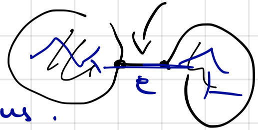
Combinatorial Polynomials – March 26.

III.2 r - rank fn. of matroid, $r: 2^E \to \mathbb{N}$, e isthmus.



$\downarrow$

$r(A \cup \{e\}) > r(A) \quad \forall A \not\ni e$

1) Recall def. of $r_{/e}$, $r_{\setminus e}$

2) Recall def. of lattice of flats ← "closed sets"

3) Isomorphism of two (finite!) partially ordered sets;

say P, Q: $\quad f: P \to Q$ bijective and order-preserving

$p_1 \leq p_2 \Rightarrow f(p_1) \leq f(p_2)$

(with an order-pres. inverse.)

III.3 "uniform matroid" $U_{n,d}$:

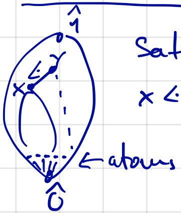
ground set $[n]$, for $A \subseteq [n]$ $\underline{r}(A) = \begin{cases} |A| & \text{if } |A| \leq d \quad \leftarrow \\ d & \text{if } |A| > d \end{cases}$

$\rightarrow$ recall def. of the **dual** r^* $\boxed{\text{(Def. 2.2.4)}}$

check that r^* is rank fn of $U_{n,n-d}$

Summary / review:

Geom. Lattices



Satisfy (G)

$x \lessdot y \Leftrightarrow \exists$ atom p, $p \leq y$, $p \not\leq x$, $y = x \vee p$

last week

Matr. rank functions:

$r : 2^E \longrightarrow \mathbb{N}$, with

r0, r1, r2

§3.3

(*) Every $\mathcal{L} \subseteq 2^E$, $E \in \mathcal{L}$, $\mathcal{L}$ geom. l. induces a matr. r.f. on E

"canonical (simple) matroid assoc. to a geom. l."

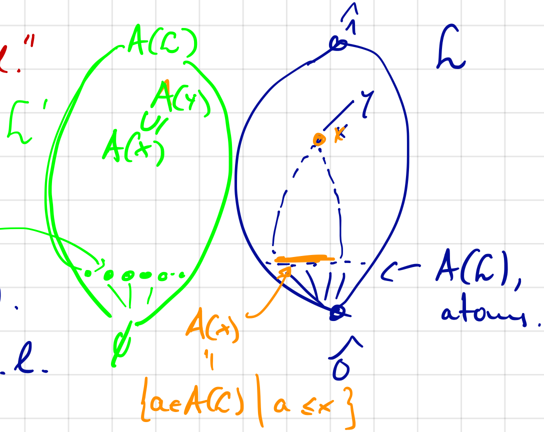
Given g.l. $\mathcal{L}$ construct a poset

$\mathcal{L}' = \{A(x) \mid x \in \mathcal{L}\}$, ordered by inclusion,

i.e. $A(x) \leq A(y)$ if $A(x) \subseteq A(y)$.

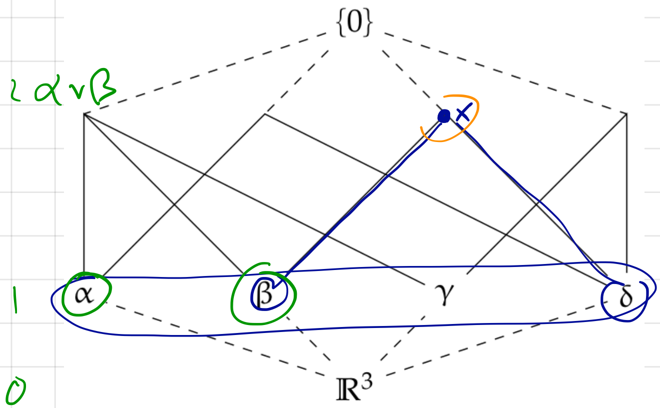
Now $\mathcal{L} \overset{\text{iso}}{\cong} \mathcal{L}'$ (e.g. take $x \longmapsto A(x)$), so $\mathcal{L}'$ geo. l.

