Today: • Rational generating functions
• Ehrhart theory

• Idea: represent any $f : \mathbb{Z}^d \to \cancel{\mathbb{K}}\ \mathbb{C}$ as a formal expression
any field.

$$\sum_{m \in \mathbb{Z}^d} f(m) \underbrace{z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}}_{z^m}$$

"formal Laurent series"

$$\mathbb{C}[[z_1, \dots z_d]] := \left\{ \underbrace{\sum_{m \in \mathbb{N}^d} f(m)\, z^m}_{\downarrow} \;\middle|\; f : \mathbb{N}^n \to \mathbb{C} \right\}$$

"formal power series"

degree $= \max \{ \sum_i m_i \mid f(m) \neq 0 \} \in \mathbb{N} \cup \{\infty\}$

Addition & Multiplication are defined:

$$\left(\sum f(m) z^m\right) + \left(\sum g(m) z^m\right) = \sum (f(m)+g(m))\, z^m$$

$$\left(\sum f(m) z^m\right)\left(\sum g(m) z^m\right) = \sum_m \left(\sum_k f(k)\, g(m-k)\right) z^m$$

<u>Note</u>: The following identity holds in $\mathbb{C}[[z]]$

$$(1-z)^{-1} = \sum_{k\in\mathbb{N}} z^k$$

In fact: $(1-z) \sum_{k\in\mathbb{N}} z^k = (1+z+z^2 \dots) - (z+z^2+z^3 \dots) = 1$

Def Call "rational" any FPS that equals a rational expression

IDEA

$f: \mathbb{N} \to \mathbb{C}$ agrees with a polynomial $\longleftrightarrow$ Rationality of FPS $\sum_{n\in\mathbb{N}} f(n) z^n$

Let me explain...

Theorem Fix $\alpha_1 \cdots \alpha_d \in \mathbb{C}$, $d \geq 1$, $\alpha_d \neq 0$. Then, for any $f: \mathbb{N} \to \mathbb{C}$ TFAE:

(i) $\boxed{\sum_{n \in \mathbb{N}} f(n) z^n = \frac{p(z)}{q(z)}}$ "rationality" where $\begin{cases} p \text{ polynomial degree } < d, \\ q(z) = 1 + \alpha_1 z + \cdots + \alpha_d z^d \end{cases}$

(ii) For all $n \in \mathbb{N}$: $\boxed{\alpha_0 f(n+d) + \alpha_1 f(n+d-1) + \cdots + \alpha_d f(n) = 0}$ "recursion"

(iii) For all $n \in \mathbb{N}$: $\boxed{f(n) = \sum_{i=1}^{k} p_i(n) \gamma_i^k}$ "polynomial", where all γ_i are nonzero complex numbers s.t. $\underbrace{1 + \alpha_1 z + \cdots + \alpha_d z^d}_{\text{"}q(z)\text{"}} = (1 - \gamma_1 z)^{d_1} \cdots (1 - \gamma_k)^{d_k}$, each p_i is a polynomial of degree less than d_i

Proof: Fix $q(z) := 1 + \alpha_1 z + \cdots + \alpha_d z^d = (1 - \gamma_1 z)^{d_1} \cdots (1 - \gamma_k z)^{d_k}$

Idea: define 4 vectorspaces:

Theorem Fix $\alpha_1 \cdots \alpha_d \in \mathbb{C}$, $d \geq 1$, $\alpha_d \neq 0$. Then, for any $f: \mathbb{N} \to \mathbb{C}$ TFAE:

"rationality"

(i) $\boxed{\sum_{n\in\mathbb{N}} f(n) z^n = \frac{p(z)}{q(z)}}$ where $\begin{cases} p \text{ polynomial degree } < d, \\ q(z) = 1 + \alpha_1 z + \cdots + \alpha_d z^d \end{cases}$

(ii) For all $n \in \mathbb{N}$: $\boxed{\alpha_0 f(n+d) + \alpha_1 f(n+d-1) + \cdots + \alpha_d f(n) = 0}$ "recursion"

"polynomial"

