Combinatorial polynomials - 9 april 2020

IV.1 Prove: (r2) $r(A) + r(B) \geq r(A\cap B) + r(A\cup B) \quad \forall\, A, B \subseteq E$

$\Updownarrow$

(r2') $r(\underline{X\cup e}) + r(\underline{X\cup f}) \geq r(X) + r(X\cup\{e,f\}) \quad \forall X \subseteq E,\ e,f\in E$

"$\Downarrow$" trivial

"$\Uparrow$" Assume (r2'), consider $A, B \subseteq E$. Induction on $|A\cup B|$

- If $\underline{A \subseteq B}$ or $B\subseteq A$, nothing to show. $A\cap B = A$, $A\cup B = B$
- Otherwise let $a\in A\setminus B$, set $A' := A\setminus\{a\}$; let $b\in B\setminus A$, set $B' := B\setminus\{b\}$

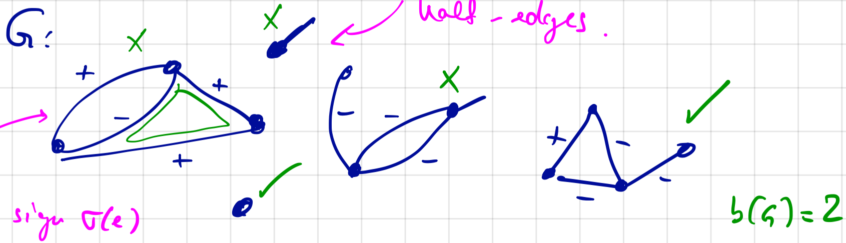
(r2') says $r(A'\cup B) + r(A\cup B') \geq r(\underbrace{A'\cup B'}_{(A'\cup B)\cap(A\cup B')}) + r(\underbrace{A\cup B}_{(A'\cup B)\cup(A\cup B')})$

$\Longrightarrow$ Case study:

1. $r(A\cup B) - r(A\cup B') = 0$
2. $r(A\cup B) - r(A'\cup B) = 0$
3. Otherwise . . . .

IV.2 $\longrightarrow$ Later.

Review: Signed graphs



$b(G)$: # of balanced components ← no half-edges, every circuit "positive"

For $A \subseteq E$ $\quad b(A) := b(G[A])$

Surprise: $r_\Sigma : 2^E \to \mathbb{N}$ $\quad r_\Sigma(A) = (\#\text{vertices of } G) - b(A)$ is a matroid rank function!

k-colorings: "colors" $\{-k, \dots, -1, 0, 1, \dots, k\}$, rule:

- Ends of half-edges not colored 0,
- $\mathrm{color}(h(e)) \neq \sigma(e)\,\mathrm{color}(t(e))$ for all regular edges e

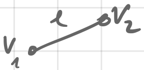
(Signed) chromatic polynomial:

$$\chi_\Sigma(t) = \sum_{A \subseteq E} (-1)^{|A|} t^{b(A)}$$

$$\overset{?}{=} (-1)^{\#V} t^{b(E)} T_{r_\Sigma}(1-t, 0)$$

usual yoga

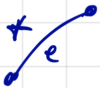
Arrangements from signed graphs

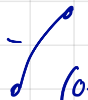
Regular graphs: Edge 

$\longrightarrow$ vector $(0,\dots,0,\overset{h(e)}{1},0,\dots,0,\overset{t(e)}{-1},0,\dots,0)\in\mathbb{R}^V$

Definition Let Σ be a loopless signed graph on vertex set V with set of edges E. For each $e\in E$ define a vector $m_e\in\mathbb{R}^V$ via:

$$(m_e)_v := \begin{cases} 1 & \text{if } v=h(e)\\ \sigma(e)(-1) & \text{if } v=t(e)\\ 0 & \text{otherwise}\end{cases}$$

Example: 

$(0\cdots0,\overset{h(e)}{1},0\cdots0,\overset{t(e)}{-1},0\cdots0)$ | 

$(0\cdots0,\overset{h(e)}{1},0\cdots0,\overset{t(e)}{1},0\cdots0)$ | 

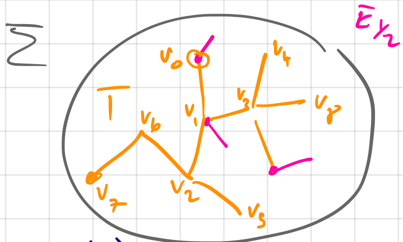
$(0,\cdots0,\overset{v}{1},0\cdots0)$

Definition: Let M_Σ the $V\times E$ matrix with colums m_e: $M_\Sigma = \left[\, m_e \;|\;|\;|\;\right]$ and for $A\subseteq E$ let $M_\Sigma[A]$ be M_Σ restricted to the columns A.