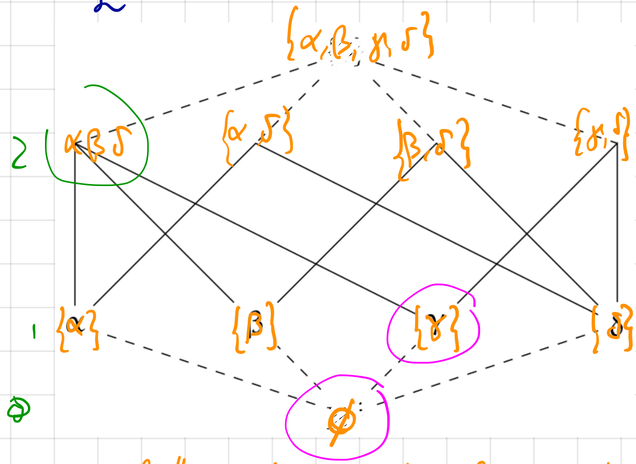
Matroid on $E = A(\mathcal{L})$: $\forall X \subseteq A(\mathcal{L})$ set $r(X) := \rho(\vee X)$



Example:

$\mathcal{L}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\mathcal{L}'$





$A(\mathcal{L}) = \{\alpha, \beta, \gamma, \delta\}$

$A(x) = \{\beta, \delta\}$

$A(\alpha) = \{\alpha\}$

$A(\mathbb{R}^3) = \emptyset$

(geom.) lattice of subsets of $E = \{\alpha, \beta, \gamma, \delta\}$, with $E \in \mathcal{L}'$.

Matroid rank fn. on E

Take $A \subseteq E$, either:

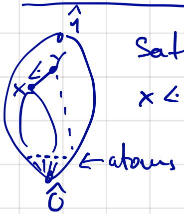
1) look at the smallest $X \in \mathcal{L}'$ with $A \subseteq X$, set $r(A) := \rho'(X)$

2) take $\vee A$ in $\mathcal{L}$, set $r(A) = \rho(\vee A)$

$A = \{\alpha, \beta\}$, $X = \{\alpha, \beta, \gamma\}$, $r(A) = 2$ | $A = \{\alpha, \beta\}$ $\quad r(A) = 2$

§3.3

Geom. Lattices



satisfy (G)

$x \lessdot y \Leftrightarrow \exists$ atom p, $p \le y$, $p \not\le x$, $y = x \vee p$

last week

Matr. rank functions:

$r : 2^E \to \mathbb{N}$, with r0, r1, r2

§3.3

corresp. up to loops, parallel elts.

$r : 2^E \to \mathbb{N}$

"Closure" of any $A \subset E$:

$cl(A) := \{ e \in E \mid r(A \cup e) = r(A) \}$

Call $F \subseteq E$ closed if $F = cl(F)$

Think of the case where E set of vectors in V and $r(A) = \dim \operatorname{span}_V(A)$ (Thm. 2.2.15)

Then $F \subseteq E$ "closed" iff linearly closed.

$\mathcal{L}_r := \{ F \subseteq E \mid cl(F) = F \}$

ordered by inclusion

- this is, in general, a geometric lattice.

with: $F_1 \vee F_2 = cl(F_1 \cup F_2)$

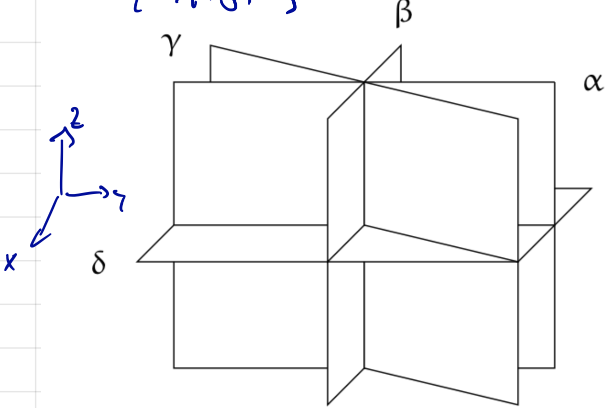
$F_1 \wedge F_2 = F_1 \cap F_2$

AND: $g(F) = r(F) \quad \forall F \in \mathcal{L}_r$

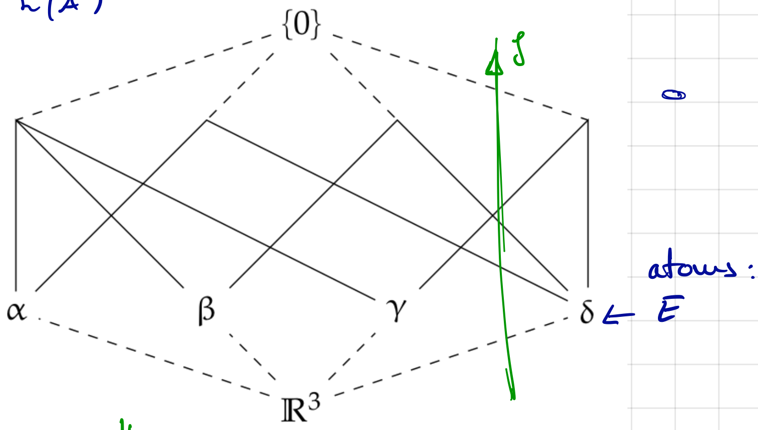
Back to arrangements – two matroid rank functions

Consider normal vectors: $n_\alpha = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $n_\beta = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $n_\gamma = \begin{pmatrix}-1\\1\\0\end{pmatrix}$, $n_\delta = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$



$\mathcal{L}(\mathcal{A})$



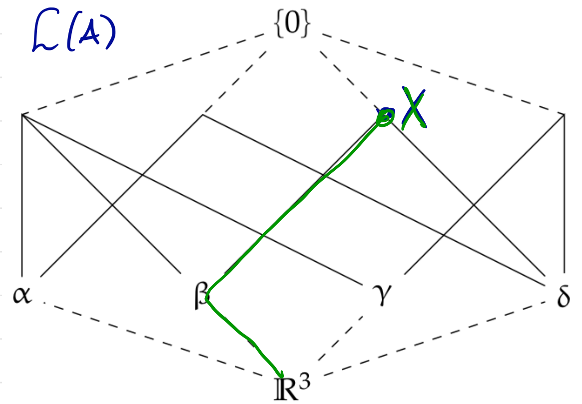
$r_{dep} : 2^E \to \mathbb{N},\ A \mapsto \dim \operatorname{span}\{n_i \mid i \in A\}$

matroid rank by Ch. 2

$r_{lat} : 2^E \to \mathbb{N},\ A \mapsto \rho(\vee A)$

matr. r.f. by §3.2

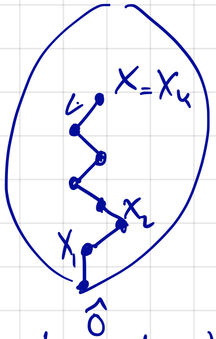
In fact: $\boxed{r_{dep} \equiv r_{lat}}$



Lemma For every $X \in \mathcal{L}(\mathcal{A})$ we have $\rho(X) = \operatorname{codim}(X)$ (in $\mathbb{R}^d$)

Proof: $\rho(X) = k$ for

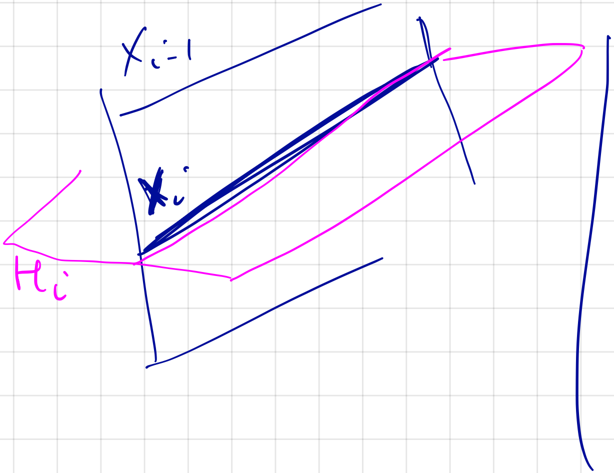
$\hat{0} \lessdot X_1 \lessdot \cdots \lessdot X_k = X$



By (G) $X_{i-1} \lessdot X_i$ means

$X_i = X_{i-1} \vee H_i$ ← atom (i.e. hyperplane)

$= X_{i-1} \cap H_i$ as subspaces. $\quad H_i \not\leq X_{i-1}$



Now.

$$\dim(X_{i-1}) + \underbrace{\dim(H_i)}_{d-1} = \dim(\underbrace{X_{i-1} \cap H_i}_{\downarrow}) + \underbrace{\dim(\underbrace{X_{i-1} + H_i}_{\mathbb{R}^d})}_{d}$$

$$\Rightarrow \dim(X_{i-1}) - 1 = \dim(X_i)$$

$$\Longrightarrow \dim(X) = \dim(X_k) = \dim(\hat{0}) - k = d - k = d - \rho(X) \quad \boxtimes$$

Proposition: $r_{lat} \equiv r_{dep}$.