(iii) For all $n \in \mathbb{N}$: $\boxed{f(n) = \sum_{i=1}^{k} p_i(n) \gamma_i^k}$, where all γ_i are nonzero complex numbers s.t. $\underbrace{1 + \alpha_1 z + \cdots + \alpha_d z^d}_{\text{"}q(z)\text{"}} = (1-\gamma_1 z)^{d_1} \cdots (1-\gamma_k z)^{d_k}$, each p_i is a polynomial of degree less than d_i

$V_1 := \{ f: \mathbb{N} \to \mathbb{C} \text{ satisfying (i)} \}$

$\dim V_1 = d$

$V_2 := \{ f: \mathbb{N} \to \mathbb{C} \text{ satisf. (ii)} \}$

$\dim V_2 = d$

$V_3 := \{ f: \mathbb{N} \to \mathbb{C} \text{ satisfying (iii)} \}$ $\dim V_3 = d$

$$V_4 := \left\{ f: \mathbb{N} \to \mathbb{C} \;\middle|\; \begin{array}{l} \sum_{n\in\mathbb{N}} f(n) z^n = \sum_{i=1}^{k} \sum_{j=1}^{d_i} \beta_{ij} (1-\gamma_i z)^{-j}, \text{ for some } \beta_{ij} \in \mathbb{C}, \\ \text{where } \gamma_i, d_i \text{ are as in (iii)} \end{array} \right\}$$

$\dim V_4 = ?$ Write $R_{ij}(z) := (1-\gamma_i z)^{-j}$ $i = 1 \cdots k$, $j = 1 \cdots d_i$.

V_4 is generated (over $\mathbb{C}$) by all $R_{ij}(z)$, and there are $\sum_{i=1}^{k} d_i = d$ of them.

This already shows $\dim V_4 \leq d$.

Need to prove: R_{ij} are linearly independent. DONE (next page)

$\Rightarrow \dim V_4 = d$

<u>Claim:</u> The $R_{ij}(z) := (1-\gamma_i z)^j \quad i=1\dots k,\ j=1\dots d_i$ are lin. indep.

<u>Proof:</u> Consider linear dependency $\sum c_{ij} R_{ij}(z) = 0$ for some c_{ij}, not all equal 0, choose some i_0 with $c_{i_0 j} \neq 0$, take j_0 maximal s.t. $c_{i_0 j_0} \neq 0$

Then $$(1-\gamma_{i_0} z)^{j_0} \sum c_{ij} R_{ij}(z) = 0,$$ and setting "$z = \frac{1}{\gamma_{i_0}}$" implies $c_{i_0 j_0} = 0$ $\lightning$

$V_1 := \{ f : \mathbb{N} \to \mathbb{C} \text{ satisfying } (i) \}$

$\dim V_1 = d$

$V_2 := \{ f : \mathbb{N} \to \mathbb{C} \text{ satisf. } (ii) \}$

$\dim V_2 = d$

$V_3 := \{ f : \mathbb{N} \to \mathbb{C} \text{ satisfying } (iii) \}$

Theorem Fix $\alpha_1, \dots, \alpha_d \in \mathbb{C}$, $d \geq 1$, $\alpha_d \neq 0$. Then, for any $f : \mathbb{N} \to \mathbb{C}$ TFAE:

"rationality"

(i) $\boxed{\sum_{n \in \mathbb{N}} f(n) z^n = \frac{p(z)}{q(z)}}$ where $\begin{cases} p \text{ polynomial degree } < d, \\ q(z) = 1 + \alpha_1 z + \dots + \alpha_d z^d \end{cases}$

(ii) For all $n \in \mathbb{N}$: $\boxed{\alpha_0 f(n+d) + \alpha_1 f(n+d-1) + \dots + \alpha_d f(n) = 0}$ "recursion"

"polynomial"