IDEA: r_Σ "linear dependency-matroid" of vectors $\{m_e\}_{e\in E}$

PRECISE: Prove For every $A\subseteq E$, $r_\Sigma(A) = \operatorname{rank} M_\Sigma[A]$

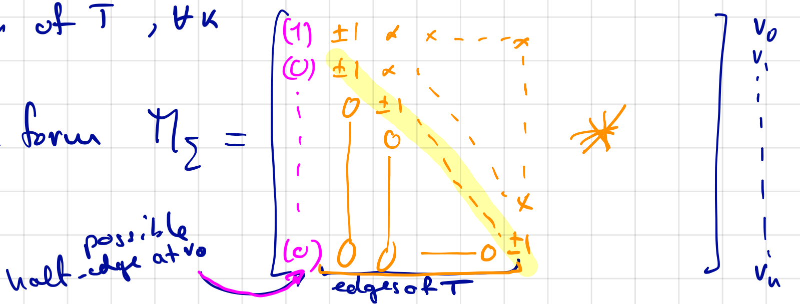
Lemma: Let Σ be a connected signed graph on n vertices.
Then $\operatorname{rank} M_\Sigma = \begin{cases} n-1 & \text{if } \Sigma \text{ balanced} \\ n & \text{otherwise} \end{cases}$



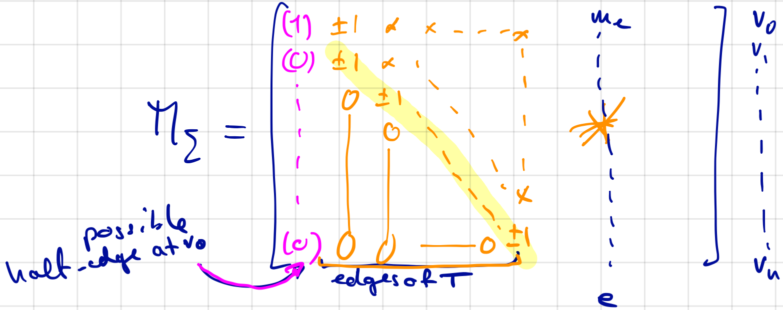
Proof: Choose T spanning tree (set of regular edges that is acyclic and incident to every vertex)

Then T has $n-1$ edges, can assume w log that if Σ has an half-edge, then some half-edges are attached to leaves of T.
Choose v_0 a leaf of T (incident to an half-edge, if one exists)
Order the vertices starting with v_0, so that $v_0, v_1, \dots, v_k$ spans a connected subgraph of T, $\forall k$

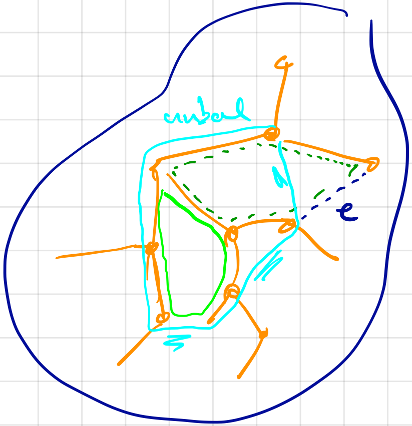
Then, M_Σ has the form $M_\Sigma =$



$\Rightarrow$ rank is at least $(n-1)$, at most n



↓ rank is at least $(n-1)$, at most n



- If Σ is <u>balanced</u>: there are no half-edges (so: no first column !) & every circuit positive.
  $\longrightarrow$ Prove that for all $e \notin T$, m_e is linear combination of edges of T (Exercise)
  $\Rightarrow$ rank $M_\Sigma = n-1$ ✓ $1 = b(A)$

- If Σ is <u>not balanced</u>, then either there is a half-edge, thus M_Σ contains the ! column $\Rightarrow$ rank $M_\Sigma = n$,

<u>OR</u>: There are no half-edges and there is at least an unbalanced circuit
$\longrightarrow$ Prove that there is some $e \in E$ s.t. m_e lin. indep. from vectors associated to edges of T } Exerc. IV.2

<u>ANYWAY</u>: rank $M_\Sigma = n \longleftarrow b(A) = 0$

∎

Theorem Let Σ be a loopless signed graph with edge-set E.