$\Big($ Notation: write $\mathcal{A} = \{H_1, \dots, H_m\}$, normal vectors $n_1 \dots n_m$,

and take $r_{lat} : 2^{[m]} \to \mathbb{N}, \quad I \mapsto \rho(\bigvee_{i\in I} H_i) = \rho(\bigcap_{i\in I} H_i)$

$r_{dep} : 2^{[m]} \to \mathbb{N}, \quad I \longmapsto \dim \operatorname{span}\{n_i \mid i \in I\}$ $\Big)$

Proof: Take any $I \subseteq [m]$, recall (Lemma) $r_{lat}(I) = \operatorname{codim}(\underline{\bigcap_{i\in I} H_i})$

Consider $r_{dep}(I) = \operatorname{rank}\begin{bmatrix} | & | & \\ n_{i_1} & n_{i_2} & \cdots \\ | & | & \end{bmatrix}$ $d \times |I|$-matrix, call it M_I, notice:

$$\ker(M_I)^T = \bigcap_{i\in I} n_i^\perp = \underline{\bigcap_{i\in I} H_i}$$

Summarizing

$$r_{dep}(I) = \operatorname{rank}(M_I)^T = d - \ker(M_I)^T = \operatorname{codim}\Big(\bigcap_{i\in I} H_i\Big) = r_{lat}(I)$$

$\boxtimes$

<u>Plan</u>: gearing towards studying dissections of $\mathbb{R}^d$ by hyperplanes
(i.e.: use <u>Tutte polynomial</u> in order to compute nr. of "pieces" in $\mathbb{R}^d \setminus \cup A$)
↳ deletion - contraction

⇝ Last preparation needed: interpret deletion/contraction in terms of $\mathcal{L}$

<u>Proposition</u> Let $r: 2^E \to \mathbb{N}$ a matroid rank f., let $e \in E$. Call $cl, cl_{/e}, cl_{\setminus e}$ the closure operators associated to $r, r_{/e}, r_{\setminus e}$. Then for $A \subseteq E \setminus e$:

① $cl_{\setminus e}(A) = cl(A) \setminus \{e\}$ ② $cl_{/e}(A) = cl(A \cup \{e\}) \setminus \{e\}$

<u>Proof</u>: Recall def. of $cl^{§3.3}$ & of $r_{/e}, r_{\setminus e}$ ↙ Chapter 2, ① Exercise.

② $cl_{/e}(A) = \{x \in E \setminus e \mid \underbrace{r_{/e}(A \cup x)}_{r(A \cup x \cup e) - \cancel{r(e)}} = \underbrace{r_{/e}(A)}_{r(A \cup e) - \cancel{r(e)}}\} = \{x \in E \setminus e \mid r(\underline{A \cup e \cup x}) = r(\underline{A \cup e})\}$

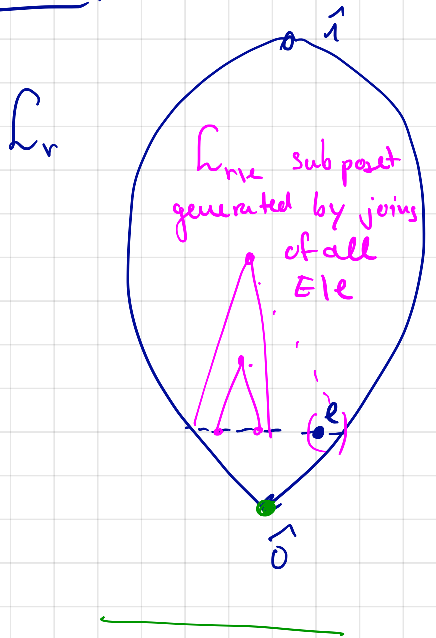
$=: cl(A \cup e) \setminus \{e\}$

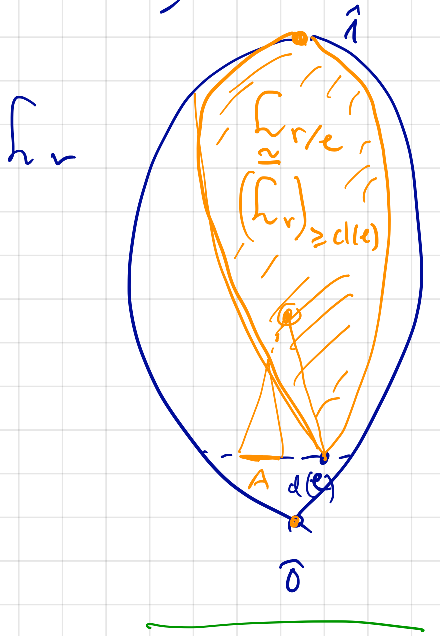
∎

Proposition Let $r: 2^E \to \mathbb{N}$ a matroid rank f., let $e \in E$. Call $cl, cl_{/e}, cl_{\setminus e}$ the closure operators associated to $r, r_{/e}, r_{\setminus e}$. Then for $A \subseteq E \setminus e$:

① $cl_{\setminus e}(A) = cl(A) \setminus \{e\}$ ② $cl_{/e}(A) = cl(A \cup \{e\}) \setminus \{e\}$

Corollary: Let r matroid rank fn. on E, $e \in E$. Then $\mathcal{L}_{r\setminus e}$, $\mathcal{L}_{r/e}$



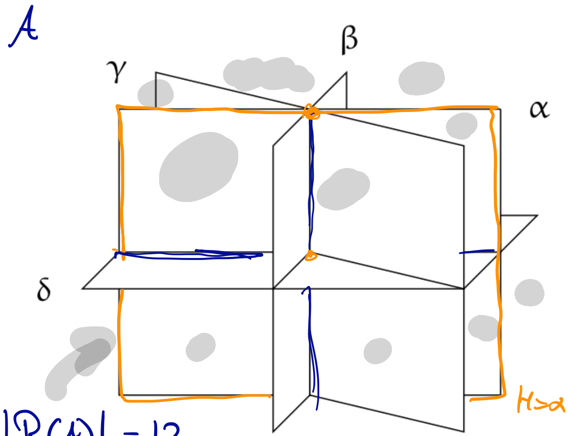


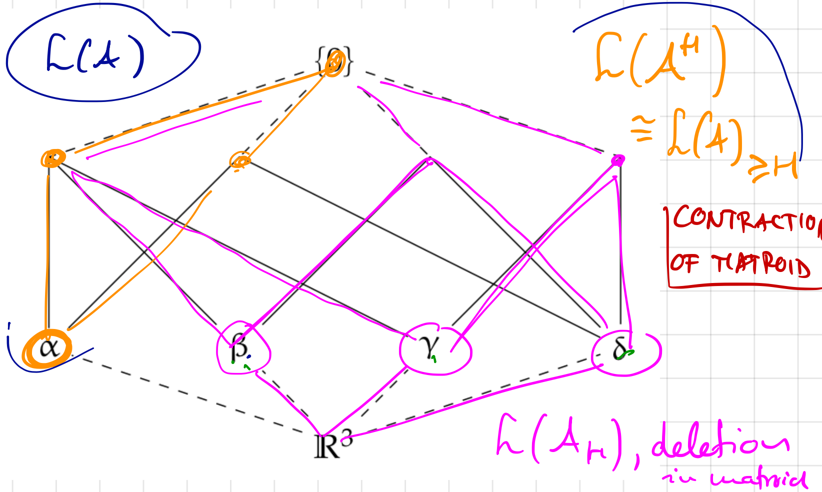
Dissections (finally!)

Let $\mathcal{A}$ arr. of hyp. in $\mathbb{R}^d$,

We have an assignment $r: \boxed{\text{Arrgts}} \longrightarrow \boxed{\text{Matroids}}$, $\mathcal{A} \longmapsto r_{\mathcal{A}}$ — either from $\mathcal{L}(\mathcal{A})$ or from normal vectors.

Consider $\mathcal{R}(\mathcal{A}) := \underbrace{\pi_0(\mathbb{R}^d \setminus \cup \mathcal{A})}_{\text{set of conn. comp.}}$, set of "regions" cut out by $\mathcal{A}$.





$|\mathcal{R}(\mathcal{A})| = 12$

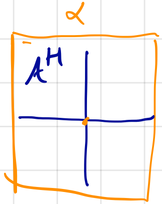
Goal: compute $|\mathcal{R}(\mathcal{A})|$ via the matroid.

Definition: For $H \in \mathcal{A}$ define $\mathcal{A}_H := \mathcal{A} \setminus \{H\}$, $\mathcal{A}^H = \{H \cap H' \mid H' \in \mathcal{A}_H\}$

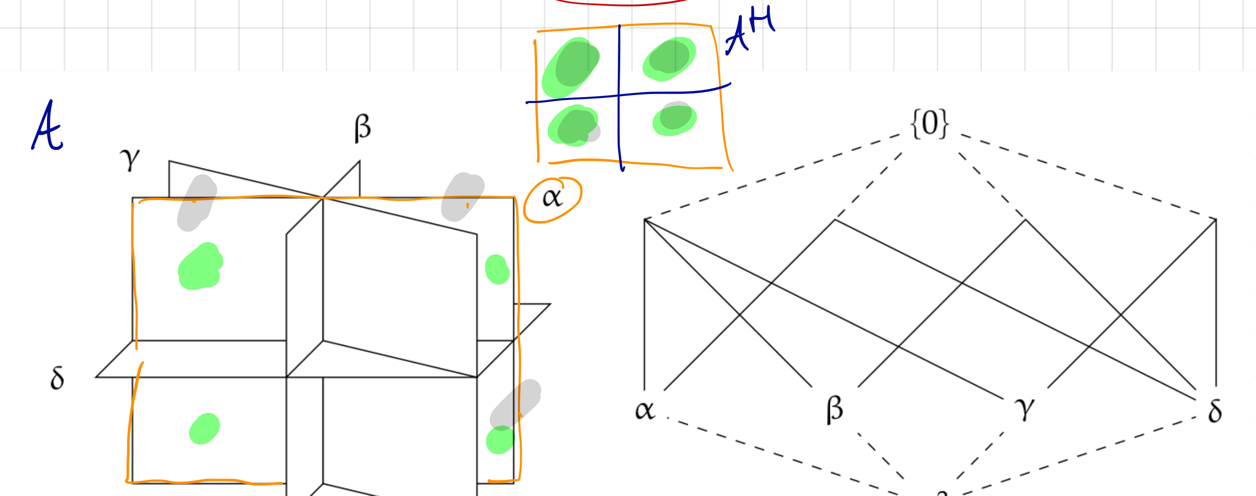
"arrgt. generated by $\mathcal{A}\setminus\{H\}$",

"arrgt. induced on H"

$\boxed{\mathcal{L}(\mathcal{A}_H) \sim \text{deletion}}$



Recall: want to count $|\mathcal{R}(\mathcal{A})|$



$|\mathcal{R}(\mathcal{A})| = 12$

By removing α, some regions are "merged"

Passing from $\mathcal{A}_H$ to $\mathcal{A}$ such regions are "cut" again – and those cuts correspond exactly to regions in $\mathcal{A}^H$!

Thus. $$|\mathcal{R}(\mathcal{A})| = |\mathcal{R}(\mathcal{A}_H)| + |\mathcal{R}(\mathcal{A}^H)|$$

If $A = \{H\}$, then $\boxed{|R(A)| = 2}$ every time when $\mathcal{L}(A) = \begin{array}{c} \hat{1} = \{H\} \\ | \\ \hat{0} \end{array}$

$\hookrightarrow r_A = r_I$

$\Rightarrow |R(A)|$ only depends on the isomorphism class of $\mathcal{L}(A)$.

In other words:

$$|R(A)| = f(r_A), \text{ where}$$

$f : \boxed{\text{Matroids}} \longrightarrow \mathbb{Z}$ is a function that is

(*) • constant on every set of matroids with isom. lattice of flats

$$\begin{tikzcd} \text{Mat} \arrow[r, "f"] \arrow[d, "\mathcal{L}_r"'] & \mathbb{Z} \\ \text{Geom. Lat} \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

(**) • Satisfies $f(r) = f(r_{/e}) + f(r_{\setminus e})$

• $f(r_I) = 2$ ① $f(r_L) = ? = 0$

• (III) Exercise.

$\hookrightarrow$ Def. 2.3.5!

Philosophy: adding loops does not change isom. class of $\mathcal{L}$

For ①: Notice that any f with (*) & (**) must have $f(r_L) = 0$.

Proof: Let $r : 2^{\{a,b\}} \to \mathbb{N}$, $r(\emptyset) = 0$, $r(a) = r(b) = r(a,b) = 1$

$\mathcal{L}_r$: $\begin{array}{c} \{a,b\} \\ | \\ \emptyset \end{array}$

Then $r_{/a}(\emptyset) = 0$, $r_{/a}(b) = 0$, so $r_{/a} = r_L$

$r_{\setminus a}(\emptyset) = 0$, $r_{\setminus a}(b) = 1$, so $\mathcal{L}_{r \setminus a}$: $\begin{array}{c} \{b\} \\ | \\ \emptyset \end{array}$

$$\underbrace{f(r) = f(r_{\setminus a})}_{=} + f(\underbrace{r_{/a}}_{r_L}) \Rightarrow f(r_L) = 0$$

In summary: our f satisfies the defin. of a Tutte-Grothendieck invariant with values in $\mathbb{Z}$, and with $f(r_1) = 2$, $f(r_2) = 0$.

By universality:

$\sim T(1-\underset{q}{t}, 0)$

$$\boxed{|R(\mathcal{A})| = T_{r_{\mathcal{A}}}(2,0)}$$

Illustration: "Graphic arrangements"

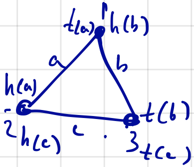
GRAPHIC ARRANGEMENTS

Let G be loopless graph on edge-set E, vertex set V.

The graphic arrangement $\mathcal{A}_G$ in $\mathbb{R}^V$ is

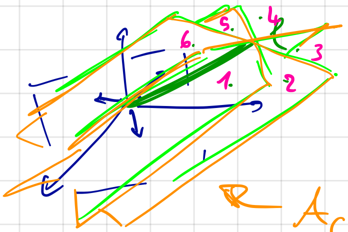
$$\mathcal{A}_G = \{H_e \mid e \in E\} \quad \text{with} \quad H_e = n_e^\perp, \quad n_e = (0 \cdots 0, \overset{\text{at position } h(e)}{1}, 0 \cdots 0, \overset{\text{at position } t(e)}{-1}, 0 \cdots 0)$$

Example: K_3



in $\mathbb{R}^3$

$$\left.\begin{aligned} n_a &= (-1, 1, 0) \\ n_b &= (1, 0, -1) \\ n_c &= (0, 1, -1) \end{aligned}\right\} \begin{array}{l}\text{all in} \\ \text{plane} \\ x+y+z=0\end{array}$$



Note: line $t\begin{pmatrix}1\\1\\1\end{pmatrix}$ in $n_a^\perp \cap n_b^\perp \cap n_c^\perp$

$\mathbb{R}$ — $\mathcal{A}_G$

We know that r_{dep} = "graphic" rank function" from chapter 1

Moreover: the chromatic polynomial of G is

$$\chi_G(t) = (-1)^{r(E)} t^{c(G)} T_G(1-t, 0)$$

↳ done with $r_{dep} \equiv r_{lat}$

Therefore $|\mathcal{R}(\mathcal{A}_G)| = T_G(2, 0) = |\chi_G(-1)|$

before

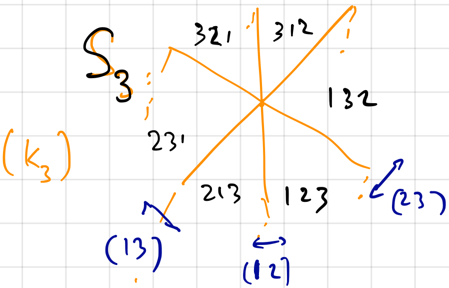
$$\chi_{K_3}(t) = t(t-1)(t-2), \quad \chi_{K_3}(-1) = -6$$

$-1 \quad -2 \quad -3$

In general, for $G = K_n$ complete graph we will have

$\chi_{K_n}(t) = t(t-1)\cdots(t-n)$, so $|R(A_{K_n})| = n!$

<u>Note</u>: in this case the hyperplanes of $A_G = A_{K_n}$ are exactly the reflecting hyperplanes of the (standard repres. of the) Coxeter group of type A_{n-1} ---> i.e. Permut. group S_n



S_3 acts freely & transitively on the regions, so $|R(A)| = |S_3|$

Coxeter theory.

QUESTION:

CAN WE DO THIS FOR OTHER FINITE COXETER TYPES?