(iii) For all $n \in \mathbb{N}$: $\boxed{f(n) = \sum_{i=1}^{k} p_i(n) \gamma_i^k}$, where all γ_i are nonzero complex numbers s.t. $\underbrace{1 + \alpha_1 z + \dots + \alpha_d z^d}_{\text{"}q(z)\text{"}} = (1-\gamma_1 z)^{d_1} \cdots (1-\gamma_k z)^{d_k}$, each p_i is a polynomial of degree less than d_i

$$V_4 := \left\{ f : \mathbb{N} \to \mathbb{C} \;\middle|\; \sum_{n \in \mathbb{N}} f(n) z^n = \sum_{i=1}^{k} \sum_{j=1}^{d_i} \beta_{ij} (1-\gamma_i z)^{-j} \;(\#), \text{ for some } \beta_{ij} \in \mathbb{C}, \text{ where } \gamma_i, d_i \text{ are as in (iii)} \right\}$$

Now: enough to prove $V_1 = V_2 = V_3 = V_4$.

① $\underline{V_3 = V_4}$. For dimension reasons, enough to show $V_4 \subseteq V_3$

Notice: $\frac{1}{(1-\gamma_i z)^j} = \left(\sum_{n \in \mathbb{N}} \gamma_i^n z^n \right)^j = \sum_{n \in \mathbb{N}} \underbrace{\binom{j+n-1}{j-1}}_{\text{polynomial in } n, \text{ degree } j-1} \gamma_i^n z^n$

coeff. of z^n in $(\#)$

Now for $f \in V_4$: $f(n) = \sum_{i=1}^{k} \left[\underbrace{\left(\sum_{j=1}^{d_i} \beta_{ij} \binom{j+n-1}{j-1} \right)}_{\text{polynomial degree } < d_i} \gamma_i^n \right] \in V_3$!

$V_1 := \{ f: \mathbb{N} \to \mathbb{C} \text{ satisfying (i)} \}$

$\dim V_1 = d$

$V_2 := \{ f: \mathbb{N} \to \mathbb{C} \text{ satisf. (ii)} \}$

$\dim V_2 = d$

$V_3 := \{ f: \mathbb{N} \to \mathbb{C} \text{ satisfying (iii)} \}$

Theorem Fix $\alpha_1 \cdots \alpha_d \in \mathbb{C}$, $d \geq 1$, $\alpha_d \neq 0$. Then, for any $f: \mathbb{N} \to \mathbb{C}$ TFAE:

"rationality"

(i) $\boxed{\sum_{n \in \mathbb{N}} f(n) z^n = \frac{p(z)}{q(z)}}$ where $\begin{cases} p \text{ polynomial degree} < d, \\ q(z) = 1 + \alpha_1 z + \cdots + \alpha_d z^d \end{cases}$

(ii) For all $n \in \mathbb{N}$: $\boxed{\alpha_0 f(n+d) + \alpha_1 f(n+d-1) + \cdots + \alpha_d f(n) = 0}$ "recursion"

"polynomial"

(iii) For all $n \in \mathbb{N}$: $\boxed{f(n) = \sum_{i=1}^{k} p_i(n) \gamma_i^k}$, where all γ_i are nonzero complex numbers s.t. $\underbrace{1 + \alpha_1 z + \cdots + \alpha_d z^d}_{\text{"}q(z)\text{"}} = (1 - \gamma_1 z)^{d_1} \cdots (1 - \gamma_k z)^{d_k}$, each p_i is a polynomial of degree less than d_i

$V_4 := \ldots$

(1) $V_3 = V_4$

(2) $V_4 = V_1$ (enough $V_4 \subseteq V_1$) $\rightarrow$ Lecture notes.

(3) $V_1 = V_2$ - enough to show $V_1 \subseteq V_2$, but

For $f \in V_1$: $q(z) \sum_{n \in \mathbb{N}} f(n) z^n = p(z)$ (as in (i)).