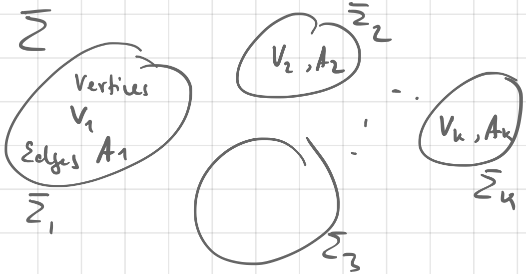
Then for every $A \subseteq E$, $r_\Sigma(A) = \mathrm{rank}\, M_\Sigma[A]$

Proof: Wlog enough to prove for $A = E$ (since $r_\Sigma(A) = r(G[A])$, $M_\Sigma[A] = M_{\Sigma[A]}$)

So we have to prove $r_\Sigma(E) = \mathrm{rank}\, M_\Sigma$

The matrix M_Σ has the form

$$M_\Sigma = \begin{array}{c|c}
\begin{bmatrix}
M_{\Sigma_1} & 0 & 0 & \\
0 & M_{\Sigma_2} & 0 & \\
0 & 0 & \ddots & \\
0 & 0 & & M_{\Sigma_k}
\end{bmatrix} &
\begin{matrix} \}V_1 \\ \}V_2 \\ \}V_3 \\ \vdots \\ V_k \end{matrix} \\
\begin{matrix} \underbrace{\quad}_{A_1} & \underbrace{\quad}_{A_2} & \quad & A_k \end{matrix} &
\end{array}$$



Here: $\Sigma_1 .. \Sigma_k$ are the conn. comp. of Σ,

Σ_i: vertex set V_i, edge set A_i.

We know $\mathrm{rank}\, M_\Sigma = \sum_{i=1}^{k} \mathrm{rank}\, M_{\Sigma_i} \overset{\text{Lemma}}{=} \sum_{i=1}^{n} \big(|V_i| - b(A_i)\big) = |V| - b(E) \overset{\text{def}}{=} r_\Sigma(E)$

$\boxtimes$

Thus we know that the matroid r_Σ is the matroid associated to the arrangement of hyperplanes $\mathcal{A}_\Sigma := \{ m_e^\perp \mid e \in E \}$,

for instance: number of regions determined by $\mathcal{A}_\Sigma$ is $T_{r_\Sigma}(2,0)$.

Observe: $$T_{r_\Sigma}(2,0) = (-1)^{|V| - b(E)} \underbrace{\chi_\Sigma(-1)}$$

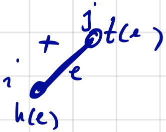
can be determined by studying colorings of Σ

Examples: root systems / finite Coxeter groups.

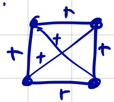
$$\varepsilon_i := (0 \cdots 0 \overset{i\text{-th place}}{1} 0 \cdots 0)$$

TYPE A_{n-1}: arrangement with normals $\underline{\varepsilon_i - \varepsilon_j}$ in $\mathbb{R}^n$, for all $i<j$

$$(0 0 \cdots 0, \underset{i}{1}, 0 \cdots 0, \underset{j\text{-th}}{-1}, 0 \cdots 0)$$



K_4^+:



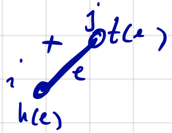
$\Rightarrow$ these normals are the vectors m_e for a signed, "all-positive" complete graph K_n^+ on n vertices:

$\varepsilon_i = (0\cdots 0\,1\,0\cdots 0)$

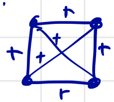
TYPE A_{n-1}

: arrangement with normals $\underbrace{\varepsilon_i - \varepsilon_j}$ in $\mathbb{R}^n$, for all $i<j$

$$\begin{array}{c} (00\cdots 0,\underset{\uparrow\, i}{1},0\cdots 0,\underset{\uparrow\, j\text{-th}}{-1},0\cdots 0) \end{array}$$



K_4^+:



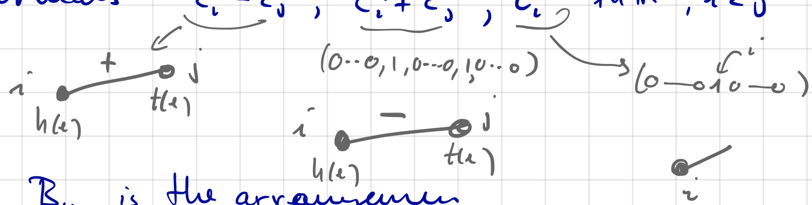
$\Rightarrow$ these normals are the vectors m_e for a signed, "all-positive" complete graph K_n^+ on n vertices.

