there is a circuit $C \subseteq B_1 \cup \{f\}$ with $f \in C$. If $e \notin C$, then $C \subseteq B_2$, a contradiction. Therefore $\{e, f\} \subseteq C$ as desired.

That $\sim_M$ is an equivalence relation follows from its formulation in terms of circuits by $(\mathcal{C}3)$. □

$c(M)$ (connectivity)

**Definition 2.2.** The equivalence classes of $\sim_M$ are called *connected components* of $M$. The *connectivity* of the matroid $M$ is the number $c(M)$ of equivalence classes of $\sim_M$. We call $M$ *connected* if $c(M) = 1$, disconnected otherwise.

**Remark 2.3.**

(1) Since equivalence classes are always nonempty, for the empty matroid $M = (\emptyset, \{\emptyset\})$ we have $c(M) = 0$, thus the empty matroid is disconnected according to Definition 2.2.

(2) It is important to notice that the notion of connectivity for matroids is *not* a generalization of the notion of connectivity for graphs (rather, if $G$ is a graph, connectivity of $M(G)$ has to do with "2-connectedness" of $G$, see §4.1 in Oxley's book). Can you find a connected graph $G$ such that $M(G)$ is disconnected as a matroid?

The notion of connectivity is directly related to the dimension of the matroid polytope.

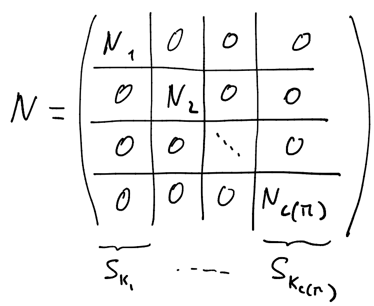
**Proposition 2.4.** *Let $M$ be a matroid on the ground set $[n]$. Then*

$$\dim P_M = n - c(M)$$

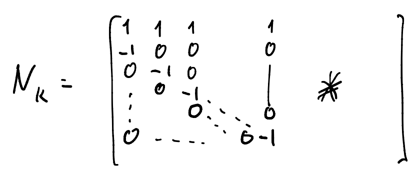
*Proof.* Lemma 2.1 shows that, for any $i \neq j$, $i \simeq_M j$ if and only if there are vertices $e_{B_1}$ and $e_{B_2}$ of $P_M$ that are connected by a segment in the direction of $e_i - e_j$. In particular, the linear space in $\mathbb{R}^n$ that is parallel to the affine span of $P_M$ is generated by

$$S := \{e_i - e_j \mid i \sim_M j\} = \biguplus_{K \in E/\sim_M} \underbrace{\{e_i - e_j \mid i, j \in K\}}_{=:S_K}$$

where $K$ runs over all connected components of $M$. Now the dimension of $P_M$ is the (column-)rank of the matrix $N$ whose columns are the elements of $S$. If we number $K_1, \ldots, K_{c(M)}$ the connected components of $M$ and rearrange the coordinate labels so that elements of $K_h$ come before elements of $K_k$ whenever $h < k$, the matrix $N$ has the block-diagonal form



Where, up to permuting and negating columns, the block $N_k$ can be written



and has thus rank $|K_k| - 1$. It follows that the rank of $N$, and hence the dimension of $P_M$, equals $\sum_k (|K_k| - 1) = n - c(M)$.

□

2.2. **Direct sums and cartesian products.**

**Lemma 2.5.** *Let $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$ be two matroids given in terms of independent sets and with disjoint ground sets ($E_1 \cap E_2 = \emptyset$). Then*

$$\mathcal{I}_1 \oplus \mathcal{I}_2 := \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1,\, I_2 \in \mathcal{I}_2\}$$

*is the family of independent sets of a matroid on $E_1 \uplus E_2$.*

*Proof.* Exercise (see Worksheet 2). □

Direct sum

**Definition 2.6.** Let $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$ be as in Lemma 2.5. The matroid $M_1 \oplus M_2 := (E_1 \uplus E_2, \mathcal{I}_1 \oplus \mathcal{I}_2)$ is called the *direct sum* of $M_1$ and $M_2$.