$\leftarrow$ coeff. of z^d is 0

$\underbrace{1 + \alpha_1 z + \cdots + \alpha_d z^d}_{\text{coefficient of } z^d \text{ is the recursion in (ii)}}$

Theorem Fix $\alpha_1, \dots, \alpha_d \in \mathbb{C}$, $d \geq 1$, $\alpha_d \neq 0$. Then, for any $f : \mathbb{N} \to \mathbb{C}$ TFAE:

(i) $\boxed{\sum_{n \in \mathbb{N}} f(n) z^n = \frac{p(z)}{q(z)}}$ "rationality" where $\begin{cases} p \text{ polynomial degree } < d, \\ q(z) = 1 + \alpha_1 z + \dots + \alpha_d z^d \end{cases}$

(ii) For all $n \in \mathbb{N}$: $\boxed{\alpha_0 f(n+d) + \alpha_1 f(n+d-1) + \dots + \alpha_d f(n) = 0}$ "recursion"

(iii) For all $n \in \mathbb{N}$: $\boxed{f(n) = \sum_{i=1}^{k} p_i(n) \gamma_i^n}$ "polynomial", where all γ_i are nonzero complex numbers s.t. $\underbrace{1 + \alpha_1 z + \dots + \alpha_d z^d}_{\text{"}q(z)\text{"}} = (1 - \gamma_1 z)^{d_1} \cdots (1 - \gamma_k z)^{d_k}$, each p_i is a polynomial of degree less than d_i.

Example: Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_{n+2} - F_{n+1} - F_n = 0$

By theorem (i)$\Leftrightarrow$(ii): $\boxed{\Phi := \left(\sum_{n \in \mathbb{N}} F_n z^n \right) = \frac{p(z)}{1 - x - x^2}}$ $= (1 - \varphi z)(1 - \bar{\varphi} z)$

γ_1, γ_2 "$k=2$"

$d_1 = 1$, $d_2 = 1$

Initial conditions $F_0 = 0$, $F_1 = 1$ imply $p(z) = z$

By theorem (iii) $F_n = \alpha \gamma_1^n + \beta \gamma_2^n$ for some $\alpha, \beta \in \mathbb{C}$.

(can be found using $F_0 = 0$, $F_1 = 1$)

Note We call $f : \mathbb{N} \to \mathbb{C}$ polynomial if there is a polynomial $p \in \mathbb{C}[z]$ with $f(n) = p(n)$ for all $n \in \mathbb{N}$.

Corollary An $f : \mathbb{N} \to \mathbb{C}$ is polynomial of degree $\leq d \in \mathbb{N}$ iff

$$\sum_{n \in \mathbb{N}} f(n) z^n = \frac{P(z)}{(1-z)^{d+1}}$$

Compare with (i)

very special q!

for some polynomial $p(z)$ of degree at most d

Moreover, f is polynomial of degree exactly d iff $p(1) \neq 0$

$\to$ Proof: L Notes.

Ehrhart theory → Book: Beck & Robins

Convex polytope P in $\mathbb{R}^d$

$$L_P(t) := \#(tP \cap \mathbb{Z}^d) = \#\left(P \cap \tfrac{1}{t}\mathbb{Z}^d\right) \quad \text{for } t \in \mathbb{N}$$

lattice-point enumerator function of P.

Associated to function $L_P : \mathbb{N} \to \mathbb{C}$ is the Ehrhart series of P

$$\mathrm{Ehr}_P(z) := \sum_{t \in \mathbb{N}} L_P(t)\, z^t$$

GOAL / THEOREM For every integral, d-dimensional polytope P, $L_P(t)$ is a polynomial of degree d.

Idea: "enumerate" all $m \in P \cap \mathbb{Z}^d$ as "monomials z^m" in a series

Def Let $X \subseteq \mathbb{R}^d$. The integer point transform of X is

$$\sigma_X(z_1, \dots z_d) = \sum_{m \in X \cap \mathbb{Z}^d} z^m$$

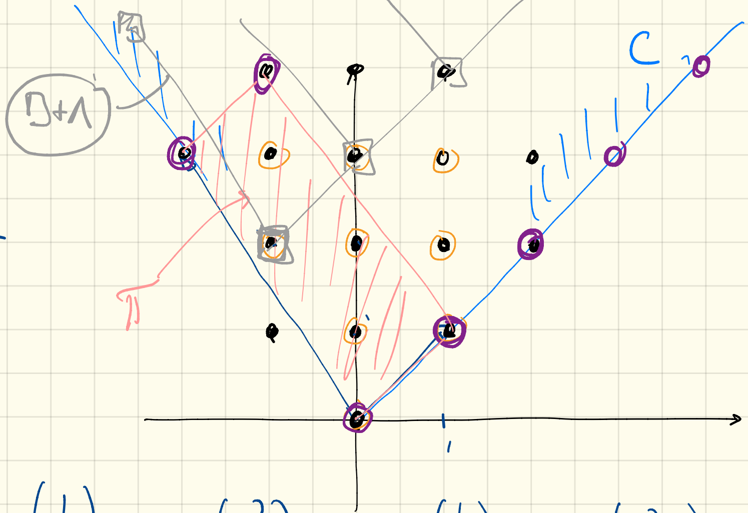
Example 1 $X := [0, \infty[$ in $\mathbb{R}$ (a cone)

$$\sigma_X(z) = \sum_{m \in \mathbb{Z} \cap X} z^m = \sum_{m \in \mathbb{N}} z^m = \frac{1}{1-z}$$

Example 2 $\quad C := \mathrm{cone}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\}$

simplicial cone

$\hookrightarrow$ generators are a basis of $\mathbb{R}^2$



Thus: every point $m \in C \cap \mathbb{Z}^2$ has

a unique expression

$$m = \underbrace{\lambda}_{\geq 0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \underbrace{\mu}_{\geq 0} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \lfloor \lambda \rfloor \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lfloor \mu \rfloor \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \{\lambda\} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \{\mu\} \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

integer parts $\qquad 0 \leq \{\lambda\}, \{\mu\} < 1$

points in $\Lambda = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\rangle_{\mathbb{N}}$ $\qquad$ must be an integer vector inside

In particular:

$$\mathbb{Z}^2 \cap C = \Lambda + (\mathbb{Z}^2 \cap \Pi)$$

$$\Pi := \left\{ \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 3 \end{pmatrix} \;\middle|\; 0 \leq \alpha, \beta < 1 \right\}$$

"fundamental parallelotope"

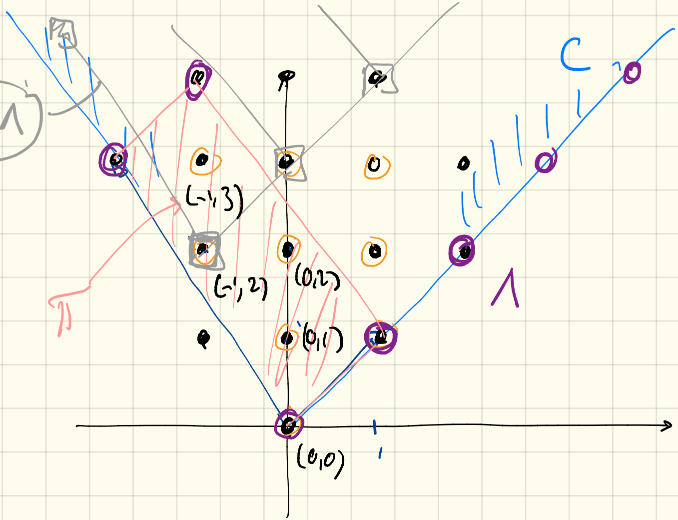
$$\boxed{\sigma_C(z) = \sigma_\Lambda(z) \cdot \sigma_\Pi(z)}$$

Example 2

$C := \mathrm{cone}\left\{\binom{1}{1}, \binom{-2}{3}\right\}$

$\Lambda = \mathbb{N}\binom{1}{1} + \mathbb{N}\binom{-2}{3}$

$\Pi = \left\{ \alpha\binom{1}{1} + \beta\binom{-2}{3} \;\middle|\; 0 \le \alpha, \beta < 1 \right\}$



$$\sigma_\Lambda(z_1, z_2) = \sum_{m \in \Lambda} z^m$$

$$= \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} z^{j\binom{1}{1}} z^{k\binom{-2}{3}} = \left(\sum_{j \in \mathbb{N}} \left(z^{\binom{1}{1}}\right)^j \right) \left(\sum_{k \in \mathbb{N}} \left(z^{\binom{-2}{3}}\right)^k \right)$$

$$= \frac{1}{(1 - z_1 z_2)(1 - z_1^{-2} z_1^3)}$$

$$\sigma_\Pi(z_1, z_2) = \underset{(0,0)}{1} + \underset{(0,1)}{z_2} + \underset{(0,2)}{z_2^2} + \underset{(-1,2)}{z_1^{-1} z_2^2} + \underset{(-1,3)}{z_1^{-1} z_2^3}$$