So write $\Sigma(A_{n-1})$ = complete all-positive signed graph on n-vertices

$\chi_{\Sigma(A_{n-1})}(t) = t(t-1)\cdots$, thus # of regions is $n!$

(= order of the Coxeter group of type A_{n-1}, i.e., S_n)

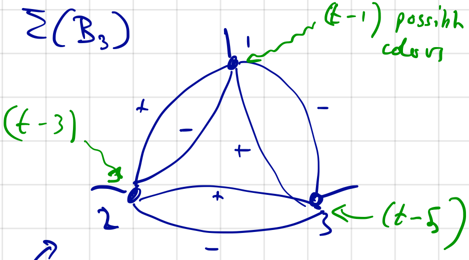
Type B_n Arrangement has normals $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j$, ε_i in $\mathbb{R}^n$, $i<j$



Thus: reflection arrangement of type B_n is the arrangement

$\mathcal{A}_{\Sigma(B_n)}$, with $\Sigma(B_n)$: signed graph

on n vertices, with
- one half-edge at every vertex
- one positive & one negative regular edge between any two vertices



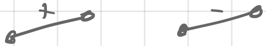
What is $\chi_{\Sigma(B_n)}(t)$? For odd t, color with colors $\left\{-\left\lfloor\frac{t}{2}\right\rfloor, \dots, \left\lfloor\frac{t}{2}\right\rfloor\right\}$
- half-edges everywhere $\Rightarrow$ can never use 0 as a color
- double ("+&-") edges everywhere: neighboring vertices cannot have neither equal nor opposite colors!

$$\Rightarrow \chi_{\Sigma(B_n)}(t) = (t-1)(t-3)\cdots(t-(2n-1))$$

Number of regions (= # of the group): $|\chi_{\Sigma(B_n)}(-1)| = 2n(2n-2)(2n-4)\cdots = \boxed{2^n n!}$

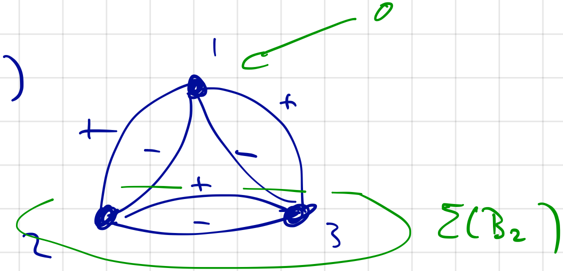
Type D_n

Normals to arrangement: $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j$, $i<j$, in $\mathbb{R}^n$



Graph $\Sigma(D_n)$: on n vertices,
any two vertices joined
by a pair of "+/-" edges

$\Sigma(D_3)$



Colorings with t (odd) signed colors, of $\Sigma(D_n)$

$n \cdot \# \{\text{signed } t\text{-colorings of } \Sigma(B_{n-1})\}$ ← if 0 is used

here 0 forbidden

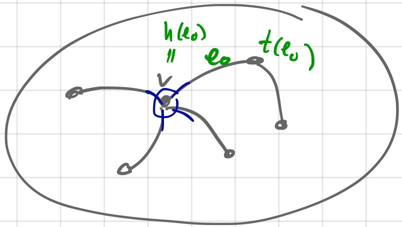
$+ \quad \# \{\text{signed } t\text{-colorings of } \Sigma(B_n)\}$ ← if 0 never used

$$= \chi_{\Sigma(D_n)}(t)$$

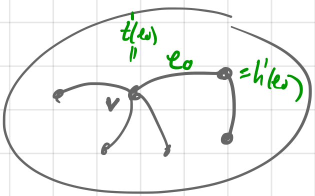
⇒ number of regions / order of the group: $2^{n-1} n!$

Ch. 5 - Integral flows (and polytopes)

