30.04 Faces of polyhedra (Unimodularity)

07.05 Unimodularity, Triangulations

14.05 → Ehrhart's theorem (integer-point-counting polynomials)

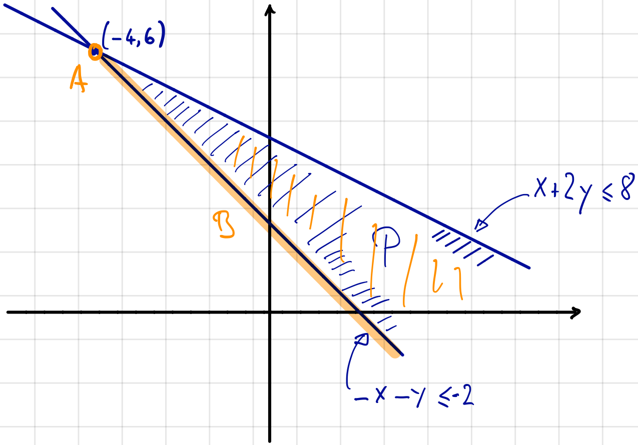
(21.05)

28.05 Reserve / review / questions &c.

Polyhedra / polytopes

$$P = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 8 \\ -2 \end{bmatrix} \right\}$$

"$Ax \leq b$"



Def: A face of a polyhedron $P = \{Ax \leq b\}$ is any subset of P of the form

$$F = \{ x \in P \mid A'x = b' \}$$

where "$A'x \leq b'$" is a sub-system of $Ax \leq b$ (i.e. only some rows)

$$A = \left\{ x \in P \,\middle|\, \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix} \right\}$$

$$B = \left\{ x \in P \,\middle|\, \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -2 \right\}$$

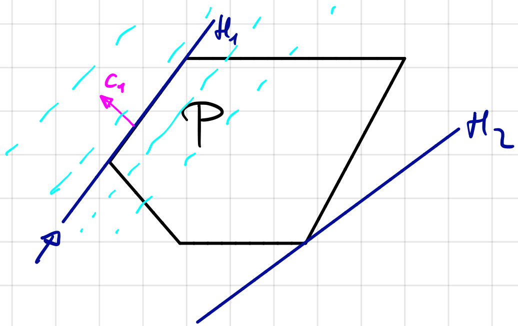
$$P = \{ x \in P \mid 0x = 0 \}$$

Def Let $P = \{Ax \le b\}$ polyhedron.

A hyperplane $H = \{c \cdot x = \delta\}$ is **supporting hyperplane** for P if

$$\delta = \max\{c \cdot x \mid x \in P\}$$

"H touches P, P lies wholly on one side of H"

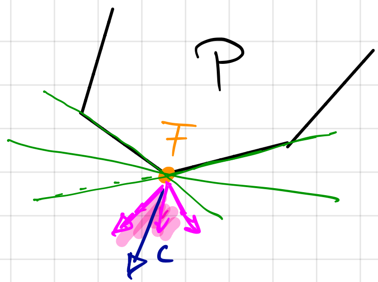


Rem: Every proper face of P is $P \cap H$ for some supporting hyp. H.

If $F = \{x \in P \mid A'x = b'\}$ proper,

take $c =$ sum of all rows of A',

then $F = \left\{ \begin{array}{l} \text{points in } P \\ \text{attaining maximum} \\ \max\{c \cdot x \mid x \in P\} = \sum b' \end{array} \right\}$



$F = \{x \in P \mid \ldots = \ldots\}$

Coming up: "$P \cap H$" $\Rightarrow$ "face"

Lemma ("Farkas' Lemma") Let A matrix, b vector. Then:

There exists $x \geq 0$ s.t. $Ax = b$ $\iff$ $\left\{ \begin{array}{l} yb \geq 0 \text{ for each row vector} \\ y \text{ with } yA \geq 0 \end{array} \right.$

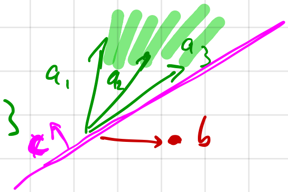
Proof: "$\Rightarrow$": if $x \geq 0$, $Ax = b$, $yA \geq 0$, then $yb = \underbrace{yA}_{\geq 0}\underbrace{x}_{\geq 0} \geq 0$

"$\Leftarrow$": Contraposition: suppose there is

$\underline{\text{no}}$ $x \geq 0$ with $Ax = b$

$$\begin{array}{c} A \\ \left[\begin{array}{cccccc} | & | & | & | & | & | \\ | & | & | & | & | & | \end{array} \right] \\ a_1\, a_2 \cdots\cdots \end{array} \overset{\geq 0}{x} = \overset{b}{\left[\quad \right]} \iff b \in \text{cone}\{a_1, a_2, \dots\}$$

$b \notin \text{cone}\{a_1, a_2 \dots\}$



Then, by "C^T"

there is y with $yb < 0$ (**)

and $\underbrace{yA \geq 0}$

$$[\ y\ ] \left[\begin{array}{ccc} | & | & \cdots \\ a_1 & a_2 & \end{array} \right] \geq 0$$

$\iff y a_i \geq 0$ (*) ∎

Last week

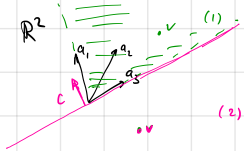
Theorem: Let $a_1, \dots, a_m, v \in \mathbb{R}^n$. Then either

(1) $v = \lambda_1 a_1 + \dots + \lambda_m a_m$ with $\lambda_i \geq 0$

or

(2) There is $c \in \mathbb{R}^n$ s.t. $c^\perp$ contains $(t-1)$ lin. indep. vectors from $\{a_1 \dots a_m\}$

and s.t. $c \cdot v < 0$ (**) & $c \cdot a_1, \dots, c \cdot a_m \geq 0$ (*), where $t = \dim\{a_1, \dots a_m, v\}$



Corollary: Let A matrix, b vector.

$$\{ Ax \leq b \text{ has solution} \} \iff \{ yb \geq 0 \ \forall\, y \geq 0 \text{ with } yA = 0 \}$$

Proof: iff "

$$(*) \quad [\mathbb{1} \mid A \mid -A] \begin{bmatrix} z_0 \\ \hline z_1 \\ \hline z_2 \end{bmatrix} = b$$

has nonnegative solution"

Indeed: ① let $\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} \geq 0$ solution, then $\overset{\geq 0}{z_0} + Az_1 - Az_2 = b$

$\Rightarrow A(z_1 - z_2) \leq b$

② let $Ax \leq b$

$$[\mathbb{1} \mid A \mid -A] \begin{bmatrix} b - Ax \\ \hline x^+ \\ \hline -x^- \end{bmatrix} \begin{matrix} \geq 0 \\ \geq 0 \\ \geq 0 \end{matrix} = (b - Ax) + \overbrace{\underbrace{Ax^+ + (-A)(-x^-)}_{Ax^+ + Ax^-}}^{Ax}$$

where, if $x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, let $x_i^+ := \begin{cases} x_i & \text{if } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$ $\quad A(x^+ + x^-) = Ax$

$x_i^- := \begin{cases} x_i & \text{if } x_i < 0 \\ 0 & \text{otherwise} \end{cases}$

Apply Farkas to see that $*$ has solution iff

$yb \geq 0$ whenever $\underbrace{y[\mathbb{1} \mid A \mid -A]}_{= [y \mid yA \mid -yA]} \geq 0$.

$= [y \mid yA \mid -yA] \geq 0 \iff \overbrace{y \geq 0, yA \geq 0, -yA \geq 0}^{\Leftrightarrow\ y \geq 0,\ yA = 0}$

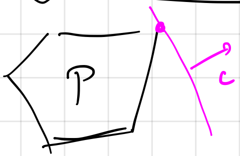
☒

The Corollary was meant to prove:

Proposition Let A matrix, b, c vectors. Then

"duality for LP"

$$\underbrace{\max\{c\cdot x \mid Ax \leq b\}}\ =\ \underbrace{\min\{yb \mid y \geq 0,\ yA = c\}}$$



whenever both sets non-empty

Proof: • "max $\leq$ min" is a computation (~~as~~ in previous corollary)

• For "max $\geq$ min" need to show: there are x, y with:

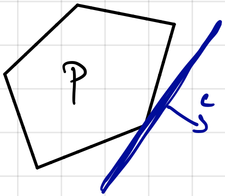
$$\left\{\begin{array}{l} Ax \leq b, \\ y \geq 0 \\ yA = c, \\ c\cdot x \geq yb \end{array}\right\} \quad \text{equivalently: solve} \quad \begin{bmatrix} 0 & -1 \\ A & 0 \\ -c & b^t \\ 0 & A^t \\ 0 & -A^t \end{bmatrix} \begin{bmatrix} x \\ y^t \end{bmatrix} \leq \begin{bmatrix} 0 \\ b \\ 0 \\ c^t \\ -c^t \end{bmatrix} \begin{array}{l} \leftarrow y \geq 0 \\ \leftarrow Ax \leq b \\ \leftarrow c\cdot x \geq yb \Leftrightarrow 0 \geq yb - cx \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} yA = c \end{array}$$

---- apply previous corollary. ∎

Corollary: Let $P=\{Ax\le b\}$ polyhedron. Then F is proper face of P if and only if $F=P\cap H$, H supporting hyperplane of P

Proof One direction already proved (Remark earlier: every face is "$F=P\cap H$")

The other: Let $F=P\cap H$ with $H=\{c\cdot x=\delta\}$ $\leftarrow \max\{cx \mid x\in P\}$



Now by the Proposition: $\delta=\min\{yb \mid y\ge 0,\ yA=c\}$,

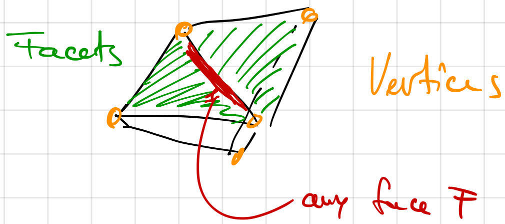
choose y_0 attaining minimum

Choose from the rows in $Ax\le b$ those corresponding to positive entries of y_0 to form the system $A'x\le b'$.

Claim: $F=\{x\in P \mid A'x=b'\}$.

Proof: in P always $Ax\le b$, thus $c\cdot x=\delta$ means $y_0Ax=y_0b$, $\Leftarrow$ which differs from $A'x=b'$ by zero rows (recall $y\ge 0$).

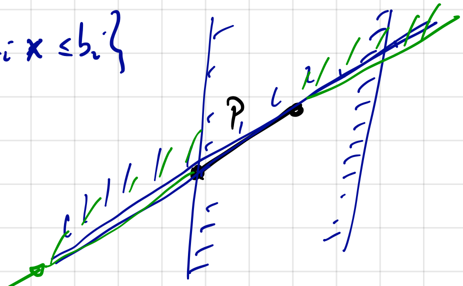
Faces of polyhedra.



Def A facet is any maximal proper face.

Let $P = \left\{ Ax \le b, \text{ say } \begin{matrix} a_1 x \le b_1 \\ \vdots \\ a_m x \le b_m \end{matrix} \right\}$, so: $P = \bigcap_{i=1}^{m} \{a_i x \le b_i\}$

a_i rows of A



An inequality $a_i x \le b_i$ is called an *implicit equality* if $P \subseteq (a_i x = b_i)$

otherwise: effective inequality.