**Remark 2.7.** A matroid is disconnected (in the sense of §2.1) if and only if it can be written as a direct sum of two nonempty matroids. See Worksheet 2 for a precise statement.

**Corollary 2.8.** *Let $M_1$, $M_2$ be matroids on disjoint ground sets. Then*

$$\mathcal{B}(M_1 \oplus M_2) = \{B_1 \uplus B_2 \mid B_1 \in \mathcal{B}(M_1),\ B_2 \in \mathcal{B}(M_2)\}.$$

**Proposition 2.9.** *Let $M_1$, $M_2$ be matroids on disjoint ground sets. Then*

$$P_{M_1 \oplus M_2} = P_{M_1} \times P_{M_2}$$

*Proof.* From our Warm-up we know that $P_{M_1} \times P_{M_2} = \operatorname{conv}\{e_{B_1} + e_{B_2} \mid B_1 \in \mathcal{B}(M_1),\ B_2 \in \mathcal{B}(M_2)\}$. Now the claim follows from Corollary 2.8 since $e_{B_1} + e_{B_2} = e_{B_1 \uplus B_2}$. □

2.3. **Flacets.** Our last step will be to eliminate the redundancy from Theorem 1.10, i.e., to understand which flats do indeed define *facets* of $P_M$ that meet the interior of $r \cdot \Delta$. For this, given a matroid $M$ and a subset $X$ of the ground set of $M$, aside from the already mentioned notion of restriction $M[X]$, we need to define the "contraction" of $X$ in $M$.

First recall that, if $\operatorname{rk}(X) = s$, the bases of $M[X]$ are

$$\mathcal{B}(M[X]) = \max \mathcal{I}[X] = \{X \cap B \mid B \in \mathcal{B}(M),\ |X \cap B| = s\}.$$

Now consider the set

$$\mathcal{B}_{/X} := \{B \setminus X \mid B \in \mathcal{B}(M),\ |X \cap B| = s\}.$$

We would like to prove that $\mathcal{B}_{/X}$ is the set of bases of a matroid. This we will do in Corollary 2.11, but first a word about duality.

$M^*$
*(Dual matroid)*

**Lemma-Definition 2.10.** *Let $M$ be a matroid on the ground set $E$. Then*

$$\mathcal{B}^*(M) := \{E \setminus B \mid B \in \mathcal{B}(M)\}$$

*is the set of bases of a matroid on $E$ that is called* dual *to $M$, and denoted $M^*$.*

*Proof.* Consider the following preparatory claim:

> *Let $M$ be a matroid on the ground set $E$. Given $B_1, B_2 \in \mathcal{B}(M)$ and $e \in B_2 \setminus B_1$, then there is $f \in B_1 \setminus B_2$ such that $(B_1 \setminus \{f\}) \cup \{e\} \in \mathcal{B}(M)$.*
> *Proof.* By maximality of $B_1$, the set $B_1 \cup \{e\}$ is dependent and thus contains a circuit $C$. Necessarily $e \in C$, otherwise $C \subseteq B_1$ and the latter would be dependent. Also, $C \neq \{e\}$, since otherwise $B_2$ would be dependent. Therefore $C \cap B_1 \neq \emptyset$, so we can choose $f$ in this set and let $B_1' := B_1 \setminus \{f\} \cup \{e\}$. Now $B_1'$ is independent: any circuit $C' \subseteq B_1'$ would contain $e$ and, with $(\mathcal{C}3)$ applied to $C$, $C'$ and $e$, it would lead to the existence of a circuit fully contained in $B_1$. Since $|B_1| = |B_1'|$, the claim follows.