G



"flip" e_0



$G' = G^{\neg\{e_0\}}$

Recall: Integral k-flow on $G=(V,E,h,t)$ is $f: E \to \mathbb{Z}$ s.t. "conservation"

$$\sum_{h(e)=v} f(e) = \sum_{t(e)=v} f(e) \quad , \forall v \in V$$

e_0

$$\varphi_G^{\mathbb{N}} = \text{nr. of nowhere-zero integral k-flows} = ?$$

Let $A \subseteq E$. Define $\boxed{G^{\neg A}} := (V, E, h', t')$

s.t. $h'(e) = \begin{cases} h(e) & \text{if } e \notin A \\ t(e) & \text{if } e \in A \end{cases}$, $t'(e) = \begin{cases} t(e) & \text{if } e \notin A \\ h(e) & \text{if } e \in A \end{cases}$

Fact: If f is integer k-flow on G, then

$f': E \to \mathbb{Z}$, $f'(e) := \begin{cases} f(e) & e \notin A \\ -f(e) & e \in A \end{cases}$

is again an integer k-flow on $G^{\neg A}$

Let $f': E \to \mathbb{Z}$

$f'(e) = \begin{cases} f(e) & e \neq e_0 \\ -f(e_0) & e = e_0 \end{cases}$

then, at v:

$$\sum_{h'(e)=v} f'(e) \overset{!}{=} \sum_{t'(e)=v} f'(e)$$

$\leftarrow e_0$

Lemma: Let f integral k-flow on G, let $A := \{e \in E \mid f(e) < 0\}$.
Then by changing sign of f at every element of A we obtain a integral k-flow $f' \geq 0$ on $G^{\neg A}$.

Def: An integral k-flow is called positive if $f(E) > 0$

Corollary: For every nowhere-zero flow f on G there is exactly one $A \subseteq E$ s.t. changing sign of f on A yields a positive k-flow on $G^{\neg A}$

We obtain:

$$\varphi_G^N(k) = \sum_{A \subseteq E} \underbrace{I_{G^{\neg A}}(k)}$$

Given graph H:

$I_H(k) = \#$ positive integral k-flows.

$\Rightarrow$ New task: determine $I_H(k)$.

$\Rightarrow$ New task: determine $I_H(k)$.

<u>Idea:</u> consider the "incidence matrix" of G, $A \in \mathbb{R}^{V \times E}$ with

$$A_{e,v} = \begin{cases} 1 & \text{if } v = h(e) \\ -1 & \text{if } v = t(e) \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{array}{cc} A = \begin{bmatrix} & \\ \cdots & \cdots \\ & \end{bmatrix} & \begin{array}{c} v_1 \\ \vdots \\ v \\ \vdots \end{array} \\ e \text{ --- } \cdots & \end{array}$$

Now, a positive k-flow on G is any $f \in \mathbb{Z}^E$ with

$$Af = 0, \quad \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \leq f \leq \begin{pmatrix} k \\ \vdots \\ k \end{pmatrix}. \quad \leftarrow \text{k-flow}$$

conservation

diophanteam system   with constraints.

"Relax" $\mathbb{Z}$ to $\mathbb{R}$, think linear algebra.

Consider $U := \{ x \in \mathbb{R}^E \mid Ax = 0,\ \underline{0} < x < \underline{1} \}$, $\bar{U} := \{ \cdots\cdots\ \underline{0} \leq x \leq \underline{1} \}$

componentwise

<u>Now:</u> $I_G(k) = \# \, \mathbb{Z}^E \cap kU$ $\leftarrow \{k \cdot y \mid y \in U\}$

"number of integer points in the k-fold dilation of U"

Tool: Ehrhart theory!

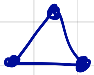
Consider $U := \{ x \in \mathbb{R}^E \mid Ax = 0,\ \underline{0} \overset{\text{componentwise}}{<} x < \underline{1} \}$, $\bar{U} := \{ \dots\dots\ \underline{0} \le x \le \underline{1} \}$

Now: $I_G(k) = \# \, \mathbb{Z}^E \cap kU$ $\quad kU = \{ k \cdot y \mid y \in U \}$

"number of integer points in the k-fold dilation of U"

Ehrhart theory – tool in order to count integer points in dilations

of "$\underbrace{\text{convex polytopes}}_{?}$ with $\underbrace{\text{integer vertices}}_{?}$"

Example: $G =$  , $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & +1 & -1 \end{bmatrix}$

$Ax = 0 \iff \begin{cases} x = (x_1, x_2, x_3) \\ x_1 = x_2 = x_3 \end{cases}$

Polytope in $\mathbb{R}^2$: