5.2. Polyhedra and polytopes

(Motivation: count integer flows $\xrightarrow{\S 5.1}$ count integer points in "dilated polytopes")

Def A <u>polyhedron</u> is any $P = \{x \in \mathbb{R}^n \mid \boxed{Ax \leq b}\} \subseteq \mathbb{R}^n$

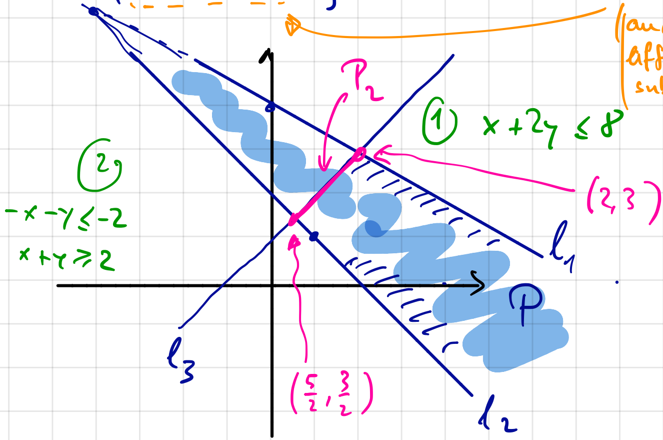
componentwise

"intersection of half-spaces (and affine subspaces)"

For some matrix A and some vector b.

Examples (in $\mathbb{R}^2$)

$$P_1 = \left\{ \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{"}x\text{"}} \leq \underbrace{\begin{bmatrix} 8 \\ -2 \end{bmatrix}}_{b} \right\} \begin{matrix} \leftarrow ① \\ \leftarrow ② \end{matrix}$$



$$P_2 = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 8 \\ -2 \\ -1 \\ 1 \end{bmatrix} \right\} \begin{matrix} ① \\ ② \\ \\ \\ \end{matrix}$$

$$\left.\begin{matrix} x - y \leq -1 \\ -x + y \leq 1 \iff x - y \geq -1 \end{matrix}\right\} x - y = -1 \quad ③$$

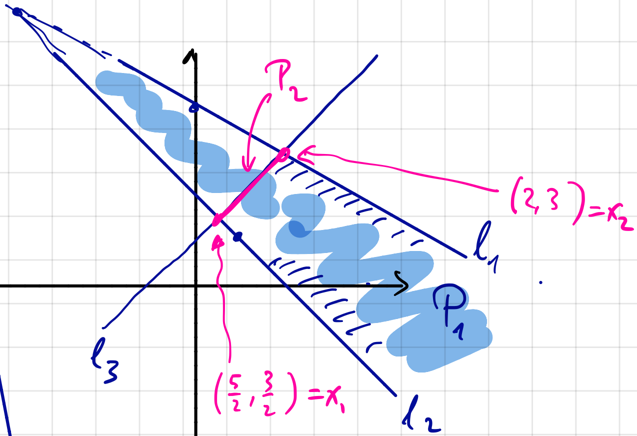
Def: A *polytope* is any subset of $\mathbb{R}^n$ of the form

$$P = \operatorname{conv}\{x_1, \dots, x_k\} := \left\{ \sum_{i=1}^{k} \lambda_i x_i \;\middle|\; \lambda_i \in \mathbb{R}_{\geq 0},\ \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

for some finite subset $\{x_1, \dots, x_k\} \subseteq \mathbb{R}^n$.

Example: $P_2 = \{ t x_1 + (1-t) x_2 \mid 0 \leq t \leq 1 \}$

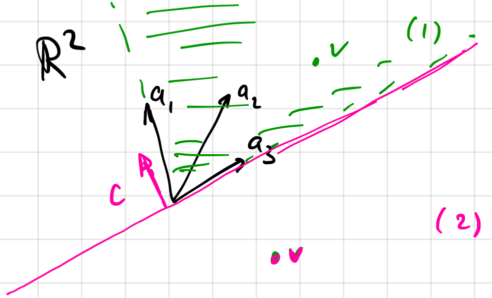
$$= \operatorname{conv}\{x_1, x_2\}$$



"Fundamental thm. of polytope theory":

A subset of $\mathbb{R}^n$ is a polytope if and only if it is a bounded polyhedron

Stepping stone:



Theorem: Let $a_1,\dots,a_m, v \in \mathbb{R}^n$. Then either

(1) $v = \lambda_1 a_1 + \dots + \lambda_m a_m$ with $\lambda_i \geq 0$

or

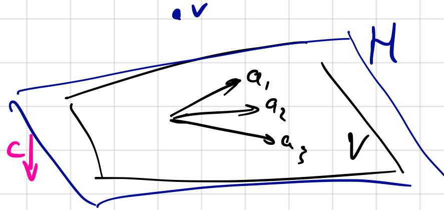
(2) There is $c \in \mathbb{R}^n$ s.t. $c^\perp$ contains $(t-1)$ lin. indep. vectors from $\{a_1 \dots a_m\}$ and s.t. $c\cdot v < 0$ & $c\cdot a_1, \dots, c\cdot a_m \geq 0$, where $t = \dim\langle a_1, \dots, a_m, v\rangle$

Proof

- (1) & (2) are mutually exclusive: if both hold at the same time:

$$0 \overset{(2)}{>} c\cdot v \overset{(1)}{=} c\cdot\Big(\sum \underbrace{\lambda_i}_{\geq 0} a_i\Big) = \sum \underbrace{\lambda_i}_{(\geq 0)} \underbrace{c\cdot a_i}_{\substack{\geq 0 \\ (2)}} \geq 0 \quad \lightning$$

- Case 1: $v \notin V := \langle a_1 \dots a_m\rangle$. Then $V \subsetneq \mathbb{R}^n$, so $V \subseteq H$ for some hyperplane H. Choose c normal to H, pointing away from v $\Rightarrow$ (2)✓



Proof

Case 2: $v \in \langle a_1 \cdots a_m \rangle \overset{\text{w.l.o.g.}}{=} \mathbb{R}^n$, so $t = n$

Set $D_0 := \{i_1 \cdots i_n\}$, $\{a_j\}_{j \in D_0}$ is any max. lin. indep. set

For all $i \geq 0$ iterate:

(i) Write (uniquely) $v = \sum_{j \in D_i} \lambda_j a_j$. If $\lambda_j \geq 0 \ \forall j \in D_i \to \boxed{\text{STOP}}$, (1)

(ii) Otherwise choose $h := \min\{j \in D \mid \lambda_j < 0\}$, let $H := \langle a_j \mid j \in D \setminus \{h\} \rangle$,

choose normal c to H s.t. $c \cdot a_h = 1$.

In particular: $c \cdot v = c \cdot \sum_{j \in D} \lambda_j a_j = 0 + c \cdot (\lambda_h a_h) = \lambda_h \overbrace{c \cdot a_h}^{=1} < 0$

(iii) If $c \cdot a_i \geq 0$ for all $a_1, \ldots, a_m \to \boxed{\text{STOP}}$, case (2).

(iv) Otherwise, let $s := \min\{j \in [m] \mid \underline{c \cdot a_j < 0}\}$, set $\to a_s \notin H$, $a_s \neq a_h$

$$D_{i+1} := (D_i \setminus \{h\}) \cup \{s\}$$

Must prove: recursion terminates!

By way of contradiction, suppose does not terminat.

(i) Write (uniquely) $v = \sum_{j \in D_i} \lambda_j a_j$. If $\lambda_j \geq 0 \ \forall j \in D_i \to \boxed{STOP}$, (1)

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$\to a_s \notin H$, $a_s \neq a_h$

Since $\binom{m}{n} < \infty$,

$$\overbrace{\dots D_k \quad D_p \quad D_q \quad D_\ell \dots}^{=}$$

D_p: remove r; D_q: add r

$D_p \cap \{r+1 \dots m\}$
$= D_q \cap \{r+1 \dots m\}$

$r :=$ max. among elements "removed" in steps between k, ℓ, say this happens at step p

q: step between k, ℓ where element r is added.

Contradiction: Let c_q vector (ii) @ step q, then:

$$0 > c_q \cdot v = c_q \cdot \left(\sum_{j \in D_p} \lambda_j a_j\right) = \sum_{j \in D_p} \lambda_j c_q \cdot a_j > 0$$

every summand is nonnegative, $\lambda_r \, c_q \cdot a_r > 0$

$\lambda_r \overset{①}{<} 0$, $c_q \cdot a_r \overset{②}{<} 0$

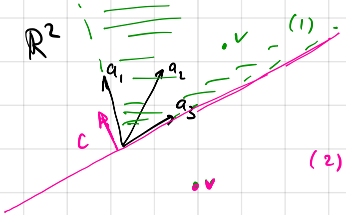
$j \in D_p, j < r$: $\lambda_j \overset{①}{\geq} 0$, $c_q \cdot a_j \overset{②}{\geq} 0$

$j \in D_p \ j > r$: $j \in D_q$, so $c_q \cdot a_j = 0$ by (ii) @ step q

① r removed from D_p, so $r = \min\{j \in D_p \mid \lambda_j < 0\}$

② r added to D_q, so $r = \min\{j \in [m] \mid c_q \cdot a_j < 0\}$

Theorem: Let $a_1,\dots,a_m, v \in \mathbb{R}^n$. Then either



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or

(2) There is $c \in \mathbb{R}^n$ s.t. $c^\perp$ contains $(t-1)$ lin. indep. vectors from $\{a_1 \dots a_m\}$ and s.t. $c\cdot v < 0$ & $c \cdot a_1, \dots, c\cdot a_m \geq 0$, where $t = \dim\langle a_1, \dots a_m, v\rangle$

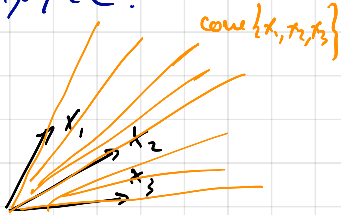
(1) $\Leftrightarrow$ $v \in \operatorname{cone}\{a_1 - a_m\}$

(2) $\Leftrightarrow$ v not in smallest polyhedral cone containing $a_1 \dots a_m$

Definition: • A <u>convex cone</u> is any $C \subseteq \mathbb{R}^n$ s.t., for all $x, y \in C$, all $\lambda, \mu \geq 0$, $\lambda x + \mu y \in C$.

• Given $x_1, \dots, x_k \in \mathbb{R}^n$, the "cone generated by x_i's" is

$$\operatorname{cone}\{x_1, \dots, x_k\} := \{\lambda_1 x_1 + \dots + \lambda_k x_k \mid \lambda_i \geq 0 \ \forall i\}$$



→ any such cone is called <u>finitely generated</u>

Def: A <u>polyhedral cone</u> is any nonempty $C \subseteq \mathbb{R}^n$ of the form

$$C = \{x \in \mathbb{R}^n \mid Ax \leq 0\}$$

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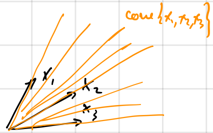
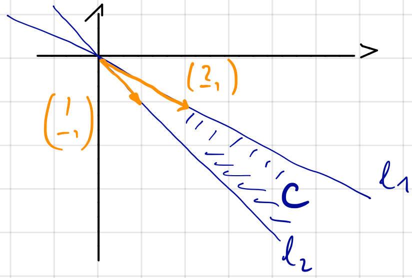


Def: A polyhedral cone is any nonempty $C \subseteq \mathbb{R}^n$ of the form

$$C = \{x \in \mathbb{R}^n \mid Ax \leq 0\}$$

Example

$$C = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq 0 \right\} \quad \begin{matrix} 1 \\ 2 \end{matrix}$$



$$C = \operatorname{cone}\{(1,-1), (2,-1)\}$$

Prop A convex cone is polyhedral if and only if it is finitely generated.

Proof "Fin. gen ⇒ polyh" Let $X = \{x_1 \dots x_k\} \subset \mathbb{R}^n$, wlog $\langle X \rangle = \mathbb{R}^n$.

Given $c \in \mathbb{R}^n$: halfspace $H_c := \{x \in \mathbb{R}^n \mid c \cdot x \leq 0\}$. Bounding hyp: $H_c^0 = \{c \cdot x = 0\}$

Let $\mathcal{H} = \left\{ \begin{array}{l} \text{all half-spaces containing } X, \\ \text{with bounding hyps. spanned by el. of } X. \end{array} \right\}$ ← finite set.

For every $H \in \mathcal{H}$ choose c_H s.t. $H = H_{c_H}$.

By Theorem:

$$\operatorname{cone}\{x_1 \dots x_k\} \underset{\text{theorem}}{=} \bigcap \mathcal{H} \underset{\text{tautology}}{=} \left\{ \begin{bmatrix} \text{---} \ c_H \ \text{---} \\ \vdots \end{bmatrix} x \leq 0 \right\}$$

- "Polyh. ⇒ fin. gen" Consider polyh. cone

$C := \{Ax \leq 0\}$, $a_1 \dots a_m$ rows of $A = \begin{bmatrix} — & a_1 & — \\ & \vdots & \\ — & a_m & — \end{bmatrix}$

cone$\{a_1 \dots a_m\}$ is polyhedral (by "other direction"),

so there are vectors $b_1 \dots b_t$ s.t.

$$\text{cone}\{a_1 \dots a_m\} = \Big\{ \underbrace{\begin{bmatrix} — & b_1 & — \\ & \vdots & \\ — & b_t & — \end{bmatrix}}_{=:B} x \leq 0 \Big\}$$

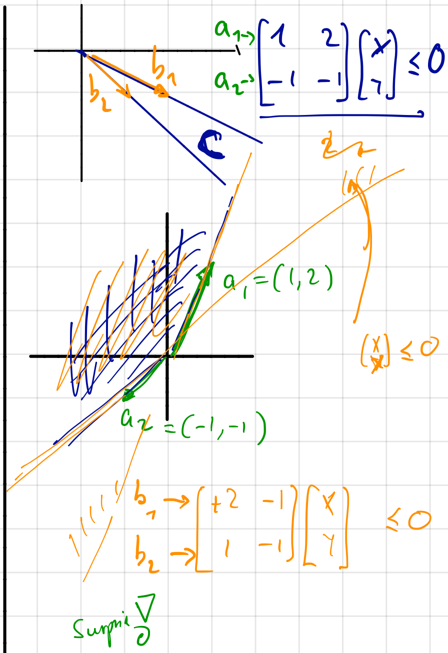
Claim $\overbrace{\text{cone}\{b_1 \dots b_t\}}^{=:K} = C$, so C fin. gen.

Pf: $K \subseteq C$ enough to prove $b_j \in C$ $\forall j$.

But $b_j \cdot a_i \leq 0$ $\forall i,j$ since a_i are in $\{Bx \leq 0\}$,

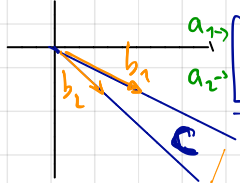
hence in particular $Ab_j = \begin{bmatrix} — & a_1 & — \\ — & a_m & — \end{bmatrix} b_j \leq 0$,

so $b_j \in C$ $\forall j$.



• "Polyh. ⇒ fin. gen" Consider polyh. cone

$C := \{Ax \le 0\}$, $a_1 \cdots a_m$ rows of $A = \begin{bmatrix} — a_1 — \\ \vdots \\ — a_m — \end{bmatrix}$



$\begin{array}{l} a_1 \to \\ a_2 \to \end{array} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le 0$

cone$\{a_1 \cdots a_m\}$ is polyhedral (by "other direction"),

so there are vectors $b_1 \cdots b_t$ s.t.

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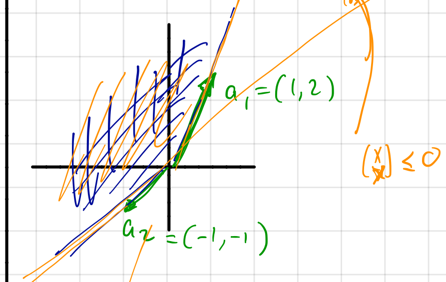


$\begin{array}{l} b_1 \to \\ b_2 \to \end{array} \begin{bmatrix} +2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le 0$

Surprise

Claim $\overbrace{\text{cone}\{b_1 \cdots b_t\}}^{=:K} = C$, so C fin. gen.

Pf: $C \subseteq K$ Suppose we can choose $y \in C \setminus K$.

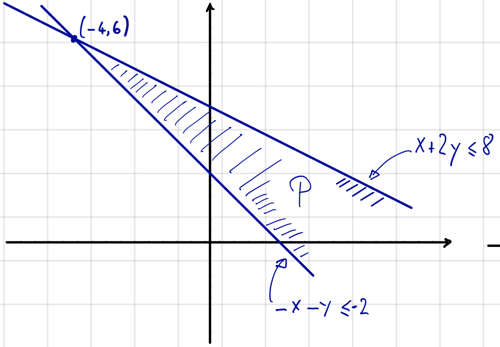
By theorem above (case (2) enters), there would be c with $c \cdot y < 0$, $c \cdot b_i \ge 0$ $\forall i$.

Then: $(-c) \cdot b_i \le 0$, so $(-c) \in \text{cone}\{a_1 \cdots a_m\}$, $B(-c) \le 0$

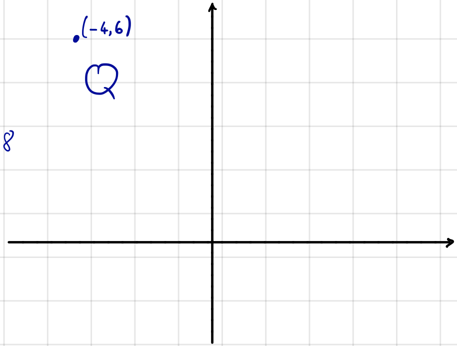
therefore $(-c) = \lambda_1 a_1 + \cdots + \lambda_m a_m$ for some $\lambda_i \ge 0$

Now for all $x \in C$: $(-c) \cdot x = (\sum \lambda_i a_i) \cdot x = \sum \lambda_i a_i \cdot x \le 0$ $\quad \ge 0 \quad x \in C = \{Ax \le 0\}$

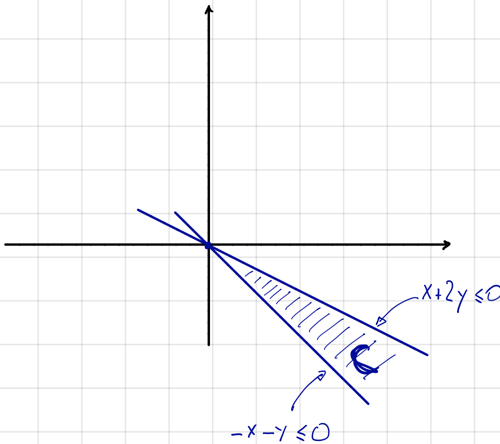
This holds in particular for y : so $(-c) \cdot y \le 0$, hence $c \cdot y \ge 0$ ⚡ ☒



P - polyhedron



$Q = \operatorname{conv}\{(-4,6)\}$ - polytope



C - polyhedral cone

We have $$P = \underbrace{Q + C}_{\{x+y \mid x \in Q,\ y \in C\}}$$

<u>Proposition</u>: Let $P \subseteq \mathbb{R}^n$ the following are equivalent:

(1) P is a polyhedron

(2) $P = Q + C$ for a polytope Q and a polyh. cone C

Proposition: Let $P \subseteq \mathbb{R}^n$ the following are equivalent:

(1) P is a polyhedron

(2) $P = Q + C$ for a polytope Q and a polyh. cone C

Proof: (2) $\Rightarrow$ (1) Let $P = Q + C$

$Q = \text{conv}\{x_1 \dots x_m\} = \{\sum \lambda_i x_i \mid \lambda_i \geq 0, \sum \lambda_i = 1\}$

$\text{cone}\{y_1 \dots y_t\} = \{\sum \lambda_i y_i \mid \lambda_i \geq 0\}$

Now $v \in P$ if and only if

$$\begin{pmatrix} v \\ 1 \end{pmatrix} \in \underbrace{\text{cone}\left\{\begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} x_m \\ 1 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} y_t \\ 0 \end{pmatrix}\right\}}_{=: K} = \left\{\begin{array}{l} \sum \lambda_i x_i + \sum \lambda_{m+i} y_i = v \\ \lambda_1 \dots \lambda_m, \lambda_{m+1} \dots \lambda_{m+t} \geq 0 \\ \underline{\lambda_1 + \dots + \lambda_m = 1} \end{array}\right.$$

$\Rightarrow \sum \lambda_i x_i \in Q$.

By Proposition there is A' with $K = \left\{ A' \begin{pmatrix} x \\ \mu \end{pmatrix} \leq 0 \right\} = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} \,\middle|\, Ax - \mu b \leq 0 \right\}$

for $A' = [A \mid -b]$

Now $v \in P$ iff $Av - b \leq 0$, so $P = \{Ax \leq b\}$, a polyhedron.

(1) $\Rightarrow$ (2) Assume $P = \{Ax \leq b\}$ polyhedron. Must prove $P = Q + C$

Consider the polyhedral cone

$$K := \left\{ \left[\begin{array}{c|c} 0 & -1 \\ \hline A & -b \end{array}\right] \begin{bmatrix} x \\ \mu \end{bmatrix} \leq 0 \right\} = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} \,\middle|\, Ax - \mu b \leq 0,\ \mu \geq 0 \right\}$$

By the Prop., K is finitely generated - ie,

$$K = \operatorname{cone}\left\{ \begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \begin{pmatrix} x_2 \\ 1 \end{pmatrix} \cdots \begin{pmatrix} x_k \\ 1 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ 0 \end{pmatrix} \cdots \begin{pmatrix} x_m \\ 0 \end{pmatrix} \right\}.$$

Now by definition $P = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} \in K \,\middle|\, \mu = 1 \right\}$, so $v \in P$ iff $\begin{pmatrix} v \\ 1 \end{pmatrix} \in K$, i.e.

$$\begin{cases} v = \sum \lambda_i x_i & \text{for some } \lambda_1 \cdots \lambda_m \geq 0, \\ 1 = \lambda_1 + \cdots + \lambda_k \end{cases}$$

Now $P = \underbrace{\operatorname{conv}\{x_1 \cdots x_k\}}_{=:Q} + \underbrace{\operatorname{cone}\{x_{k+1} \cdots x_m\}}_{=:C}$ $\boxtimes$

Summary: P polyhedron $\iff$ $P = Q + C$

polytope

polyh. cone

Corollary: P Bounded polyhedron $\iff$ P polytope (Fund. thm.)

Pf: Polyh. cone C is bounded iff $C = \{0\}$, therefore

P bounded polyhedron $\iff$ $P = Q + \{0\}$ is a polytope.