Now consider $B_1^*, B_2^* \in \mathcal{B}^*(M)$ and $e \in B_1^* \setminus B_2^*$. By definition, $B_1 = E \setminus B_1^*$ and $B_2 = E \setminus B_2^*$ are bases of $M$, and we have $e \in B_2 \setminus B_1$. Now by our "preparatory claim" we can find $f \in B_1 \setminus B_2$ so that $B_3 := (B_1 \setminus \{f\}) \cup \{e\}$ is in $\mathcal{B}(M)$. The basis $B_3^* := E \setminus B_3 \in \mathcal{B}^*(M)$ then satisfies $B_3^* = (B_1^* \setminus \{e\}) \cup \{f\}$ and witnesses $(\mathcal{B}2)$ for the chosen $B_1^*, B_2^*, e$. Aziom $(\mathcal{B}1)$ is obviously true for $\mathcal{B}^*(M)$, since this set contains the complement of any element of the (nonempty) set $\mathcal{B}(M)$. This concludes the proof. □

$M/X$
*(Contraction)*

**Corollary-Definition 2.11.** *The set $\mathcal{B}_{/X}$ is the set of bases of a matroid that we call $M/X$, the* contraction *of $X$ in $M$.*

*Proof.* This follows directly from Lemma 2.10 because $\mathcal{B}_{/X} = \mathcal{B}^*(M^*[E \setminus X])$.

To see this last fact, notice that the bases of $M^*$ are the sets $E \setminus B$ for $B$ a basis of $M$; hence the bases of $M^*[E \setminus X]$ are the sets $(E \setminus B) \cap (E \setminus X)$ with maximal possible cardinality, i.e., such that $B \cap X$ has maximal cardinality, this maximal cardinality being $s = \mathrm{rk}(X)$. Therefore, writing $X^c$ and $B^c$ for $E \setminus X$ and $E \setminus B$,

$$\mathcal{B}^*(M^*[E \setminus X]) = \{\underbrace{X^c \setminus (B^c \cap X^c)}_{=X^c \setminus B^c = B \setminus X} \mid |B \cap X| = s\}$$

and the claim follows. □

Flacet

**Definition 2.12.** A flat $F$ of a matroid $M$ is called a *flacet* if the inequality $\sum_{i \in F} x_i \leq \mathrm{rk}(F)$ defines a facet of $P_M$ that meets the interior of $r \cdot \Delta$.

**Proposition 2.13.** *Let $M$ be a connected matroid. A flat $F$ is a flacet of $M$ if and only if both $M[F]$ and $M/F$ are connected.*

**Example 2.14.** See Figure 1 for an illustration of this theorem. All three "green" facets are flacets.

*Proof of Proposition 2.13.* Let $F$ be a flat. The vertices of the face $Q$ of $P_M$ given by $P_M \cap \{\sum_{i \in F} x_i = \mathrm{rk}(F)\}$ are exactly those $e_B$ where $B \in \mathcal{B}(M)$ is such that $|B \cap F| = \mathrm{rk}(F)$. Since $F$ is a flat, these $B$ are exactly ([3]) the bases of the matroid

$$M_F := M/F \oplus M[F].$$

Now, with Proposition 2.9 we can write

$$Q = P_{M_F} = P_{M/F} \times P_{M[F]}$$

and the dimension of $Q$ can be computed as $\dim Q = \dim(P_{M/F}) + \dim(P_{M[F]}) = |E \setminus F| - c(M/F) + |F| - c(M[F])$.

Now $Q$ is a facet if and only if $\dim Q = \dim P_M - 1 = n - 2$ (recall that $M$ is connected). This is the case exactly when $c(M/F) = c(M[F]) = 1$. □

[3] The inclusion $\{B \in \mathcal{B}(M) \mid |B \cap F| = \mathrm{rk}(F)\} \subseteq \mathcal{B}(M/F \oplus M[F])$ is clear. For the reverse inclusion consider any basis $B_1 \uplus B_2$ of $M/F \oplus M[F]$ and notice that if it not a basis of $M$ then there must be some $e \in B_1 \subseteq E \setminus F$ such that $\{e\} \cup B_2$ is dependent, i.e., $\mathrm{rk}(\{e\} \cup B_2) = |B_2| = \mathrm{rk}(F)$. Now (R3) gives $\mathrm{rk}(B_2 \cup \{e\}) + \mathrm{rk}(F) \geq \mathrm{rk}(B_2) + \mathrm{rk}(F \cup \{e\})$ and so we conclude $\mathrm{rk}(F) \geq \mathrm{rk}(F \cup \{e\})$, which implies $\mathrm{rk}(F) = \mathrm{rk}(F \cup \{e\})$ via (R2), a contradiction to $F$ being a flat.