Remark ① Every face $F = \{x \in P \mid A'x \le b'\}$ can be written $F = \{\underline{x \in P} \mid \overbrace{A'^{+} x \le b'^{+}}^{\text{only effective inequalities}}\}$

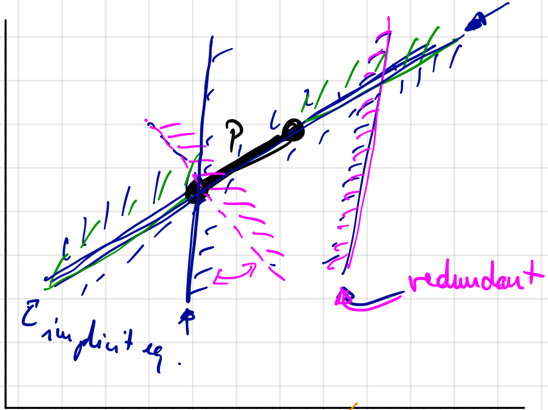
already expresses the implicit eq.

Notation: $A^{=} x \le b^{=}$ subsystem of implicit equalities

$A^{+} x \le b^{+}$ " " effective inequalities.

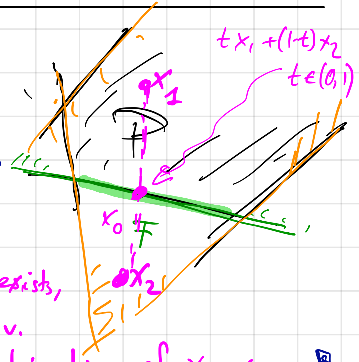
Remark (2) If $Ax \le b$ has an effective inequality $a_i x \le b_i$, then there is some $x \in P \setminus \{a_i x = b_i\}$, and in particular this x has

$$A^= x = b^= \text{ and } \underline{A^+ x < b^+}$$



Def: an inequality redundant if removing it does not change P (i.e. the ~~set~~ set of solutions)

Lemma: Let $a_i x \le b_i$ effective, irredundant. Then $F = \{x \in P \mid a_i x = b_i\}$ is facet of P



Proof. Let $A'x \le b'$ system of all effective in. other than $a_i x \le b_i$

Enough to find x_0 s.t. $A^= x_0 = b^=$, $A' x_0 < b'$, $\underline{a_i x_0 = b_i}$.

- from (2): $\exists x_1$ s.t. $A^= x_1 = b^=$, $A^+ x_1 < b^+ \rightarrow a_i x_1 < b_i$
- Since $a_i x \le b_i$ irredundant $\exists x_2$ $A^= x_2 = b^=$, $A' x_2 < b'$, $\underline{a_i x_2 > b_i}$

$\Big\}$ x_0 exists, conv. combination of x_1, x_2 $\square$

Proposition Every face of a polyhedron $P = \{Ax \le b\}$ is an intersection of facets.

say: $a_1' x \le b_1'$
$\vdots$
$a_k' x \le b_k'$, w.l.o.g. can assume all $a_i' x \le b_i'$ effective, irredundant.

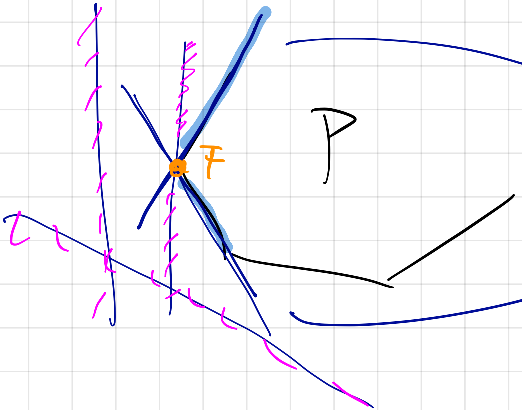
Proof: Let $F = \{x \in P \mid A' x = b'\}$

Then $F = \bigcap_{i=1}^{k} \underbrace{\{x \in P \mid a_i' x \le b_i'\}}$

all of these are facets (previous Lemma)

$\square$

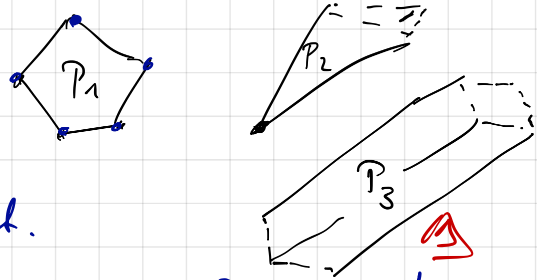
idea:



both cut out facets

$\Rightarrow$ Every face is an intn. of facets

Let's talk <u>vertices</u>.



<u>Lemma</u> A polyhedron P is an affine subspace iff it has no nonempty faces except P itself.

<u>Pf</u>: P aff. subspace $\Leftrightarrow$ $P = \{Mx = v\}$ for some matrix M, some vector v.

then $P = \left\{ \underbrace{\begin{bmatrix} M \\ -M \end{bmatrix}}_{\text{"}A\text{"}} x \leq \underbrace{\begin{bmatrix} v \\ -v \end{bmatrix}}_{\text{"}b\text{"}} \right\}$, and every point

of P satisfies any subsystem $A'x \leq b'$ with equality

$\rightarrow$ no (nonempty) faces except P itself.

Other direction: if $P = \{Ax \leq b\}$ has no proper face, then

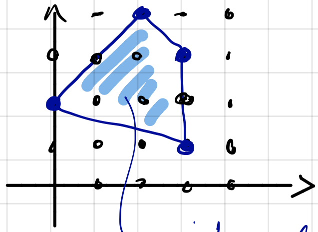
in particular $P = \{Ax = b\}$ <u>is</u> an affine subspace ☒

<u>Corollary</u> 1 The minimal non-empty faces of a polyhedron are affine subspaces!

Corollary 1 The minimal non-empty faces of a polyhedron are affine subspaces!

Corollary 2 The minimal non-empty faces of a convex polytope P (bounded!) are points (i.e. affine subspaces of dim. 0); and they are called the **vertices** of P

Definition Call a polytope $P \subseteq \mathbb{R}^n$ $\boxed{\text{integral}}$ if all its vertices are points in $\mathbb{Z}^n \subseteq \mathbb{R}^n$, rational if all vertices are in $\mathbb{Q}^n \subseteq \mathbb{R}^n$

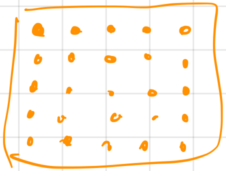


Question: Given $P = \{Ax \leq b\}$ polytope; determine whether P is integral.

→ In order to tackle this, we look at **totally unimodular matrices**

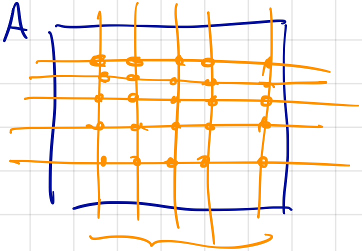
Def: A matrix A with entries in $\mathbb{Z}$ is called <u>totally unimodular</u> if every square minor of A has determinant $0, 1, -1$.

A



k rows

k columns

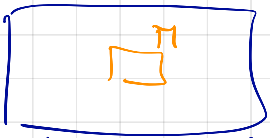


$\hookrightarrow$ square minor of A

We'll want to look at incidence matrices of graphs (these have entries $0, 1, -1$)

Proposition: Let A be a matrix with entries from $\{0, \pm 1\}$ and such that every column contains exactly one 1 and one -1. Then A is totally unimodular.

A



Proof. Choose a square minor M from A. Induction on size of M.
If M size 1, trivial. Suppose M size > 1.

(1) M has an all-zero column. ✓

(2) M has a column with exactly one nonzero entry $\Rightarrow$ done by ind. hyp. after developing w.r.t. that column

(3) Otherwise every column has exactly a $+1$ and a -1, and so the sum of all rows is 0! $\Rightarrow \det(M)$ is 0. ∎

WHY CARE?

<u>Theorem</u>: For an integral matrix A, the following are equivalent:

(i) A is totally unimodular

(ii) The polyhedron $\{Ax \leq b, x \geq 0\}$ is integral for all b.

(iii) " " $\{a \leq Ax \leq b, \quad c \leq x \leq d\}$ has only integral vertices, for all a, b, c, d.

(next time).