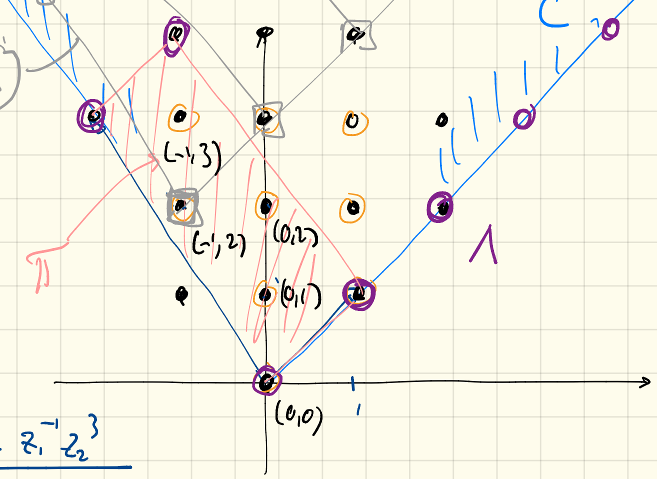
$\sigma_C(z)$

Example 2 $C := \mathrm{cone}\left\{\binom{1}{1}, \binom{-2}{3}\right\}$

$$\Lambda = \mathbb{N}\binom{1}{1} + \mathbb{N}\binom{-2}{3}$$

$$\Pi = \left\{ \alpha\binom{1}{1} + \beta\binom{-2}{3} \;\middle|\; 0 \le \alpha, \beta < 1 \right\}$$

$\Pi + \Lambda$



$$\sigma_C(z) = \underbrace{\sigma_\Lambda(z)}\, \sigma_\Pi(z) = \frac{1 + z_1 + z_1^2 + z_1^{-1} z_2^2 + z_1^{-1} z_2^3}{(1 - \underbrace{z_1 z_2}_{z^{w_1}})(1 - \underbrace{z_1^{-2} z_2^3}_{z^{w_2}})}$$

w_1, w_2 generators of C.

Theorem Let $u_1, \dots, u_d \in \mathbb{Z}^d$ be such that $C = \mathrm{cone}\{u_1, \dots, u_d\}$ simplicial cone. Then for every $v \in \mathbb{R}^n$ we have

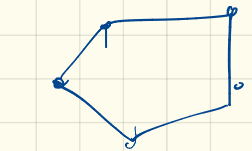
$$\sigma_{v+C}(z) = \frac{\sigma_{v+\Pi}(z)}{(1 - z^{u_1}) \cdots (1 - z^{u_d})}$$

where $\Pi = \left\{ \sum_{i=1}^{d} \lambda_i u_i \;\middle|\; 0 \le \lambda_i < 1 \right\}$

We can handle <u>simplicial cones generated by integer vectors</u>,

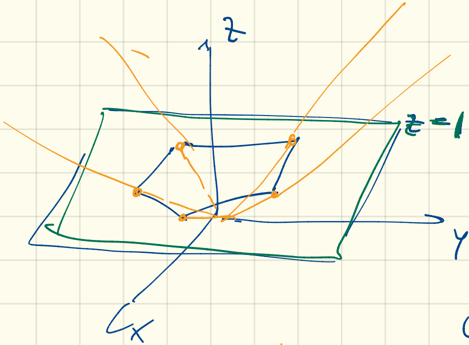
we <u>need</u> to handle <u>integral polytopes</u>.

For every polytope $P \subset \mathbb{R}^d$ with vertices $x_1 \cdots x_k$



consider the <u>standard cone</u> over P

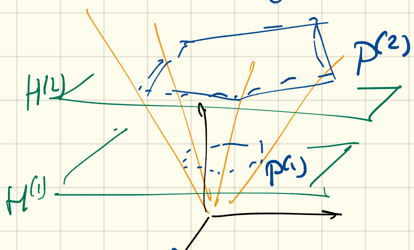
$$C(P) := \operatorname{cone}\{(x_1,1),\dots,(x_k,1)\} \subseteq \mathbb{R}^{d+1}$$



And

$$H^{(t)} = \{ y \in \mathbb{R}^{d+1} \mid \langle e_{d+1}, y \rangle = t \}.$$

Let then $P^{(t)} := H^{(t)} \cap C(P)$.



<u>Notice</u>:
$$L_P(t) = L_{P^{(t)}}(1)$$

& every integer point in $C(P)$ is an integer point of exactly one $H^{(t)}$, $t \in \mathbb{N}$

In terms of integer point transforms:

$$\sigma_{C(P)}(\underbrace{1,\dots,1}_{d \text{ times}}, \underset{\substack{\uparrow \\ d+1\text{-st} \\ \text{coord.}}}{z}) = \sum_{t \geq 0} L_{P(t)}(1) z^t = \mathrm{Ehr}_P(z)$$

$$\| $$

$$\sum_{m \in C(P) \cap \mathbb{Z}^{d+1}} \underbrace{1^{m_1} 1^{m_2} \cdots 1^{m_d}}_{=1, \text{ for each int. point}} z^{m_{d+1}}$$

GOAL / THEOREM For every <u>integral</u>, d-dimensional polytope P, $L_P(t)$ is a polynomial of degree d.

Proof: • By our corollary about FPS, it is enough to prove

$$\mathrm{Ehr}_P(z) = \frac{p(z)}{(1-z)^{d+1}} \qquad \text{for } \deg(p) \le d,\ p(1) \neq 0.$$

• By chapter 4: enough to prove this for P = <u>integral simplex</u>, since every integral P can be triangulated with no new vertices.

• Let then $P = \mathrm{conv}\{x_1 \dots x_{d+1}\}$ an integral d-dim. simplex in $\mathbb{R}^d$.

$C(P)$ simplicial, generated by $(x_1,1), \dots, (x_{d+1},1)$ ⟵ <u>integer</u>

• Now:

$$\mathrm{Ehr}_P(z) = \sigma_{C(P)}(1, \dots, 1, z_{d+1}) \underset{\Lambda,\,\Pi}{\overset{\downarrow}{=}} \frac{\sigma_\Pi(1,1,\dots,1,z_{d+1})}{(1-z_{d+1})^{d+1}}$$

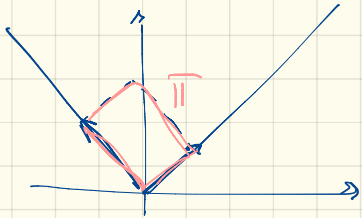
• polynomial in z_{d+1} (since Π bounded)

• Value at 1: $\sigma_\Pi(1,\dots,1) = \#\mathbb{Z}^{d+1} \cap \Pi \neq 0$ $\underset{\{0\}}{\uplus}$

$\prod_i (1 - z^{(x_i,1)})$, BUT $z_1 = \dots = z_d = 1$

Remains to prove: $\deg\left(\sigma_\Pi(1,\dots,1,z)\right) \le d$

$$\sigma_\Pi(1\dots1\ z) = \sum_{m \in \mathbb{Z}^{d+1} \cap \Pi} z^{m_{d+1}}$$



Degree $\le$ maximum possible integer value of last coordinate for a point in Π

$$\Pi = \left\{ \sum_{i=1}^{d+1} \lambda_i \underbrace{\begin{pmatrix} x_i \\ 1 \end{pmatrix}} \;\middle|\; 0 \le \lambda_i < 1 \right\}$$

last coordinate of $= \sum \lambda_i < 1+1\dots+1 = d+1$

biggest integer $< d+1$ is d ✓

Corollary: The number of integral flows on graphs is a polynomial!
(Polytope "U" of beginning of ch. V is integral since incidence matrices of graphs are totally unimodular).